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Y MATEMATICAS

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Matemáticas y Sociedad

10-21 de marzo de 2003

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Programa

Módulo 1: Matemáticas y Sociedad

10-21 de marzo, 18:00-20:00 horas

Coordinadores: *José Barrios García, María Isabel Marrero Rodríguez*

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Luis Balbuena Castellano

Catedrático de Matemáticas del IES "Viera y Clavijo" de La Laguna y miembro fundador de la Sociedad Canaria "Isaac Newton" de Profesores de Matemáticas

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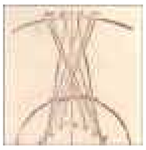

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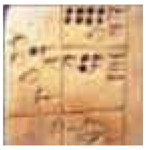

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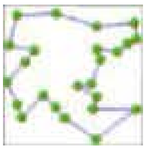

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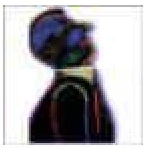

La proyección social de las Matemáticas

Mesa redonda

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Alfredo Bermúdez de Castro

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Jorge Casas Pérez

Técnico de "General Electric" y alumno de la Facultad de Matemáticas de la Universidad de La Laguna

Álvaro Dávila González

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Catedrático de Análisis Matemático de la Universidad Autónoma de Madrid y Director Gerente de Consultoría de Riesgos e I+D de "Tecnología, Información y Finanzas" (Grupo Analistas)

Laureano González Vega

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Contactos de las Matemáticas con la Sociedad

Luis Balbuena Castellano

Catedrático de Matemáticas del IES "Viera y Clavijo" de La Laguna
Miembro fundador de la Sociedad Canaria "Isaac Newton" de Profesores de Matemáticas

1. Introducción

Me parece que el tema escogido para el ciclo es muy oportuno porque la ciencia y la tecnología son siempre actualidad y porque es bueno que hablemos de estos asuntos en todo tiempo y lugar. Además, es importante que hablen los científicos, principalmente por dos razones. Por una parte, porque hay mucho aventurero de la palabra que habla de la ciencia y la tecnología sin saber bien de qué está hablando. Y, por otro lado, porque los científicos tienden con frecuencia a vivir en unas urnas de cristal que contienen bibliotecas y laboratorios dentro; están así felices y pierden el contacto con lo que hay fuera de su urna.

No quiero empezar mi intervención con definiciones y clarificaciones conceptuales porque temo no ser preciso y, por tanto, deslizar incorrecciones. Creo que casi todos tenemos unas ideas más o menos claras para dilucidar si un asunto es estrictamente científico o estrictamente técnico, pero estoy seguro de que también existe una especie de tierra de nadie en la que es difícil establecer con nitidez las fronteras entre una y otra. La historia parece que ha otorgado a la ciencia el "saber por qué" y a la técnica el "saber hacer", pero el aspecto eminentemente empírico de la técnica ha proporcionado elementos de observación para la construcción de la ciencia, y también la elaboración de teorías científicas ha proporcionado herramientas de gran potencialidad al desarrollo de la técnica. Hay quienes van más allá y afirman que una y otra han creado un nuevo producto de conocimiento, que es la tecnología, que viene a ser una especie de síntesis por cuanto que el tecnólogo, opinan, emprende investigaciones y aplica tanto los conocimientos científicos como la experiencia técnica de que dispone.

Pero no quiero seguir por ese camino porque en el ciclo intervienen personas cualificadas para clarificar todos estos términos y explicarnos las distintas teorías existentes.

El contenido de este ciclo es casi el título de una de las optativas que se han venido ofertando a los alumnos del Bachillerato surgido de la LOGSE. Supongo que la mayoría o la totalidad de los que estamos ligados a la ciencia o a la tecnología estaremos de acuerdo con la presencia de esta opción dentro del plan de estudios. Esto significa, entre otras cosas, el reconocimiento de la importancia que tienen estas áreas en la formación de las personas. No creo que sea necesario insistir en esto.

La ciencia y la tecnología suelen deslumbrar a la mayor parte de los ciudadanos no científicos y, en cierta manera, también a los científicos. Bien es verdad que el avance ha sido tan rápido y tan espectacular que la capacidad de asombro se ha ido llenando con la misma rapidez, y pocas veces ya nos emocionamos con lo nuevo. En efecto, lo último en lo que sea ya no nos lleva a hacer exclamaciones de asombro ni a tener apreciaciones ciertamente pintorescas y cercanas algunas veces a la mitología, como ocurría no hace mucho tiempo. Por ejemplo, cuando las vías férreas empezaron a invadir los campos llevando gente y mercancías de un lugar a otro, hubo en Inglaterra psiquiatras que advirtieron de los peligros de esas ruidosas máquinas y de los riesgos que conllevaría estar mucho tiempo sometido a enloquecedoras velocidades de más de 40 km/h. Poco menos que en cada estación habría que montar un servicio de urgencia para atender a quienes llegaran con los síntomas y unas ambulancias para evacuar a los más por el exceso de velocidad.

A principios del siglo XX se produjo un acontecimiento tecnológico que deslumbraba a la humanidad de manera especial porque, además, había sido algo en lo que siempre se soñó. Desde el mítico Dédalo hasta aquel momento de cambio de siglo, pasando por el genio de Leonardo da Vinci, hubo muchos intentos y se derrochó mucha fantasía en torno a ese objetivo. Me refiero a volar, a despegarse del suelo para desplazarse de un lugar a otro a capricho, esto es, dirigiendo el vuelo hasta posarse de nuevo sin peligrar la integridad del que lo hace. En ese momento, esa capacidad de levantar vuelo es lo que produce el asombro, la admiración y la veneración de la ciudadanía por la ciencia y la tecnología. Es el gran logro cuyo mérito se aplica a los hermanos Wrigth, Orville y Wilbur, dos norteamericanos que, aunque recibieron educación superior, no llegaron a licenciarse. Poseían un gran talento para la mecánica y desde jóvenes se interesaron por la aeronáutica. Empezaron montando un taller en el que vendían, reparaban e incluso llegaban a fabricar bicicletas. Esto les proporcionó medios para poder dedicarse a fondo al tema que les atraía. Y como suele ocurrir en casi todos los casos, empezaron a estudiar lo que ya otros habían elaborado y avanzado hasta ese momento. En este caso, estudiaron las obras de los ingenieros Lilienthal, Chanute y Langley, lo que nos permite afirmar, una vez más, que siempre hay hombros de gigantes en los que subirse para poder seguir avanzando.

El impacto de la aviación fue tal, que los futuristas llegaron a predecir avances realmente espectaculares. Pero siempre considerando el lado amable del invento, sus aportaciones a la calidad de vida de los ciudadanos y con un convencimiento en que se seguiría adelante con más y más inventos y, por tanto, con una progresiva mejora de esa calidad de vida.

Pero toda esa fe y toda la admiración de la sociedad en general por la ciencia y la tecnología sufrieron una fractura cuando en pocos segundos se fue capaz de matar de modo directo a más de cien mil personas en Hiroshima, o cuando el DDT empezó a dañar de manera irreversible a suelos y animales. La sociedad empezó entonces a dotarse de instrumentos para defenderse de los ataques que recibía de los intereses que empezaron a explotar la ciencia y la tecnología. Así, por ejemplo, en 1969 nació Greenpeace. Por si fuera poco el problema de la desconfianza, hay otro frente que la aumenta y es que el desarrollo tecnológico es uno de los criterios que se valoran a la hora de medir el grado de desarrollo de un país. La ONU publica con cierta periodicidad una especie de ranking utilizando para elaborarlo unos criterios que tienen que ver con la calidad de vida. La última de estas listas salió el mes de julio del pasado año. ¡Qué enormes abismos existen entre los primeros y los últimos! Pero al decir los últimos no hay que pensar que se trata de los diez del final. Desgraciadamente, se pueden considerar con ese calificativo desde más arriba de la mitad…

2. La cultura científica y su divulgación

(Voy a intentar penetrar en campos en los que me desenvuelvo mejor y en los que puedo ofrecerles experiencias personales y conclusiones a las que he podido llegar).

Se sigue considerando como algo normal que la cultura permanezca ligada casi en exclusiva a los conocimientos relacionados con los que comúnmente se llaman humanísticos. No parece que los conocimientos científicos o tecnológicos deban formar parte del perfil de persona culta, que acepta, sin más, no saber nada de Isaac Newton o de los fundamentos de la química, y no digamos nada cuando se trata de conocimientos de matemáticas. En este caso, alardear de no saber nada, a veces, parece como un mérito añadido.

Quiero aclarar, antes de seguir y para que no se me malinterprete, que no estoy tratando de decir que haya que sustituir unos conocimientos por otros. En absoluto. Se trata de un "además de", y no de un "en lugar de".

Se vienen haciendo grandes esfuerzos para que la cultura científica se difunda y pase a formar parte del bagaje cotidiano de conocimientos de los ciudadanos. Creo que aquí es donde hay que buscar gran parte de las causas de ese desconocimiento y despego de la sociedad por la ciencia, por la técnica y por lo que representa. El investigador estudia y logra resultados que raramente vende al gran público porque, desde mi punto de vista, en esa cadena de la comunicación hay una especie de eslabón perdido. Y ese eslabón lo forman los llamados comunicadores. La ciencia se comunica en muy pequeñas dosis, y no siempre de la forma adecuada. Es una evidencia de nuestra cultura actual que lo que no se comunica, lo que no aparece en los medios de comunicación, no existe. Incluso podría afinarse un poco más ese axioma estableciendo categorías entre los medios, pues no es lo mismo que lo que se quiere comunicar aparezca sólo en los periódicos o que lo haga en la televisión.

Hay que indicar, no obstante, que algunos campos del saber científico se abren camino en el mundo de la comunicación de tal forma que están permitiendo que personas no especialistas hablen de ello en conversaciones cotidianas, comparables a cuando hablan de arte o de literatura. Es el caso de ciertos documentales que ofrecen algunas cadenas de televisión, centrados especialmente en aspectos relacionados con las ciencias naturales (incluida la medicina). La sociedad se acerca así a las ciencias. Pero no todas tienen el mismo tratamiento. En efecto, una intuitiva ordenación las colocaría así: tras las ciencias naturales, se situaría la física, asociada sobre todo a la astrofísica; algo más lejos, la química y la geología; y a una distancia sensible, las matemáticas.

En el mes de marzo de 1999 se celebró en Granada un Congreso bajo el título "Comunicar la ciencia en el siglo XXI". Se presentaron ciento setenta comunicaciones, de las cuales sólo tres están dedicadas explícitamente a divulgación matemática, dos de ellas centradas en el reparto de escaños en unas elecciones. El dato es para preocuparse y demuestra, por si alguien no lo había palpado aún, el enorme abandono que tienen las matemáticas por parte de los que comunican ciencia. Quizá los que enseñamos esta disciplina deberíamos revisar nuestro rol y pensar en si no deberíamos incluir en él una parcela de divulgadores. Al fin y al cabo, muchísimos de nuestros alumnos y alumnas sólo tienen contacto con las matemáticas a través de nuestras enseñanzas. El dato que les he apuntado me dejó una honda preocupación, hasta el punto de estimularme a pasar a la acción.

Con motivo de la celebración del 2000 como Año Mundial de las Matemáticas, se celebró en la Universidad de Verano de Adeje un curso sobre "Las Matemáticas y el Periodismo". Los diferentes expertos que pasaron por aquella tribuna declararon insistentemente la dificultad de transmitir matemáticas. Nombraron también la falta de objetivos institucionales consistentes en acercar la ciencia a la ciudadanía, salvo la creación de Museos de la Ciencia que, afortunadamente, empiezan a cubrir el vacío casi total que ha existido. Manuel Calvo Hernando, presidente de la Asociación de Periodistas Científicos y uno de los ponentes del citado curso, ha publicado un decálogo del divulgador de la ciencia en el que señala como punto de partida que la misión del divulgador es "poner al alcance de la mayoría el patrimonio científico de la minoría". A nadie se le oculta lo difícil que es llevar a la práctica este objetivo, sobre todo en algunos campos del saber científico. En otro de los mandamientos de su decálogo indica que "frente a tanto temor y tanta desconfianza parece necesario humanizar la ciencia al presentarla al público, y situarla entre nosotros de modo entrañable y cordial, sin por ello restarle seriedad y trascendencia".

Este Congreso me indicó que deberíamos hacer un esfuerzo para conseguir divulgar el conocimiento matemático más allá de lo que puede conseguirse en las aulas dentro de los currículos de las enseñanzas regladas. Esto es lo que hemos hecho siempre, y ya ven con qué resultados. Por esta razón he tratado de desarrollar algunos proyectos con ese objetivo, y es lo que voy a explicar sucintamente en lo que sigue. Quiero aclarar que no me considero un especialista en la divulgación científica ni he acudido a escuela alguna para ello. Tan sólo he tratado de aprovechar las oportunidades que me han ofrecido para poner mi granito de arena en la divulgación de las matemáticas, lanzándome a ello con sólo mis intuiciones y los consejos de amigos, no todos periodistas.

3. Contactos con la sociedad a través de las matemáticas

3.1. Radio

Hace unos años el hoy director de COPE Tenerife, D. José Carlos Marrero, me sugirió la idea de preparar un programa de radio en el que tratar temas relacionados con las matemáticas. La posibilidad me resultó sugerente, pero le pedí tiempo para reflexionarlo y preparar guiones que pudieran tener interés y coherencia. Tras varias conversaciones con él, quien como especialista en la comunicación me orientó con maestría, y utilizando por mi parte el consabido método de ensayo y error, logré lanzar por las antenas un programa semanal de media hora de duración que titulamos "Un Sorbito de Ciencia". Me pareció excesivamente arriesgado que el sorbito fuese sólo de matemáticas (por eso lo de "ciencia"), pues tenía la impresión de que era demasiado tiempo para dedicarlo sólo a matemáticas. El reto propuesto salió por fin a los aires y tuvo una buena aceptación, por cuanto que las llamadas y felicitaciones llegaban de sitios y personas muy diversas. Fue una favorable acogida que nos sorprendió.

El esquema del programa consistía en lo siguiente: tras la presentación de las personas que hablaríamos y de comentar aquellas noticias de carácter científico producidas durante la semana, se proponía a los oyentes un acertijo para ser resuelto durante el tiempo que durara el programa. Se trataba de sencillos problemas de matemática recreativa del estilo de: "Si un tapón y su botella cuestan 1.10 euros y la botella es 1 euro más cara que el tapón, ¿cuánto cuesta cada una?"; o este otro: "Si en un cubo de un metro de lado caben mil litros de agua, ¿cuántos litros caben en un cubo que tenga medio metro de lado?". Cada uno de los acertijos y de los problemas planteados van acompañados de explicaciones y repeticiones suficientes como para ser comprendidos por los oyentes. Precisamente, una de las limitaciones importantes que me he encontrado es la necesidad de encontrar cuestiones en las que la imagen no fuera imprescindible.

A continuación se entrevistaba, durante unos diez minutos, a alguna persona relacionada con aspectos de la ciencia, tratando de acercar a los oyentes el trabajo y las investigaciones que realizan esas personas a las que llamamos científicos en centros de Canarias. Así, por ejemplo, fueron entrevistados D. Antonio González González, D. Francisco Sánchez (Director del IAC), D. José Méndez (catedrático de Análisis Matemático), Dª Marisa Tejedor (catedrática de Edafología), D. Manuel Ibáñez (especialista en la construcción de relojes no mecánicos), etc.

La sección llamada "El Problema de la Semana" creó cierta expectación, porque se centraba en proponer un problema un poco más complejo que los acertijos, que requería una discusión y una reflexión para llegar a la solución, la cual se daba y explicaba en el programa siguiente. Quedaba, por tanto, planteado durante una semana, y en muchas ocasiones me llegaron a llamar conocidos o llamaban oyentes a la emisora para saber si la solución a la que habían llegado era correcta. En el momento de dar la respuesta a estas cuestiones, y también en la de los acertijos, aprovechaba la oportunidad para lanzar mensajes en los que aconsejaba a los oyentes compartir la resolución del problema con otras personas, haciéndoles ver que la solución se conseguía aportando razonamientos para convencer a los demás y no utilizando el deplorable método de chillar más que los otros, que aparece en ocasiones en ciertas tertulias y programas que vemos u oímos en medios de comunicación. Como ya he indicado, la imposibilidad de presentar imágenes constituía una de las dificultades mayores a la hora de preparar los guiones, ya que sus textos tenían que ser suficientemente claros y no tener muchos datos para que se pudieran retener o apuntar con notas simples, y sin posibilidad de hacer figuras. He aquí algunos:

"Un vendedor de huevos hace su primera venta dando al cliente la mitad de los huevos que lleva en su cesta más medio huevo. Al segundo le vende la mitad de los huevos que le quedan más medio huevo. Con el tercero y con el cuarto hace lo mismo. Al final se quedó sin huevos. La cuestión que se plantea es: ¿con cuántos huevos empezó la venta? Está de más aclarar que no rompió ningún huevo para hacer este reparto".

"Ángel tarda tres horas en terminar un informe mecanográfico en su oficina, mientras que su compañera Begoña sólo tarda dos en hacer el mismo trabajo. El jefe decide que los dos realicen el mismo informe distribuyéndose adecuadamente las hojas a mecanografiar. ¿Cuánto tiempo tardarán en hacerlo?"

Se insistía mucho en la necesidad de ser constantes y perseverantes, pues del trabajo en esos problemas se empezarían a obtener frutos cuando se acumulasen estrategias, se propusiesen problemas parecidos, etc. La sección llegó a causar cierta "adicción" entre los más curiosos.

Tras esta propuesta de trabajo para la semana, se explicaban curiosidades relacionadas generalmente con las matemáticas. Se explicaban situaciones de la vida cotidiana en las que se hace uso de las matemáticas sin que, en muchas ocasiones, haya conciencia de ello.

El programa terminaba con un ranking que trataba de hacer palpable la importancia de la ciencia en la historia de la humanidad o en nuestra vida cotidiana. En una de las ediciones utilicé la lista de los 50 primeros personajes que figuran en el libro de Michael H. Hart titulado "Los 100 principales". Cada día nombraba a dos empezando en el 50º para terminar en el primero. Lo que me indujo a utilizar esta lista es que de esos 50 primeros personajes, 24 están relacionados con la ciencia o la tecnología. Así que explicaba por qué el autor del libro le atribuía el número y nombraba sus méritos. En otra edición del programa utilicé una lista de los 50 inventos de todas las épocas que más impactaron a un conjunto de cerca de 300 personas que fueron entrevistadas sobre ese particular con la ayuda de mis alumnos. Debo aclarar que en esta edición pude contar con la colaboración de mis alumnos del Taller de Matemáticas que, en grupos de dos o tres, asistían al estudio conmigo y participaban en la exposición de los temas. El trabajo desarrollado lo sinteticé en una memoria que presenté al premio "Francisco Giner de los Ríos", concretamente a la XVI edición que convoca el Ministerio de Educación y financia la Fundación Argentaria, logrando el primer premio de aquel año.

3.2. Televisión

Cierto día, el director de la emisora de televisión "Canal 7 del Atlántico", D. Francisco Padrón, me ofreció la oportunidad de preparar programas para ser emitidos por su emisora. Esto me parecieron ya "palabras mayores", porque mi inexperiencia y desconocimiento del medio eran totales. Tardé un tiempo en dar la respuesta afirmativa; lo hice cuando hube preparado unos guiones que él consideró que estaban bien. Empezó la grabación en los estudios de la calle

Numancia. No podría contar con imágenes exteriores, porque se trata de una emisora que maneja un humilde presupuesto; así que tendría que suplirlas a base de materiales que llevaba al estudio. Grabé una primera serie de quince programas bajo el título genérico de "2πR". Intentaba dar a conocer las matemáticas que se encuentran de manera cotidiana en el entorno de los telespectadores. No era mi intención impartir clases de matemáticas en sentido académico, sino divulgar ideas, conceptos y curiosidades con aquel objetivo.

3.3. Prensa escrita

También he intentado la divulgación de las matemáticas a través de la prensa escrita. En el curso 1979/80 publicamos en los periódicos "El Día" y "La Provincia" un suplemento semanal que titulamos "Números y figuras", el cual tuvo una notable acogida. La experiencia se repitió a lo largo del 2000, Año Mundial de las Matemáticas, sólo que esta vez se extendió durante todo el año.

3.4. En el aula

El aula es también un lugar idóneo para la divulgación de las matemáticas. Hay que tener en cuenta que, tal y como se ha indicado, la sociedad carece de divulgadores de la ciencia y la mayor parte de los alumnos sólo recibirán las enseñanzas científicas que les impartan sus profesores. Por eso me cuestiono si el profesor no debería de considerar este rol dentro de lo que es su misión de enseñar. Además, se da la circunstancia de que la mayoría de los libros de texto no han sido, hasta ahora, excesivamente proclives a la divulgación de la ciencia, pese a que en las intenciones de los creadores de los currículos se acuda con frecuencia a esa especie de tópico de que la ciencia hay que construirla partiendo de la realidad cotidiana de los alumnos. Un claro ejemplo de lo que trato de explicar lo constituyen las cónicas. Cuando tratan de poner un ejemplo de elipse, casi siempre se acude al movimiento de los planetas alrededor del Sol como un "ejemplo cotidiano", como algo que cualquiera "puede ver". Y no sé por qué no se ponen como ejemplos de hipérbolas las que posiblemente los alumnos vean todas las noches en su casa con la luz que sale de las lámparas situadas cerca de una pared.

En una ocasión llevé a mi aula una experiencia relatada por el profesor Ismael Roldán en la revista SUMA. Se trataba de hacer una "cata de leches". La idea me pareció original y válida para inducir otros estudios parecidos y puse los medios necesarios para reproducirla. Se abrió un debate interesante sobre la forma de desarrollar la experiencia y las medidas y productos que hay que prever. Aparecen inmediatamente una buena cantidad de elementos matemáticos, empezando por el cálculo de la leche que es necesario adquirir para desarrollar la cata. Acordamos adquirir cinco leches de cinco clases diferentes. Una de ellas era leche natural, aunque este detalle lo desconocían los catadores. Se establecieron dos valoraciones entre 1 y 5 puntos. Una mediría la intensidad del sabor y la otra la calidad global.

Como había dos alumnos a quienes no les gustaba la leche, monté todo el dispositivo con ellos. Serían servidas ocultando las más mínimas señales de identificación. Se procedió a la cata, que fue seguida con curiosidad y seriedad por todos. Las hojas de valoraciones pasaron luego a los equipos que se formaron para hacer el vaciado y tratar de sacar conclusiones sobre lo que se había hecho. En esta parte estuvo lo más interesante de la experiencia. Porque, además de los datos extraídos de las valoraciones realizadas por cada catador, se tuvieron en cuenta los parámetros que figuran en los envases de cada una de ellas relativas al valor energético, proteínas, hidratos de carbono y grasas. Así que cada equipo le adjudicó un valor de ponderación a cada uno de esos ingredientes y se construyó el "polinomio de valoración" de cada leche. Al final, cuando todas las fórmulas estuvieron preparadas, se efectuó la valoración y fueron ordenadas de mejor a peor según esos criterios. La última de la lista fue la natural…

Existen muchos materiales y situaciones cotidianas que, en general, no son utilizados por el sistema. Ocurre también que en el entorno cotidiano aparecen situaciones y elementos que tienen trasfondo matemático y el sistema no proporciona los medios para que se puedan interpretar matemáticamente. Es el caso, por ejemplo, de las celosías que se encuentran, generalmente, rematando muros de jardines, de azoteas, etc. Es muy posible que el libro de texto pida calcular el área de un círculo de ocho metros de diámetro, que raramente habrá visto nadie, y no piden que se calcule el área de la zona de luz de una pieza de celosía que tal vez el alumno tenga en su propia casa.

Y así podríamos seguir enumerando ejemplos de materiales e ideas para ser utilizados en el aula.

4. Conclusiones

Todas estas "aventuras" divulgativas fueron desarrolladas sin pretender grandes objetivos y contando con medios técnicos y humanos muy limitados. Pero me permiten extraer algunas conclusiones y enseñanzas, a saber:

- Se ha difundido "otra cara" de las matemáticas, a la que, en general, se considera como una ciencia cerrada, estrictamente académica y rígida.
- Se tiene la posibilidad de establecer vínculos de las matemáticas con otras disciplinas y, de esta manera, mostrar el carácter globalizador del conocimiento.
- La divulgación de las matemáticas es posible, porque no es imprescindible acudir a complicaciones teóricas para difundir conceptos con rigor y claridad.
- Se consigue desarrollar y mejorar las capacidades de razonamiento lógico-matemático.
- Los medios pueden y deben colaborar a aumentar la llamada "cultura científica" de los ciudadanos, procurando diversificar lo que se ofrece.

A modo de conclusión, creo que he tratado de demostrar que la divulgación de la ciencia es necesaria si queremos conocer mejor lo que nos rodea, y que las matemáticas son la "cenicienta" en esa cadena de la divulgación. La ausencia del eslabón que debería existir entre los científicos y la sociedad, que son los comunicadores, hace que los profesores nos planteemos la necesidad de cubrir el hueco, al menos en nuestras aulas. Por otra parte, con las experiencias que he relatado, trato de demostrar que esa comunicación es posible incluso manejando pocos medios. Es, por tanto, responsabilidad de todos conseguir acercar las matemáticas a los ciudadanos.

Pero también las instituciones deberían incluir entre sus objetivos de carácter cultural el acercar la ciencia a los ciudadanos, no sólo promoviendo actividades con ese objetivo, sino creando entre su personal especialistas en transmitir la ciencia, en adecuar los conocimientos y las investigaciones para ser comprendidos por el ciudadano medio.

En ese sentido, hay que reconocer el interesante papel que están jugando los Museos de la Ciencia que se vienen creando en muchas ciudades.

Bibliografía

F. Alonso et al. (editores): *Aportaciones al debate sobre las Matemáticas en los 90. Simposio de Valencia.* Mestral Libros, Valencia, 1987.

L. Balbuena Castellano, D. de la Coba García: *La matemática recreativa vista por los alumnos.* Proyecto Sur Ediciones, Granada, 1992.

S. Garfunkel: *Las matemáticas en la vida cotidiana.* Addison-Wesley/Universidad Autónoma de Madrid, Madrid, 1998.

M.I. González García, J.A. López Cerezo, J.L. Luján López: *Ciencia, tecnología y sociedad.* Tecnos, Madrid, 1996.

M. Hart: *The 100, a ranking of the most influential persons in history.* Simon & Schuster, London, 1993.

G. Howson, B. Wilson (editores): *Las Matemáticas en Primaria y Secundaria en la década de los 90. Kuwait, 1986*. Mestral Libros, Valencia, 1987.

A. Martinón (editor): *Las matemáticas del siglo XX: Una mirada en 101 artículos.* Sociedad "Isaac Newton" y Nivola, Madrid, 2000.

L. Moledo: *De las tortugas a las estrellas.* AZ Editora, Buenos Aires, 1994.

En Internet

http://www.sinewton.org
Sociedad Canaria "Isaac Newton" de Profesores de Matemáticas

Las Matemáticas y la Cultura: Matemáticas, Arte y Ciencia en los comienzos de la Revolución Científica

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Introducción

En el año de gracia de 1543, en dos ciudades del centro de Europa, Nuremberg y Basilea, distantes apenas 300 kilómetros, se publicaron dos libros que jugaron un papel tan fundamental en el desarrollo de la ciencia que es frecuente situar en ese momento el comienzo de la Revolución Científica. El primero de ellos se titulaba *De Revolutionibus Orbium Coelestium Libri Sex* y pretendía demostrar matemáticamente que, contra toda apariencia, la esfera de las estrellas está inmóvil y el Sol ocupa el centro del universo, mientras la Tierra gira a su alrededor como los demás planetas, además de girar diariamente en torno a sí misma, todo lo cual explicaría perfectamente los movimientos que vemos cuando miramos al cielo. El segundo, titulado *De Humani Corporis Fabrica Libri Septem*, presenta un estudio completo de la anatomía humana, su organización articulada y la correspondencia entre su estructura interna y la apariencia externa, manteniendo que cada una de sus descripciones se verifica en el cadáver humano y defendiendo que la disección y observación cuidadosa de cuerpos humanos es la única fuente válida de conocimiento anatómico.

Uno cambiaba decisivamente el curso de la más abstracta y matemática de las ciencias, la astronomía, no tanto introduciendo nuevos datos o descubrimientos observacionales, cuanto desarrollando hasta sus últimas consecuencias una interpretación alternativa de datos ya conocidos. El otro modificaba sustancialmente la más descriptiva de las ciencias, la anatomía, instaurando una nueva metodología asociada a nuevos métodos de enseñanza y a una reivindicación académica, profesional y cultural de la disciplina, lo que a su vez dio origen a un cúmulo de nuevos datos y descubrimientos observacionales. Fue un capricho del destino que los dos libros coincidieran en el tiempo y acabaran conduciendo a una nueva concepción del macrocosmos y del microcosmos y de las relaciones entre ellos, porque ambos libros, su gestación, sus objetivos inmediatos, la audiencia a la que iban dirigidos e incluso sus avatares posteriores fueron completamente independientes. Difícilmente podrían encontrarse en la historia de la ciencia dos libros más distintos escritos por dos autores con caracteres y trayectorias vitales más diferentes.

Copérnico

En efecto, el primero de ellos, dedicado al Papa Pablo III, era obra de Nicolás Copérnico (1473-1543), un circunspecto clérigo de la lejana Frauenburg con 70 años cumplidos que, tras haber estudiado durante su juventud Astronomía, Leyes y Medicina en Bolonia, Padua y Ferrara, llevaba cuarenta años en los confines del mundo ejerciendo como médico y secretario de su tío el obispo, cumpliendo sus funciones de canónigo catedralicio y llevando a cabo tareas diplomáticas ante los belicosos Caballeros Teutónicos. Durante todos esos años tuvo tiempo para publicar una traducción de las epístolas morales de Theophylactus de Simocatta, escribir un pequeño ensayo sobre la inflación y las funestas consecuencias de acuñar nuevas monedas disminuyendo el porcentaje de metal precioso e incluso defender la corrección de los datos observacionales de Ptolomeo ante la pretensión de Werner de introducir una nueva esfera o dotar de un nuevo movimiento a la esfera de la eclíptica para salvar el problema de la precesión de los equinoccios[1], pero sobre todo se dedicó intensamente a trabajar en la obra que le daría fama universal, el *De Revolutionibus*.

Según sus propias palabras, cuando la obra se publicó en 1543 hacía "más de cuatro veces nueve años" que tenía el libro preparado, o al menos su núcleo fundamental, y había pasado todo ese tiempo puliendo los detalles, precisando y modificando los cálculos y resistiéndose a su publicación, pese a las presiones de influyentes amigos, por miedo a la recepción que sus ideas pudieran tener por parte de los lectores y a que fueran malinterpretadas o no fueran comprendidas por quienes no eran matemáticos.

Independientemente de la conexión que estas reservas pudieran tener con una posible influencia pitagórica, con su convicción purista de que las matemáticas se escriben para los matemáticos, y no para el público en general, o con su compromiso explícito acerca de la naturaleza estrictamente matemática de la astronomía[2], lo cierto es que recogen también su

[1] La traducción de las *Epístolas morales* de Theophylactus es de 1509, la *Carta contra Wener* (*De octava sphaera* o *Carta a Wapowski*) de 1524 y el *Tratado de la moneda* (*Monetae cudendae ratio*) de 1526. También en este periodo es uno de los astrónomos consultados por el Papa acerca de la reforma del calendario juliano, aunque la respuesta de Copérnico fue desalentadora: difícilmente se podía intentar una reforma efectiva del calendario, ya que los astrónomos ni siquiera tenían clara la duración del año natural.

[2] El supuesto pitagorismo de Copérnico ha sido muy discutido, como también sus posibles relaciones neoplatónicas y herméticas. Aparte de la militancia neoplatónica de su antiguo profesor de astronomía, Domenico Maria de Novara, el principal argumento en su favor se basa en las leves referencias a los pitagóricos que habían defendido la movilidad de la Tierra (Filolao, Ecfanto, etc), a las fuentes clásicas que utiliza Copérnico para referirse a ellos (fundamentalmente *Academica priora* de Cicerón y *De placitis philosophorum* de Plutarco, ambos textos de influencia pitagórica), a sus comentarios místico-metafóricos sobre el Sol y, sobre todo, a la *Carta de Lysis a Hiparco* (no confundir con el astrónomo del mismo nombre), una carta apócrifa entre dos pitagóricos, en la que el primero reprocha al segundo haber hecho públicos los secretos del grupo y lo exhorta a guardar silencio y no divulgarlos. Esta carta se encuentra en el manuscrito original como cierre del libro I del *De Revolutionibus,* inmediatamente después de los comentarios de Copérnico sobre el triple movimiento de la Tierra en el capítulo 11, para justificar la poca difusión que los pitagóricos dieron a su creencia en la movilidad de la Tierra. La carta, sin embargo, fue retirada de la edición impresa del libro, siendo sustituida por los capítulos 12 y 13 dedicados a la trigonometría (en el manuscrito original de Copérnico aparece tachada), por lo que nunca fue publicada. En relación con este tema es sorprendente el silencio de Copérnico sobre Aristarco de Samos en lo que respecta al movimiento de la Tierra y la localización del Sol en el centro. En todo el libro sólo cita a Aristarco tres veces en el libro III en relación a temas técnicos menores: la oblicuidad de la eclíptica, la precesión de los equinoccios y la duración del año natural. Sin embargo, Copérnico tenía que saber que Aristarco había propuesto un sistema heliocéntrico, porque lo cuenta Plutarco en otro de sus escritos, *De facie in orbe Luna*, donde describe un eclipse de Luna acontecido en torno al año 71 d.C., que está incluido en sus *Moralia* junto con el *De placitis*. Curiosamente, Copérnico citaba a Aristarco en el breve párrafo en que hacía la introducción a la *Carta de Lysis,* diciendo: "Por estas y otras causas semejantes, es probable que Filolao se hubiera dado cuenta de la movilidad de la Tierra: respecto a lo cual algunos dicen que Aristarco de Samos era de la misma opinión", y luego continúa hablando del silencio pitagórico sobre las cosas que no pueden ser comprendidas por el vulgo. El párrafo fue retirado junto con la *Carta*. Una pregunta inevitable es ¿por qué no citó a Aristarco junto con los demás (Nicetas, Filolao, Ecfanto, Heráclides) en la dedicatoria al Papa? ¿y por qué no lo cita nunca al hablar del heliocentrismo y el

desconfianza hacia la imprenta recién inventada, hacia su enorme potencial no sólo como elemento divulgador del conocimiento científico, sino como fuente de convicción. Una prueba de ello es que, en una fecha indeterminada entre 1507 y 1514, no había tenido escrúpulo en dar a conocer un breve opúsculo manuscrito, el *Commentariolus*[3], copias del cual circularon de mano en mano entre matemáticos y otros interesados en el tema, en el que expone el núcleo fundamental de su teoría, pero deja fuera el desarrollo matemático que promete presentar detalladamente en un libro que sería, a la postre, el *De Revolutionibus*. En el *Commentariolus* estaría el origen de la información que tanto los amigos y admiradores de Copérnico (Gemma Frisius, el obispo T. Giese, el Cardenal Schönberg) como sus 'enemigos' (Lutero, Melanchton) tuvieron de la teoría copernicana antes de la publicación del *De Revolutionibus*[4]. Incluso parece

movimiento de la Tierra? ¿desconocía realmente la propuesta de Aristarco o prefirió no citar a un precursor demasiado parecido?

[3] El título completo es *N. Copernici de hypothesibus motuum coelestium a se constitutis commentariolus*. Un curioso detalle, que puede observarse ya en el título, es que está escrito en tercera persona, lo que redunda en la idea de la prevención de Copérnico a hacer públicas sus ideas. Grandes astrónomos de la época, como G. Frisius, T. Brahe, etc., tuvieron copia del manuscrito. Aunque en el *Commentariolus* no incluye datos técnicos, ya deja claros dos de los principales argumentos de Copérnico a favor de la movilidad de la Tierra: la representación del movimiento de los planetas como uniforme, dejando de lado el ecuante ptolemaico que tan artificioso y repugnante a la razón le parecía a Copérnico, y la relación entre el orden y distancia de los planetas y la duración de sus revoluciones, que permitía situar las órbitas de Mercurio y Venus en su lugar natural sin recurrir a decisiones arbitrarias (lo que había levantado quejas no sólo contra la astronomía, sino incluso contra la entonces influyente astrología, como en el caso de Pico della Mirandola que en sus *Disputationes adversus astrologiam divinatricem* de 1496 comentaba que los juicios de los astrólogos no eran creíbles porque ni siquiera estaban de acuerdo en la posición que ocupaban Mercurio y Venus; como es sabido la solución final la daría la tercera ley de Kepler).

[4] Los personajes citados no son cualesquiera. Gemma Frisius era uno de los científicos más renombrados de la época y estaba al servicio del emperador Carlos V; Tiedeman Giese era obispo de Kulm y en 1536 había escrito una pequeña obra, el *Hiperaspisticon*, en la que elogiaba a Copérnico; y el Cardenal Schönberg era nada menos que general de los dominicos, lo que lo convertía indirectamente en responsable último de la Inquisición. Por su parte, el núcleo duro del protestantismo, radicado en Wittenberg, conoció la teoría a través del *Commentariolus* y su recepción fue más burlona que agresiva. Así, Lutero comenta en sus *Diálogos de sobremesa* (*Tischreden*) de 1539: "Se habla de un nuevo astrólogo que pretende demostrar que la Tierra se mueve y que gira en círculo, en lugar de hacerlo el cielo, el Sol y la Luna, exactamente como si alguien que viajara en un vehículo o barco sostuviera que él está sentado e inmóvil en tanto que los campos y árboles se mueven. Pero así son las cosas hoy día: cuando un hombre desea ser más avisado, tiene que inventar algo especial, y la manera en que lo hace tiene que ser la mejor. Ese necio desea trastocar todo el arte de la astronomía de arriba abajo. Sin embargo, como nos dicen las Sagradas Escrituras, Josué mandó al Sol que se detuviera y no a la Tierra". En este párrafo aparece por primera vez el famoso texto de las Escrituras que sería utilizado hasta la saciedad por la Iglesia Católica para condenar el copernicanismo en 1616 y que obligaría a Galileo a darle una retorcida y refinada interpretación en la *Carta a Cristina de Lorena* para intentar eliminar la acusación (cosa que no consiguió, pues él mismo fue condenado en 1632). Por su parte, Melanchton, amigo y seguidor de Lutero, humanista y profesor en Wittenberg, también rechazó el heliocentrismo copernicano en términos semejantes a los de Lutero, pero aceptó sus desarrollos técnicos dando pie a la llamada 'interpretación de Wittenberg': que la teoría copernicana era simplemente una buena hipótesis de trabajo para mejorar los cálculos y como tal debía usarse, aunque su afirmación fundamental sobre el movimiento de la Tierra no debía considerarse verdadera. Esta interpretación es la que se recoge en el prólogo de Osiander y se puede encontrar también tras las *Tablas Prusianas* de Reinhold, calculadas utilizando la teoría de Copérnico. La reacción luterana, no obstante, no pasó de ahí y la mejor prueba es que Rheticus, luterano, alumno y protegido de Melanchton y profesor de matemáticas en Wittenberg desde 1536, no tuvo ningún impedimento para trasladarse a Frauenburg y acabar siendo el único discípulo

haber llegado a oídos del Papa Clemente VII, que en 1533 pedía a Widmanstadt que se lo explicara, e igualmente fue la causa de que Rheticus, un joven profesor de matemáticas nada menos que en la muy protestante universidad de Wittenberg, se trasladara a Frauenburg para convertirse en el único discípulo que tuvo Copérnico.

Pese a todo, Copérnico no publicó su libro hasta después que Rheticus publicara en 1540 la *Narratio Prima*[5], donde exponía la teoría copernicana, e incluso entonces se desentendió de la edición del libro hasta tal punto que la obra apareció con una carta al lector a modo de prólogo en la que se contradecía no ya el espíritu, sino las afirmaciones expresas del autor en la dedicatoria al Papa Pablo III que iba a continuación. Incluso es posible que el título inicial, simplemente *De Revolutionibus*, fuera sustituido por el mucho más correcto políticamente *De Revolutionibus Orbium Coelestium* y la impresión del libro no se llevó a cabo a partir del manuscrito original de Copérnico, sino de una copia que poseía Rheticus. En efecto, Copérnico delegó en Rheticus todo lo concerniente a la impresión y éste, teniendo que incorporarse a su nuevo puesto en la universidad de Leipzig, dejó a cargo de la edición a Andreas Osiander, polémico teólogo luterano que acabaría siendo expulsado de Nuremberg como sospechoso de inclinaciones papistas, el cual escribió por su cuenta la polémica carta al

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que tuvo Copérnico durante toda su vida e incluso acometer la edición del *De Revolutionibus*. Por el contrario, la reacción católica fue mucho más tardía, aunque también mucho más agresiva. Todavía a finales del siglo XVI, Clavius, el famoso astrónomo jesuita director del Colegio Romano, mostraba sus simpatías hacia la teoría copernicana entendida como modelo matemático, en línea con la interpretación de Wittenberg, aunque acabó siendo el impulsor de la aceptación 'oficial' de la teoría mixta de Tycho Brahe como sustituta de la teoría de Ptolomeo. No sería hasta que Galileo comenzó sus ataques al aristotelismo apoyándose en la teoría copernicana que la respuesta de la Iglesia Católica se volvió violenta, primero los dominicos y luego los jesuitas, hasta culminar en la condena de Galileo, que no sería revocada hasta el s. XX.

[5] El título completo es *De libri revolutionum N. Copernici narratio prima*. A lo largo de toda la exposición, Rheticus se presenta como discípulo y defensor de Copérnico, aunque se había formado en Wittenberg y pertenecía al grupo de Melanchton, Peucer, etc… Rheticus escribió también una defensa de Copérnico para salvar la contradicción con las Sagradas Escrituras, defensa que desgraciadamente se ha perdido. En cualquier caso, parece que tal defensa iba dirigida más a los protestantes, especialmente al círculo de Wittenberg al que Rheticus pertenecía, que a la Iglesia Católica, cuya reacción violenta no tuvo lugar hasta mucho después con la condena del copernicanismo en 1613. Sorprendentemente, Copérnico no hace ni una sóla referencia a Rheticus en su dedicatoria al Papa, pese a que cita al resto de sus amigos, quizá porque Rheticus era protestante o porque pensaba que un ayudante no merecía ser citado. Extraña forma de agradecerle que se hubiera encargado de la edición del libro o que hubiera defendido el copernicanismo frente a su protector, el poderoso Melanchton. Hay otros dos detalles interesantes en la *Narratio Prima*. El primero es que parece indicar que el título del libro de Copérnico sería simplemente *De Revolutionibus* (*De libri revolutionum* dice el título de la *Narratio Prima*), sin especificar si quien gira es la Tierra o las esferas celestes, de ahí la carga de la acusación a Osiander de haber cambiado el título en la imprenta para hacerlo políticamente correcto (el título definitivo impuesto por Osiander sugiere que se mueven las esferas celestes y la Tierra permanece fija, al menos para quienes no estuvieran al corriente de la teoría copernicana). La segunda es que incluye un capítulo dedicado a la astrología y a las mejoras en la precisión de los horóscopos que introduciría la teoría de Copérnico, destacando la peculiar interpretación que hace Rheticus del nacimiento y decadencia cíclicos de los grandes imperios, fenómenos que estarían relacionados con los momentos de máxima proximidad y máximo alejamiento entre la Tierra y el Sol, los cuales a su vez dependerían del movimiento excéntrico achacado por Copérnico a la Tierra en *el De Revolutionibus*. De ahí que a la excéntrica de la Tierra se la llamara la "rueda de la fortuna". Un tercer detalle, más anecdótico, es que la *Narratio prima* es uno de los primeros textos en que aparece la metáfora del universo como un reloj y Dios como el relojero que tan popular se haría en épocas posteriores.

lector y la incluyó sin firma en el libro. Quiere la leyenda, encarnada en T. Giese, que Copérnico recibiera el libro instantes antes de morir y fuera testigo de la superchería. Ni las airadas protestas de Giese, ni la rabia de Rheticus lograron que el editor quitara el prólogo de Osiander o identificara públicamente al autor, de manera que la mayoría de los lectores lo consideraron original del propio Copérnico hasta que Kepler en la *Astronomia Nova* de 1609 descubrió el fraude e identificó a su autor.

Vesalio

Por su parte, el segundo libro, titulado *De humani corporis fabrica libri septem* y dedicado al emperador Carlos V, es completamente distinto. Su autor era un joven y ambicioso profesor de la universidad de Padua de apenas 28 años llamado Andreas Vesalio (1514-1564), antiguo estudiante de medicina en las muy tradicionales universidades de Lovaina y París, que se había hecho popular por sus métodos de enseñanza poco ortodoxos, especialmente su obsesión en realizar disecciones por su propia mano al mismo tiempo que impartía la clase y describía detalladamente las operaciones que iba realizando[6].

Su descaro y su interés en la disección como prueba empírica habían quedado patentes desde su época de estudiante en París, donde no dudaba en ofrecerse como ayudante voluntario para llevar a cabo la disección mientras el profesor dictaba la clase[7], y una vez en Padua llevó a cabo su primera disección pública dos días después de recibir allí el grado de doctor en 1537. De la misma manera, dejó constancia de su audacia sin límites y de su habilidad para vender su imagen desde esas mismas fechas, pues nada más terminar sus estudios en París y antes de incorporarse a Padua escribió un *Comentario sobre Rhazes* (*Paraphrasis in nonum librum Rhazae*), publicado poco después en Basilea, en el que según dice pretendía "comparar cuidadosamente la terapia de los árabes con la de los griegos", pero que dado su casi nulo conocimiento del árabe consistía simplemente en una ligera corrección estilística de la antigua traducción latina de la obra del famoso médico árabe y en identificar las drogas y remedios citados en términos de la farmacopea de la época. Más atrevida aún fue su publicación en 1538 de una edición de las *Institutiones anatomicae* de su antiguo profesor en París, Günther de Adernach, sin permiso del autor, al que ni cita, y corrigiendo lo que él consideraba erróneo por su cuenta y riesgo.

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[6] El sistema tradicional consistía en que el profesor impartía la clase sentado en la cátedra leyendo o comentando algún texto mientras la disección era llevada a cabo por los prosectores u ostensores o, en el mejor de los casos, por algún estudiante ayudante. Esto hacía que, con frecuencia, lo que el profesor iba contando tuviera poco que ver con lo que el encargado de la disección estaba haciendo. En el prefacio al De Fabrica, Vesalio critica ácidamente esa costumbre e insiste en que el único método de enseñanza razonable consiste que sea el propio profesor quien lleve a cabo la disección al tiempo que imparte la clase. La propuesta era realmente revolucionaria porque, como es sabido, los médicos de la época no solían ejercer la cirugía, que era llevada a cabo por los cirujanos-barberos, una profesión reconocida gremialmente cuyos practicantes la adquirían mediante el sistema de aprendices y sin estudiar en la universidad. En algún caso, como el de la universidad de París, el ingreso en la facultad de medicina incluía el juramento de no dedicarse a la cirugía.

[7] Poco dado a la modestia, el propio Vesalio lo cuenta en el prefacio al *De Fabrica*: "En la tercera disección en la que estuve presente y a requerimiento de mis compañeros estudiantes y de los profesores, la llevé a cabo en público y de manera más completa de lo que era habitual" y en la página 538 se recrea en los detalles diciendo que el cadáver era de "una prostituta de hermosa figura y en la flor de la vida, que se había ahorcado y con la que hice mi primera disección en una anatomía pública". El profesor era Günther de Adernach.

Curiosamente, tanto en esta, como en todas sus publicaciones anteriores al *De Fabrica*, Vesalio sigue respetuosamente los planteamientos de Galeno, sea en sus obras dedicadas a la docencia, como las láminas de las *Tabulae sex* (o *Tabulae anatomicae*) de 1538, una innovación pedagógica fundamental pero en la que siguen apareciendo supuestos galénicos tan importantes como la 'rete mirabile' o el hígado con cinco lóbulos[8], sea en sus obras más teóricas, como la *Carta sobre la venesección* de 1539, en la que se involucra en la disputa sobre la sangría entre la escuela 'musulmana' y la nueva escuela griega, por la que toma partido[9]. Incluso en 1541 participa en la edición de Giunta de las obras completas de Galeno encargándose de tres de sus obras: *Sobre la disección de los nervios*, *Sobre la disección de las venas y arterias* y *Sobre los procedimientos anatómicos*.

Era difícil imaginar que, mientras tanto, entre 1540 y 1542 Vesalio estaba trabajando intensamente en el *De Fabrica*, un libro con más de 700 páginas y 70 láminas con 200 ilustraciones que pretendía ser una descripción completa de la anatomía humana en la que todas las descripciones pudieran verificarse con referencia al cadáver humano, sea mediante disecciones directas, sea a través de las cuidadosas láminas incluidas en la obra, en lugar de la práctica habitual de Galeno consistente en diseccionar cadáveres de animales y extrapolar sus observaciones a la anatomía humana.

Esto implicaba otras muchas consecuencias que Vesalio desgrana prolijamente en el prólogo del libro: mostrar que Galeno estaba equivocado en sus descripciones y, sobre todo, en su método; defender que la anatomía tenía que basarse en la experiencia directa obtenida de la disección de seres humanos y no de animales; recuperar la disección como parte de la profesión médica; cambiar sustancialmente los métodos de enseñanza y, muy especialmente, dignificar la anatomía como ciencia y mostrar su importancia tanto para la práctica médica como para el conocimiento general del cuerpo humano. De ahí que el libro esté destinado no sólo a anatomistas, médicos o estudiantes, sino también a artistas preocupados por la representación naturalista y exacta del cuerpo humano y, en general, a todos aquellos interesados en conocer su estructura interna y su articulación como un todo. Por ello, Vesalio no dudó en publicar junto con el *De Fabrica* un breve compendio resumido del gran tratado, el *Epitome*, en el que las láminas juegan un papel aún más importante y que dedica al hijo del emperador, el entonces príncipe Felipe II. Para conseguir esta multiplicidad de objetivos Vesalio explotó exhaustivamente todos los recursos de la imprenta, incluyendo los más subliminales, como hace con las letras capitulares en las que aparecen *putti* y enanos llevando a cabo prácticas habitualmente atribuidas a los estudiantes de medicina, como robar cadáveres, hacer disecciones e incluso vivisecciones de animales, etc[10].

A diferencia de Copérnico, Vesalio fue consciente desde un primer momento del enorme potencial de la imprenta no sólo como medio de difusión, sino como instrumento de convicción mediante la combinación adecuada de textos e imágenes, que permitían integrar en un conjunto articulado argumentos lógicos, pruebas empíricas, sugerencias retóricas y elementos estéticos, dando lugar a un soberbio despliegue difícil de rechazar aunque no se

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[8] Las *Tabulae sex* son, como su nombre indica, seis grandes láminas destinadas a 'enseñar' anatomía a los cirujanos-barberos. Dibujadas por van Kalkar bajo la dirección de Vesalio, incluyen 3 diagramas del sistema vascular y tres esqueletos (de frente, de perfil y de espaldas).

[9] La polémica se refería a si la incisión para la sangría en los casos de pleuresía debía hacerse en el costado izquierdo o en el costado derecho. En el mismo texto Vesalio observa y comenta el problema de las válvulas en las venas, con lo que estuvo muy cerca de descubrir la circulación de la sangre.

[10] Estas actividades no eran nada extrañas. El propio Vesalio habla en la Carta sobre la raíz china de sus frecuentes 'excursiones' al Cementerio de los Inocentes de París durante su época de estudiante para buscar huesos y cómo robó su primer cadáver en 1536.

estuviera de acuerdo con él. Por ello su control de la edición fue absoluto y minucioso durante el año largo que duró, hasta el punto que no se sabe a ciencia cierta quien fue el autor de las ilustraciones, si Johannes van Kalkar, que ya había trabajado con Vesalio en las *Tabulae Sex*, o varias personas del taller de Tiziano.

En cualquier caso la supervisión de Vesalio fue tan estricta que pagó de su bolsillo las planchas grabadas en madera, envió instrucciones detalladas al editor con cada uno de los grabados y no tuvo escrúpulo en trasladarse él mismo a Basilea a finales de 1542, dejando de cobrar su salario de la universidad de Padua, hasta que el libro estuvo impreso en el verano de 1543. Todavía entonces esperó más de un mes hasta obtener una copia especialmente buena del *De Fabrica* y el *Epitome* para encuadernarla lujosamente en terciopelo rojo y llevarla personalmente al emperador. Nada más entregar el regalo fue nombrado médico de Carlos V y sólo volvió a Padua para reclamar el salario de los meses pasados en Basilea, dejando la universidad definitivamente para dedicarse a su nueva tarea como médico del emperador y posteriormente de su hijo Felipe II.

En contraste con la frenética actividad desarrollada entre 1537 y 1543, Vesalio sólo escribió dos pequeños trabajos durante el resto de su vida: uno, la *Carta sobre la raíz china* de 1546, dedicada a defender su libro contra los seguidores de Galeno y a narrar detalles autobiográficos, muchos de los cuales encierran venganzas personales, y otro, el *Examen de Fallopio*, en el que revisa y replica de manera harto amable para su carácter pendenciero a las *Observaciones anatómicas* de Fallopio.

Bibliografía

N. Copérnico: *Sobre las revoluciones de las esferas celestes.* Ed. Nacional.
N. Copérnico: *Commentariolus.* Alianza.
M.J. Crowe: *Theories of the world from Antiquity to the Copernican Revolution.* Dover.
A. Durero: *Geometrie.* Ed. du Seuil.
S.Y. Edgerton: The *Renaissance rediscovery of linear perspective*. Basic Books.
S.Y. Edgerton: *The heritage of Giotto's geometry*. Cornell University Press.
J.V. Field: *The invention of infinity*. Oxford University Press.
J.V. Field, J.J. Gray: *The geometrical work of G. Desargues*. Springer.
T. Hall: *La revolución científica*. Crítica.
P. Hamou: *La vision perspective*. Payot.
N.R. Hanson: *Conjeturas y constelaciones*. Alianza.
T.S. Kuhn: *La revolución copernicana*. Ariel.
C. O'Malley: *Andreas Vesalius of Brussels*. University of California Press.
C.M. Saunders, C. O'Malley: *Illustrations from the works of A. Vesalius of Brussels*. Dover.
A. Vesalio: *De humani corporis fabrica*. Edisa.

En Internet

http://www.bj.uj.edu.pl/bjmanus/revol/titlpg_e.html
Nicholas Copernicus: "De Revolutionibus"
Edición facsímil del manuscrito original de Copérnico.
http://webexhibits.org/calendars/year-text-Copernicus.html
Full text - Nicholas Copernicus, "De Revolutionibus"
Traducción inglesa del texto de Copérnico.

http://www.frombork.art.pl/Ang01.htm

Nicolaus Copernicus Museum in Frombork

Museo de Copérnico en su ciudad natal, con datos biográficos.

http://www.hps.cam.ac.uk/starry/starrymessenger.html

Starry Messenger

Página de historia de la astronomía hasta la Revolución Científica (biografías, problemas, instrumentos).

http://www.hao.ucar.edu/public/education/sp/images/derevolutionibus.html

Copernicus' De Revolutionibus

Breve historia de la física solar (esta es la parte dedicada a Copérnico).

http://www.dartmouth.edu/~matc/readers/renaissance.astro/0.intro.html

As the World Turned

Recepción del Copernicanismo (con textos muy curiosos de Dee, Bruno, etc.).

http://www.octavo.com/products/index.html

Octavo Products

Los libros de Copérnico y Vesalio están editados en CD por Octavo (como también Newton, Harvey, etc.). Esta es la página de la editorial, y pueden verse fragmentos de los libros.

http://vesalius.northwestern.edu

Andreas Vesalius' De Humani Corporis Fabrica

Excelente página dedicada a Vesalio de la Universidad Northwestern.

http://www.nlm.nih.gov/exhibition/dreamanatomy/da_intro.html

Dream Anatomy

Exposición de imágenes de anatomía, muchas contemporáneas de Vesalio (para comparar con las de su libro).

http://www.dartmouth.edu/~matc/math5.geometry/unit11/unit11.html

Geometry in Art and Architecture

Curso sobre Arte y Ciencia, especialmente matemáticas (esta es la parte de perspectiva) de la Universidad de Dartmouth.

http://www.crs4.it/Ars/arshtml/arstoc.html

The Art of Renaissance Science

Página muy interesante sobre Arte y Ciencia en el Renacimiento mantenida por J. Dauben, un historiador de la ciencia especializado en Galileo.

http://mathforum.org/sum95/math_and/perspective/perspect.html#discussion

Perspective Drawing

Breve discusión de la perspectiva con enlaces a obras pictóricas.

Naturaleza del conocimiento matemático y sus implicaciones en la Enseñanza de las Matemáticas en la Educación Secundaria

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Resumen

La integración de nuestro país en la Comunidad Europea planteó a nuestro Sistema Educativo nuevas demandas; entre otras, se encuentran los esfuerzos para mejorar la calidad de la enseñanza en todos sus niveles, la necesaria reforma de la Educación Secundaria para ampliar el período de enseñanza obligatoria hasta los dieciséis años, y la necesidad de que desaparezcan las distancias y desigualdades educativas debidas a causas sociales, culturales o económicas. Es dentro de este marco que las Matemáticas no deben aparecer sólo como una disciplina formal que se construye lejos de nosotros y de nuestros intereses, sino más bien como un lenguaje que se manifiesta en todas las formas de expresión humana y que emerge como un derecho cultural esencial para todos los sujetos de la sociedad.

Hablar de la Enseñanza de las Matemáticas es hablar de las Matemáticas como parte importante de la tarea docente. Conocer y dominar las Matemáticas es una condición necesaria para enseñarlas de forma adecuada, es decir, el conocimiento matemático debe constituir el punto de partida básico para empezar a hablar de los aspectos educativos. Muchas de las determinaciones didácticas que se adopten estarán condicionadas por las características de dicho conocimiento, el cual llega a imprimir al proceso educativo una serie de presupuestos peculiares y diferenciados de los que corresponden a otras disciplinas.

La Matemática constituye una disciplina multiforme, que tiene un uso plural, que se ha manifestado en la enseñanza, como señala Romberg (1991), con rasgos diferentes, dependiendo de las épocas y de los autores. Es, en general, considerada de formas diversas: conjunto de técnicas para aprobar un examen, cuerpo de conocimientos para ser aprendido, lenguaje específico con una notación particular, estudio de las estructuras lógicas subyacentes, juego artificial jugado por un matemático, construcción de modelos útiles en la ciencia, procedimientos de cálculo necesarios para aplicar el conocimiento... Lo importante no son los distintos aspectos de la Matemática en los que se puede o no incidir, sino el conocimiento de los elementos principales que conforman la disciplina, y hacer recaer la actividad matemática en el desarrollo de estos elementos principales.

La racionalidad de la Matemática no la podemos supeditar a la consistencia lógica de sus resultados expresados en un lenguaje formalizado. Su racionalidad es inseparable de la actividad matemática, de la conjetura, del ensayo, del error, de la construcción de lenguajes, de resultados susceptibles de completarse y mejorarse, ... La Matemática como empresa humana y racional se mueve entre dos posiciones: por un lado, su naturaleza histórica, que nos muestra la potencialidad de la creación humana; y, por otro, los objetos matemáticos, los elementos de esa cultura que llamamos culturización matemática, que nos permite hablar de descubrimiento.

Vemos cómo el lenguaje, como elemento mediador en la cultura matemática, nos va a permitir hablar a la vez de creación y descubrimiento.

Los problemas relativos a la Filosofía de la Matemática pueden ser abordados, en la actualidad, desde las dos grandes posiciones que han caracterizado la naturaleza del conocimiento matemático durante las distintas épocas: la *prescriptiva* (o *normativa*) y la *descriptiva* (o *naturalista*). La primera parte procede de una posición absolutista de la Matemática, y la segunda analiza el conocimiento matemático desde la práctica matemática y sus aspectos sociales. La relación entre la enseñanza de las Matemáticas y estos dos grandes enfoques en la Filosofía de la Matemática es una cuestión evidente (Ernest, 1994).

En esta ponencia se realiza una reflexión sobre la naturaleza de las Matemáticas en sus diferentes aspectos, así como las implicaciones que se derivan en relación con las propuestas curriculares para Matemáticas en la Educación Secundaria.

Bibliografía

P. Ernest: The philosophy of mathematics and the didactics of mathematics. En R. Biehler et al. (editores): *Didactics of Mathematics as a Scientific Discipline,* pp. 335-349. Kluwer, Dordrecht, 1994.

C. Cañón: *La matemática: Creación y descubrimiento.* Publicaciones de la Universidad Pontificia de Comillas, Madrid, 1993.

P.J. Davis, R. Hersh: *Experiencia matemática*. MEC-Labor, Madrid, 1988. [Título original: *The Mathematical Experience*. Birkhäuser, Boston, 1982].

P. Kitcher: *The Nature of Mathematical Knowledge*. Oxford University Press, Oxford, 1984.

I. Lakatos: *Matemáticas, ciencia y epistemología.* Alianza, Madrid, 1981.

S. Restivo: *Mathematics in Society and History*. Kluwer, Dordrecht, 1992.

T.A. Romberg: Características problemáticas del currículo escolar de matemáticas. *Revista de Educación* **294** (1991), 323-406.

UNIVERSIDAD DE LA LAGUNA

SOCIEDAD, CIENCIA, TECNOLOGÍA Y MATEMÁTICAS

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NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

Martín M. Socas

Hablar de la enseñanza de las Matemáticas es hablar de las Matemáticas como parte importante de la tarea docente.

Conocer y dominar las Matemáticas es una condición necesaria, para enseñarlas de forma adecuada.

Muchas de las determinaciones didácticas que se adopten estarán condicionadas por dicho conocimiento, el cual llega a imprimir al proceso educativo una serie de características peculiares y diferenciadas de las que corresponden a otras disciplinas.

Desde esta perspectiva, la profesión de docentes de Matemáticas, tiene un punto de partida ineludible: la Cultura Matemática.

El futuro profesor deberá entender el proceso de "Matematización de la Cultura" para adaptar y procesar su conocimiento teórico con el fin de ayudar a sus futuros alumnos a construir su propio conocimiento matemático.

La Matemática constituye de hecho una disciplina multiforme, que tiene un uso plural, que se ha manifestado en la enseñanza, con rasgos diferentes (Romberg, 1991).

Es considerada de formas diversas: conjunto de técnicas para aprobar un examen, cuerpo de conocimientos para ser aprendido, lenguaje específico con una notación particular, construcción de modelos útiles en la ciencia, procedimientos de cálculo necesarios para aplicar el conocimiento...

Lo que debemos resaltar son los elementos principales de la disciplina matemática y hacer recaer la actividad matemática en el desarrollo de estos elementos principales.

¿Cuáles son esos elementos principales de la disciplina Matemática? ¿Qué influencia han tenido en los currículos de matemáticas de las diferentes reformas educativas?

Responder a las preguntas anteriores es la intención de esta ponencia en la que se reflexiona sobre la naturaleza de las Matemáticas y se analiza sus implicaciones en la enseñanza de las Matemáticas en la Educación Secundaria tomando en consideración las diferentes reformas educativas que han tenido lugar en este país en los últimos treinta años:

Ley General de Educación (LGE, 1970), Ley Orgánica de Ordenación General del Sistema Educativo (LOGSE, 1990) y Ley Orgánica de Calidad de la Educación (LOCE, 2002).

Estructuramos la presentación en cuatro apartados.

- **Naturaleza del conocimiento matemático**
  (Tomamos como punto de partida el finalizado siglo XX y diferenciamos la primera y segunda mitad del mismo).
- **Los currículos de matemáticas en las diferentes reformas educativas: LGE, LOGSE y LOCE**
- **Calidad de las matemáticas que se estudian en la Educación Secundaria (12-18 años)**
  Se analizan y comparan los resultados obtenidos en Matemáticas por alumnos de 12, 13, 14, 15 y 18 años en distintas pruebas nacionales e internacionales.
- **Formación del profesorado de Matemáticas de Secundaria**

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO

Las escuelas que han caracterizado la naturaleza del conocimiento matemático se pueden organizar, en dos grandes grupos: prescriptiva (o normativa) y descriptiva (o naturalista) (Ernest 1994).

- **La concepción prescriptiva de las Matemáticas considera la tradición absolutista y el platonismo como corriente filosófica. El conocimiento matemático es fijo y objetivo y está constituido por verdades absolutas y representa el único sustento del conocimiento verdadero, base del conocimiento humano y de la racionalidad.**

- **La concepción descriptiva de las Matemáticas incluye en su análisis un aspecto importante del conocimiento matemático: la práctica matemática y sus aspectos sociales.**

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO

La racionalidad de la Matemática la podemos supeditar a la consistencia lógica de sus resultados, expresados en un lenguaje formalizado, o, por el contrario su racionalidad es inseparable de la actividad matemática, de la conjetura, del ensayo, del error, de la construcción de lenguajes, de resultados susceptibles de completarse y mejorarse...

En cualquier caso parece razonable aceptar que la Matemática como empresa humana y racional se mueve entre dos posiciones, la de su naturaleza histórica que nos muestra la potencialidad de la creación humana, y la de los objetos matemáticos, los elementos de esa cultura que llamamos culturización matemática, que nos permite hablar de descubrimiento. El lenguaje como elemento mediador en la cultura matemática nos va a permitir hablar a la vez de creación y descubrimiento.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: PRIMERA MITAD DEL SIGLO XX

Encontramos tres grandes enfoques: el logicista, el formalista y el intuicionista, que intentaron cimentar el edificio matemático y mostrar la racionalidad de esta disciplina.

El Logicismo tiene su origen en Leibniz: el conocimiento matemático es un conocimiento verdadero, los objetos de la Matemática son verdades necesarias y universales, y los principios lógicos juegan un papel determinante para fundamentar los resultados matemáticos. Este planteamiento racionalista y logicista encuentra su apoyo en Frege, quién en 1884 lo inicia como una escuela en busca de los fundamentos de la Matemática.

Las dificultades que surgen dentro de la escuela logicista, en relación a los fundamentos, llevan a dos grandes matemáticos, Hilbert y Brouwer, a realizar sendas propuestas a partir de los presupuestos de la filosofía kantiana, el formalismo y el intuicionismo, respectivamente.

La Matemática es para Hilbert (Formalismo) producto del pensamiento humano y constituye un juego desprovisto de significado y constituido por axiomas, definiciones, teoremas y fórmulas. Desde este punto de vista no tiene sentido hablar de la naturaleza de los objetos matemáticos, dado que no existen. Sólo hay reglas y cadenas de símbolos.

El planteamiento de Brouwer (Intuicionismo) es mostrar la exactitud de la Matemática con independencia del lenguaje y de la lógica. En relación con el logicismo su planteamiento es radicalmente distinto. Para los logicistas, la Matemática clásica no podía contener errores, sin embargo, para los intuicionistas ocurría todo lo contrario, lo que les lleva a reconstruir las Matemáticas desde su base, a partir del concepto de número natural.

Las consideraciones que sobre las Matemáticas hace el intuicionismo pueden enmarcarse tanto dentro del planteamiento prescriptivo como descriptivo.

Es prescriptivo en cuanto que trata de asegurar los fundamentos de las Matemáticas sobre una base constructiva.

Es descriptivo al reconocer la importancia humana de la actividad matemática en la construcción de las demostraciones y en la creación de los nuevos conocimientos.

Dos concepciones ontológicas:
- Las acciones de descubrir e inventar nos lleva en la actividad matemática a dos concepciones ontológicas diferentes. Platonismo: que supone aceptar que los objetos matemáticos y las relaciones entre ellos tienen un carácter objetivo, y Constructivismo, que por el contrario, dota de subjetividad a estos objetos y sus relaciones.
- Para Platón los objetos matemáticos no están en continuidad con los objetos sensibles, su existencia es independiente de ellos. Tampoco son producto del pensamiento humano. Los objetos matemáticos pertenecen a un tercer mundo de naturaleza diferente a los dos anteriores, Popper (1974).
- El trabajo del matemático platónico es un trabajo empirista, dado que no inventa sino que descubre los conceptos matemáticos. Utiliza para ello la percepción y la intuición matemática.

Dos concepciones ontológicas
El formalismo y el intuicionismo comparten el carácter exacto, independiente de toda experiencia, de las leyes matemáticas.
Lo que provoca la separación entre las dos escuelas es el papel que los formalistas otorgan a la lógica y al lenguaje en la actividad matemática y en la fundamentación de los resultados.

Al pensar en los objetos de la Matemática podemos: considerar el lenguaje en un nivel secundario en relación con los objetos (Intuicionista) o pensar que la objetividad de la Matemática está inseparablemente unida a su formulación lingüística (Formalista).

El formalismo mantiene una posición absolutista mientras el intuicionismo mantiene una posición relativista en relación con el conocimiento matemático.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

Reducir la actividad matemática a justificaciones lógicas expresadas en teoría de conjuntos e ignorando otros modos de expresión y otras formas de razonamiento, no han producido los resultados esperados.

Una vez abandonada la búsqueda de fundamentos para las Matemáticas:

- La filosofía de las matemáticas puede comenzarse de nuevo examinando las prácticas reales de los matemáticos y de los que usan las matemáticas.

- Si contemplamos la Matemática sin prejuicios, aparecen muchos hechos relevantes que los fundamentalistas ignoraron: demostraciones informales, desarrollo histórico, la posibilidad del error matemático, comunicación entre matemáticos, el uso de ordenadores en la matemática y muchos más,...Tymozcko (1986)

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

Es necesario una reconceptualización de la Filosofía de las Matemáticas (Ernest, 1989 y 1991), que dé respuestas a cuestiones como las siguientes:

¿Cuál es el propósito de las Matemáticas? ¿Qué papel posee el ser humano dentro de las Matemáticas? ¿Cómo el conocimiento subjetivo del individuo llega a ser el conocimiento objetivo de las Matemáticas? ¿Cómo se refleja la Historia en la Filosofía de las Matemáticas? ¿Cuál es la relación de las Matemáticas con las otras áreas de experiencia y el conocimiento humano? ¿Por qué las teorías probadas por la Matemática pura llegan a ser tan potentes y útiles en sus aplicaciones a la ciencia y a los problemas prácticos?

El análisis de todos estos factores, permitirá considerar, además de los problemas internos de las Matemáticas -ontológicos y epistemológicos- exclusivamente tratados por el absolutismo, los aspectos externos, como su historia, la génesis, su práctica, etc.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

- En la segunda mitad del siglo XX, surgen nuevas corrientes acerca de la naturaleza de las Matemáticas que recuperan las posiciones no absolutistas (Intuicionismo) de la primera mitad del siglo.

- Dentro de estas corrientes que contemplan las Matemáticas desde una perspectiva descriptiva o naturalista, se sitúan una serie de tendencias más modernas que surgen desde una visión falibilista de las Matemáticas y que contemplan las necesidades e implicaciones sociales de las matemáticas, y examinan críticamente la estructura del conocimiento matemático adquirido por el ser humano inmerso en la sociedad.

- Estas tendencias son: el empirismo, el cuasi-empirismo, el convencionalismo y el naturalismo.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

El empirismo tiene sus raíces en diferentes autores del los siglos XVII y XVIII, Locke, Berkeley y Hume entre otros. La idea central es conceder a la experiencia humana la validez exclusiva como fuente del conocimiento.

Representa la opción más extrema de la consideración descriptiva de las Matemáticas. Esta corriente filosófica admite una visión de la naturaleza de las Matemáticas que descansa sobre la consideración de que las verdades matemáticas son generalizaciones empíricas. Así, los conceptos matemáticos tienen orígenes empíricos y las verdades matemáticas se derivan de las observaciones del mundo físico. Sus justificaciones provienen también de estas observaciones.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

El cuasi-empirismo es una corriente, relativamente reciente, surge de la enérgica oposición de su fundador -Imre Lakatos- al Logicismo y Formalismo.

- Esta corriente filosófica incluye la dimensión histórica de las Matemáticas, a partir de la cual se puede mostrar por qué se desarrollaron los conceptos y resultados particulares de las Matemáticas (Lakatos, 1978, 1981).

- Tiene más importancia para esta corriente filosófica la Matemática informal y práctica que la formal o acabada, y considera que la dialéctica conjetura-refutación, así como el uso constante de contraejemplos, constituyen la clave para la elaboración de teorías matemáticas informales.

- Davis y Hersh (1988) aportan al cuasi-empirismo de Lakatos la naturaleza cultural de las Matemáticas, tanto a los aspectos internos como a los externos de la misma.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO: SEGUNDA MITAD DEL SIGLO VEINTE

El convencionalismo tiene como principal representante a Wittgenstein (1978, op. cit. en Ernest 1991), quién ofrece una importante visión social de las Matemáticas y considera que el conocimiento matemático y la verdad están basados en convenios lingüísticos; en particular, que las afirmaciones de la lógica y las Matemáticas son analíticas, verdaderas en virtud del significado de los términos que utilizan. Su contribución clave estriba en reconocer las bases sociales y subjetivas de la certidumbre.

El naturalismo: sitúa el análisis de la naturaleza del conocimiento matemático no en los sistemas formales, sino en la actividad humana, capaz de hacer frente a situaciones nuevas y de generar procedimientos y conceptos que permitan el avance.

El modelo evolutivo de Wilder (1981), concibe las Matemáticas como una construcción humana enraizada en las culturas diversas, que se ha desarrollado en ellas un sistema según el modelo antropológico de un sistema cultural.

La perspectiva realista y ecologista de Kitcher (1984), muestra las Matemáticas como algo complejo, no abordable desde los entramados formales de conceptos y sistemas de teorías; muestra en la actividad matemática el carácter racional de los cambios en el desarrollo histórico de las Matemáticas.

El constructivismo social

Postura filosófica sobre las Matemáticas, concebida con el fin de aglutinar las características esenciales de las corrientes filosóficas “sociales”. Pretende servir de base para la conceptualización de una filosofía de la Educación Matemática (Ernest, 1989, 1991).

El objetivo central está en la génesis del conocimiento matemático más que en su justificación.

Desde el punto de vista del constructivismo social, el desarrollo del nuevo conocimiento matemático y la comprensión subjetiva de las matemáticas se derivan del diálogo y las negociaciones interpersonales, esto es, hacer y aprender matemáticas deben surgir a partir de procesos similares.

Resumen:

- Los aspectos de racionalidad matemática que subyacen en la actividad matemática se conciben de formas diferentes en las dos grandes perspectivas adoptadas: la absolutista y la relativista.

- En la primera la racionalidad matemática surge como una propiedad de los sistemas formales.
- En la segunda se entiende como una propiedad de la empresa humana, y abre el horizonte de una racionalidad fuera de los ámbitos de la lógica formal y sustentada en la actividad de los matemáticos, en la historia y en el contexto socio-cultural.

- En el último cuarto del siglo XX, se ha desplazado el centro de interés desde las teorías matemáticas como productos acabados hacia la actividad matemática, entendida como una práctica social (Wittgenstein, 1987; Lakatos, 1981 y 1986; Davis y Hersh 1988; Ernest, 1991, 1994 y 1998).

Tres aspectos esenciales de la Matemática:

- La Matemática es un sistema conceptual lógicamente organizado y socialmente compartido. Esta organización lógica explica un gran número de dificultades y obstáculos en el aprendizaje.
- La Matemática es una actividad de resolución de problemas socialmente compartida. Problemas que pueden tener relación con el mundo natural o social o ser problemas internos de la propia disciplina. La respuesta a estos dos tipos de problemas explican la evolución y desarrollo progresivo de los objetos matemáticos (conceptos, teorías,...). La actividad de resolución de problemas es un proceso cognitivo complejo que ocasiona dificultades en el aprendizaje de la Matemática.
- La Matemática es un lenguaje simbólico característico y constituye un sistema de signos propios en el que se expresan los objetos matemáticos, los problemas y las soluciones encontradas. Como todo lenguaje, tiene funciones básicas y reglas de funcionamiento que dificultan el aprendizaje.

Los currículos de matemáticas en las diferentes reformas educativas: LGE (1970), LOGSE(1990) y LOCE (2002).

El Sistema Educativo español se ha caracterizado en los últimos treinta años por sucesivas reformas y cambios.

En todas las reformas, la Matemática aparece como una referencia obligada en el estudio y determinación de las finalidades de la educación en una etapa educativa.

Ahora bien, su carácter histórico y su consideración como un sistema de prácticas y de realizaciones conceptuales, ligadas a un contexto social e histórico concreto, son los elementos indispensables para este estudio y determinación de las finalidades de la educación matemática.

Las filosofías prescriptivas se reflejan en desarrollos de los currículos de matemáticas de algunos países.

Las matemáticas descansan en ciertos fundamentos -como, por ejemplo, la lógica- y ascienden desde la abstracción a la generalidad. La historia está separada del conocimiento matemático y de su justificación; el conocimiento matemático es un conocimiento puro y aislado que pasa a considerarse útil debido a su validez universal.

La enseñanza está basada en la transmisión de los conocimientos, considerando como básica la metáfora de la comunicación del conocimiento. El énfasis se pone en los contenidos y las dificultades que impiden un aprendizaje óptimo de los alumnos; surgen, de una pobre comprensión por parte de éstos, de los conocimientos que se le transmiten o de las exposiciones poco claras de los profesores.

Un ejemplo: el Curriculum Nacional Británico, organizado mediante jerarquías que sirven para clasificar a los alumnos en clases sociales, razas, etc.

Las filosofías descriptivas de la matemática tienen consecuencias didácticas.

La corriente intuicionista pone el énfasis en la exploración y resolución de problemas, la discusión de las tareas matemáticas, el desarrollo de investigaciones en las aulas, el respeto por las creaciones realizadas por los alumnos. Ernest (1991).

Los intuicionistas consideran que el alumno debe construir activamente sus significados, basándose en procesos constructivo y de conjetura, además de seguir considerando que existe un cuerpo correcto de conocimientos matemáticos que surgen de la construcción.

El papel del profesor es el de "facilitador" de la adquisición de los conocimientos y de "corrector" de las malas realizaciones de los alumnos.

Algunos aspectos negativos son: la excesiva protección de los alumnos, la ausencia de problemas relacionados con la vida real y extraídos del entorno social donde se desenvuelve el alumno.

El constructivismo social como concepción integradora señala que para llevar a cabo la enseñanza y aprendizaje de las matemáticas es necesario:

- Respetar tanto los conocimientos previos de los alumnos como los significados que adquieren.

- Construir el conocimiento a partir de los métodos que utilizan los alumnos, mediante una negociación.

- Considerar la inseparabilidad de las Matemáticas con sus aplicaciones y la importancia de la motivación y la relevancia.

Ernest (1994)

El currículo de matemáticas de la LGE (1970)

Es una reacción a: "unos cuestionarios de matemáticas que estaban descontextualizados, transmitían una matemática polvorienta e invertebrada".

Pretende:

Retomar "una de las funciones fundamentales de la Matemática que es la de ordenar conocimientos y crear estructuras formales que las resuman y expresen"...

Centrar "La enseñanza de las Matemáticas en todos los niveles en el proceso de matematización de problemas, creación de sistemas formales, utilización de las leyes de estos sistemas para obtener unos resultados e interpretación de los mismos..."

Opta por presentar la Matemática como:

- Una disciplina fuertemente organizada y sistemática, con un excesivo predominio de los aspectos formales y estructurales.

- Basada en una metodología adaptada del programa logicista.

En resumen:

El currículo de matemáticas de la LGE se fundamenta en el modelo tecnológico con tendencias conductistas sobre el aprendizaje, en el que lo esencial es la consecución de una serie de objetivos operativos y contenidos matemáticos susceptibles de ser observados y medidos.

Se trata de un currículo: Cerrado y Obligatorio, con contenidos fijos y que tienen finalidad en sí mismos.

La enseñanza – aprendizaje está centrada en el producto.

La evaluación está centrada en los resultados y se dirige especialmente a comprobar el nivel de adquisición de contenidos por parte de los alumnos.

...

En este marco de la LGE surgen en España muchos movimientos de innovación que formulan nuevas propuestas que pretenden superar algunos de los rasgos más significativos de los currículos de Matemáticas anteriores: fundamentación conductista del aprendizaje, valoración esencialista del conocimiento, autoridad indiscutible del profesor, objetividad de la evaluación mediante las Matemáticas y, por tanto, legitimidad de la selección social fundada en ellas; sin embargo, esta visión crítica no se logró transmitir del todo al Sistema Educativo.

Para solucionar el problema era preciso replantearse las finalidades del currículo de Matemáticas, ajustándolas a las necesidades del ciudadano y de la sociedad actual....

El currículo de Matemáticas de la LOGSE (1990)

La integración en la Comunidad Europea planteó a nuestro Sistema Educativo nuevas necesidades y demandas:

- **Esfuerzos para mejorar la calidad de la enseñanza**
- **Reforma de la Educación Secundaria (ESO y Bachillerato)**
- **Disminuir las desigualdades educativas, debidas a causas sociales, culturales o económicas.**

...

En este marco las Matemáticas además de una disciplina formal debe presentarse como un lenguaje que se manifiesta en todas las formas de expresión humana y como un derecho cultural esencial para todos los ciudadanos.

- El proceso de matematización de la cultura devuelve a la comunidad unas matemáticas que no son de ninguna manera ni propiedad, ni exclusividad de un sector o grupo cultural, es por ello, que la función tradicional asignada a las Matemáticas en el Sistema Educativo se modifica profundamente.

- El papel tradicional de las Matemáticas aparece cuestionado como instrumento para legitimar estatus sociales que generan divisiones entre el trabajo intelectual y manual.

- **Emerge la función formadora de la Matemática como un conocimiento básico compartido, al menos hasta los dieciséis años.**
- **Es en este contexto donde surgió el movimiento "Matemáticas para todos". Este movimiento tiene su origen en las reformas para la enseñanza de las Matemáticas emprendido por los Estados Unidos y Gran Bretaña en los años cincuenta, y que se extiende progresivamente a los demás países occidentales.**

Resumen:

- El currículo de Matemáticas, especialmente para la ESO, hace una apuesta decidida por el aprendizaje significativo de los alumnos, donde el constructivismo se convierte en el modelo de referencia curricular. La construcción de sus aprendizajes la realiza el alumno de una manera integrada desde tres tipos de contenidos: conceptos, procedimientos y actitudes.

- El currículo permite un grado máximo de apertura y flexibilidad, convirtiendo a la vez en obligatorios determinados objetivos y contenidos (currículo básico), preservando la atención a la diversidad de los alumnos, a sus diferencias y singularidades, y potenciando la evaluación formativa como instrumento para dinamizar el progreso de los alumnos, orientando y facilitando la construcción de nuevos aprendizajes a partir de los conocimientos previos.

- El currículo de Matemática propone el desarrollo de capacidades de orden superior como la identificación y resolución de problemas, el desarrollo del pensamiento crítico y el uso de estrategias de naturaleza metacognitiva.

- La evaluación se dirige además de comprobar el nivel de adquisición de contenidos por parte de los alumnos a analizar todos los elementos del currículo para armonizar su desarrollo (alumnos, centro, profesores, entorno, ...).

- Igualmente la metodología está organizada no sólo con la finalidad de optimizar la adquisición de contenidos sino que pretende conseguir situaciones significativas de aprendizaje y de comunicación, favoreciendo la creatividad y autonomía del alumno.

El currículo de matemáticas de la LOCE (2002)

Tomaremos como referencia estos currículos:

- *Real Decreto 3473/2000, de 29 de Diciembre*, por el que se establecen las enseñanzas mínimas correspondientes a la Educación Secundaria Obligatoria.

- *Real Decreto 3474/2000, de 29 de Diciembre*, por el se establecen las enseñanzas mínimas del Bachillerato.

Motivo:

Las Matemáticas de la Educación Secundaria se encuentran en una fase de cambio motivada, en parte, por las reacciones y reajustes que tienen lugar en la propia Matemática, y, en especial, como consecuencia directa del empuje innovador que ofrecen las herramientas informáticas y de la información, tan presentes en nuestra realidad más inmediata.

En esta propuesta destaca:

- Equiparación entre contenidos de conceptos y de procedimientos, los cuales han de ser tratados con suficiente rigor formal a lo largo de la Etapa, no así los contenidos de actitudes que se eliminan.

- Incorporar al Currículo de Matemáticas el uso de todos aquellos recursos tecnológicos (calculadoras y programas informáticos) adecuados para interpretar y para analizar situaciones diversas relacionadas con los números, con el álgebra lineal, con el análisis funcional o con la estadística.

- Adaptar los contenidos matemáticos a las necesidades que requieren otras materias para su desarrollo, especialmente del ámbito científico–tecnológico.

- Considerar la resolución de problemas como una práctica constante y paralela al proceso de enseñanza/aprendizaje, no como un Bloque de Contenidos.

Modificaciones que afectan a los objetivos:

Se eliminan objetivos relacionados con el desarrollo de la personalidad del alumno, la actitud crítica, y su afectividad e interés por la disciplina.

Por ejemplo los Objetivos

9): "Apreciar el desarrollo de las Matemáticas como un proceso cambiante y dinámico, íntimamente relacionado con otras ramas del saber, mostrando una actitud flexible y abierta ante las opiniones de los demás".

3): Elaborar juicios y formar criterios propios sobre fenómenos sociales y económicos, utilizando tratamientos matemáticos, y expresar críticamente opiniones, argumentos con precisión y rigor y aceptando las discrepancias y los puntos de vista diferentes.

Se sustituyen por otros vinculados al empleo de las nuevas tecnologías y vías de información.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

Las modificaciones que afectan a los contenidos:

- La resolución de problemas deja de ser considerada como un Bloque y pasa a ser tratada de forma transversal.

- En las Matemáticas I del Bachillerato algunos contenidos son suprimidos, manipulación de expresiones algebraicas básicas, notación científica, números racionales e irracionales, probabilidad compuesta, significado y uso de variables estadísticas discretas y continuas, e integrados en el Currículo de 4º de la ESO.

- Se introducen nuevos contenidos: aplicación del Método de Gauss a la resolución e interpretación de sistemas de ecuaciones lineales sencillos; se propone un tratamiento más profundo de la topología de la recta real.

- Se trasladan a Matemáticas I contenidos que antes se abordaban en el Currículo de Matemáticas II: derivada de una función, lugares geométricos en el plano y cónicas.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

Estas modificaciones pretenden incorporar al Currículo de Matemáticas aquellos conceptos y procedimientos que tienen más valor para la época actual. Resaltando los aspectos utilitarios de la matemática e incorporando los recientes avances tecnológicos.

A costa de sacrificar los aspectos formativos y formular una propuesta más deshumanizada del currículo, propiciando más el desarrollo de las capacidades cognitivas (donde se incluyen las que tienen que ver con el manejo de las nuevas tecnologías y vías de información) en detrimento de las afectivas, las relaciones interpersonales y las de actuación e inserción social, pese a ser estas últimas las que favorecen la autorrealización del sujeto.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

En resumen encontramos:

- Dos formas diferentes de entender las matemáticas corriculares: "como la apropiación de un saber constituido y acabado" o "como un proceso de construcción y de abstracción de relaciones, progresivamente más complejas, elaboradas en, y, a partir, de la actividad del alumno".

- Que el currículo de la LGE nos presenta unas Matemáticas formada por objetos ya construidos que hay que dominar.

- Que el currículo de la LOGSE nos presenta unas matemáticas que se configuran como una forma de pensamiento abierto, en el que se deja cierto margen a la creatividad personal fomentando su ejercitación individual.

- Que el currículo propuesto por la LOCE pretenden dar una visión de la matemática más acorde con la realidad actual, en la que las nuevas tecnologías tienen un protagonismo especial a costa de restar importancia a la adquisición de capacidades vinculadas al desarrollo personal del alumno.

LA ENSEÑANZA DE LA MATEMÁTICA EDUCACIÓN PRIMARIA

Calidad de las matemáticas que se estudian en la Educación Secundaria (12-18 años)

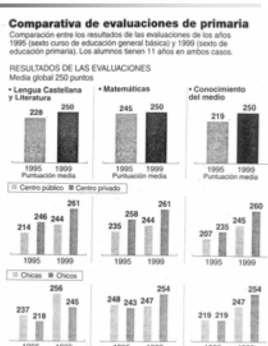
En relación con la Educación Primaria, se analizan y comparan los resultados obtenidos en Matemáticas por alumnos de doce años, sexto de EGB (INCE, 1995), y sexto de Primaria (INCE, 1999).

- INCE (1999). "Evaluación de la Educación Primaria. Fallos y dificultades de los alumnos en la Prueba de Matemáticas". http://www.ince.mec.es

- INCE (1995). "Evaluación de la Educación Primaria. Lo que aprenden los alumnos de 12 años".

http://www.ince.mec.es/prim/matenuop.htm

LA ENSEÑANZA DE LA MATEMÁTICA EDUCACIÓN PRIMARIA



LA ENSEÑANZA DE LA MATEMÁTICA EDUCACIÓN PRIMARIA

Las deficiencias que presentan estos alumnos en Matemáticas son muy notorias, como por ejemplo:

- El 50% de los alumnos considera que el número formado por 5 unidades, 6 decenas y 2 centenas es mayor que el formado por 3 centenas y 2 unidades. El 43% responde correctamente.

- Un 50% de alumnos tiene dificultades para trabajar con números decimales y con porcentajes, y casi las tres cuartas partes tienen dificultades para comprender el concepto de fracción y operar con fracciones.

- Más del 60% de los alumnos tiene dificultades para transformar tres unidades diferentes de una misma magnitud en una sola y realizar, posteriormente, una operación sencilla de suma, resta, multiplicación o división.

-Análogos niveles de dificultad se encuentran en los otros bloques de contenidos.

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN PRIMARIA

La comparación de las dos pruebas además de constatar nuevamente las dificultades que presentan nuestros alumnos en el aprendizaje de las Matemáticas en esta Etapa Educativa, nos muestra, sin embargo, que globalmente los alumnos de sexto de Primaria en 1999, mejoran en sus aciertos en Matemáticas en relación con los resultados que obtenían los alumnos de sexto de EGB en 1995, aunque esta mejora no es ciertamente significativa y supone de hecho pasar de un 58% de aciertos en la Primera prueba a un 59% de aciertos en la segunda.

Igualmente la comparación de estas pruebas muestra también que los centros públicos han experimentado una cierta mejoría en relación a los centros privados en las tres áreas objeto del estudio: Lengua Castellana y Literatura, Matemáticas y Conocimiento del Medio.

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA

- **Lapointe, A.E.; Mead, N.A. y Philips, G.W. (1989).** ***Un mundo de Diferencias. Un Estudio Internacional de Evaluación de las Matemáticas y las Ciencias.*** **Madrid: MEC-CIDE.**

- **López, J.A. y Moreno, M.L. (1997).** ***Resultados de Matemáticas. Tercer Estudio Internacional de Matemáticas y Ciencias*** **(TIMSS). MEC. Madrid.**

http://www.ince.mec.es/timss/completo.htm

- NCES (2001). ***Outcomes of Learning. Results From the 2000 Program for International Student Assessment of 15-Year-Olds in Reading, Mathematics, and Science Literacy.*** **PISA** 2000. http://nces.ed.gov/pubs2002/2002115.pdf

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
Un mundo de Diferencias. Un Estudio Internacional (1988)

Participan: 5 países (Corea, España, Estados Unidos, Irlanda y Reino Unido), y 4 provincias canadienses

Edad: 13 años

Muestra: 2000x12, de 100 escuelas

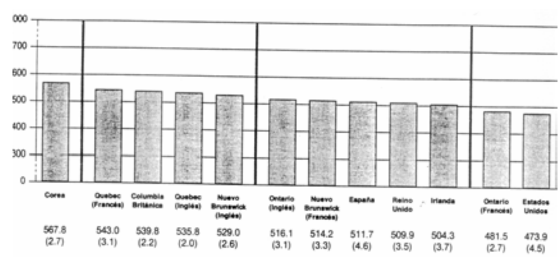
Prueba: 1986, National Assessment of Educational Progress (NAEP) de los Estados Unidos

Organismo: International Assessment of Educational Progress (IAEP):

2ª Evaluación con alumnos de 9 y 13 años (1991)

Habilidad Media en Matemáticas: Se expresa en una escala que va de 0 a 1000, con una media de 500 y una desviación típica de 100.

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA.
Un mundo de Diferencias. Un Estudio Internacional (1988)



* Las diferencias de rendimiento entre los cuatro grupos son estadísticamente significativas al nivel .05; las diferencias de rendimiento lentro de los grupos no son estadísticamente significativas. Los errores típicos objetidos por el método jackknife se muestran entre parénesis.

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA.
Un mundo de Diferencias. Un Estudio Internacional (1988)

España se sitúa en torno a la media 511.7 (4.6), frente a Corea que ocupa el punto más alto con 567.8 (2.7) y Estados Unidos se sitúa en el punto más bajo 473.9 (4.5).

En España el 99% de los alumnos de esta edad realiza operaciones de sumar y restar números naturales, el 91% realiza problemas simples de un sólo paso, el 57% problemas aritméticos de dos pasos y solo el 12% está en disposición de entender los conceptos propios de este nivel educativo (séptimo de EGB) (Lapointe, Mead y Philips, 1989).

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA.
TIMSS - 1995

Participan: 41 países

Duración del Estudio: 1991-1997

ESPAÑA

Edad: 7º y 8º (13 – 14 años)

Muestra: 7596= 3855+3741, de 153 Centros

Organismo: The International Association for the Evaluation of Educactional Achievement (IEA):

-FIMS (1964)

- SIMS (1980-82)

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA.
TIMSS - 1995

España participa en el año 1995, en el Tercer Estudio Internacional de Matemáticas y Ciencias (Third International Mathematics and Science Study), brevemente TIMSS, con alumnos de 13 y 14 años (7º y 8º de EGB) y con otros 40 países más. Los resultados de España se sitúan en esta ocasión por debajo de la media, el porcentaje internacional de aciertos en 8º de EGB es de 55% y de 7º es de 49%, mientras que en España son 51% y 42%, respectivamente (López y Moreno, 1997 y 1998). Se constata, nuevamente, las dificultades que presentan los alumnos de estas edades en Matemáticas.

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
TIMSS - 1995

Alumnos de 8.º		*Alumnos de 7.º*	
País	*%*	*País*	*%*
Singapur	79	Singapur	73
Japón	73	Japón	67
Corea	72	Corea	67
Hong Kong	70	Hong Kong	65
Bélgica (Fl)	66	Bélgica (Fl)	65
Rep. Checa	66	Rep. Checa	57
Eslovaquia	62	Austria	56
Suiza	62	Bulgaria	55
Austria	62	Holanda	55
Hungría	62	Bélgica (Fr)	54
Francia	61	Eslovaquia	54
Eslovenia	61	Hungría	54
Rusia	60	Irlanda	53
Holanda	60	Suiza	53
Bulgaria	60	Rusia	53
Canadá	59	Eslovenia	53
Irlanda	59	Australia	52
Bélgica (Fr.)	59	Tailandia	52
Australia	58	Canadá	52
Tailandia	57	Francia	51
Israel	57	Alemania	49
Suecia	56	EE UU	48
Alemania	54	Inglaterra	47
N. Zelanda	54	Suecia	47
Noruega	54	N. Zelanda	46
Inglaterra	53	Escocia	44
EE UU	53	Noruega	44
Dinamarca	52	Letonia	44
Escocia	52	Dinamarca	44
Letonia	51	Rumanía	43
España	**51**	Islandia	43
Islandia	50	**España**	**42**
Grecia	49	Chipre	42
Rumanía	49	Grecia	40
Lituania	48	Lituania	38
Chipre	48	Portugal	37
Portugal	43	Irán	32
Irán	38	Colombia	26
Kuwait	30	Sudáfrica	23
Colombia	29		
Sudáfrica	24		
Internacional	55	Internacional	49

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
PISA - 2000

Participan: 28 + 4 países

Año del Estudio: 2000

Edad: 15 años

Muestra: 265000 alumnos (ESPAÑA, 6214)

Organismo: Organización para la Cooperación y Desarrollo Económico (OCDE).

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
PISA - 2000

Los alumnos de 15 años
Puntos logrados (de 334 a 557)

		COMPRENSIÓN DE LA ESCRITURA		CULTURA MATEMÁTICA		CULTURA CIENTÍFICA	
POR ENCIMA DE LA MEDIA	1	Finlandia	546	Japón	557	Corea	552
	2	Canadá	534	Corea	547	Japón	550
	3	N. Zelanda	529	N. Zelanda	537	Finlandia	538
	4	Australia	528	Finlandia	536	R. Unido	532
	5	Irlanda	527	Australia	533	Canadá	529
	6	Corea	525	Canadá	533	N. Zelanda	528
	7	R. Unido	523	Suiza	529	Australia	528
	8	Japón	522	R. Unido	529	Austria	519
	9	Suecia	516	Bélgica	520	Irlanda	513
	10	Austria	507	Francia	517	Suecia	512
	11	Bélgica	507	Austria	515	Rep. Checa	511
	12	Islandia	507	Dinamarca	514	Francia	500
MEDIA	13	Noruega	505	Islandia	514	Noruega	500
	14	Francia	505	Liechtens.	514	EE UU	499
	15	EE UU	504	Suecia	510	Hungría	496
	16	Dinamarca	497	Irlanda	503	Islandia	496
	17	Suiza	494	Noruega	499	Bélgica	496
POR DEBAJO DE LA MEDIA	18	**España**	493	Rep. Checa	498	Suiza	496
	19	Rep. Checa	492	EE UU	493	**España**	491
	20	Italia	487	Alemania	490	Alemania	487
	21	Alemania	484	Hungría	488	Polonia	483
	22	Liechtens.	483	Rusia	478	Dinamarca	481
	23	Hungría	480	**España**	476	Italia	478
	24	Polonia	479	Polonia	470	Liechtens.	476
	25	Grecia	474	Letonia	463	Grecia	461
	26	Portugal	470	Italia	457	Rusia	460
	27	Rusia	462	Portugal	454	Letonia	460
	28	Letonia	458	Grecia	447	Portugal	459
	29	Luxemb.	441	Luxemb.	446	Luxemb.	443
	30	México	422	México	387	México	422
	21	Brasil	396	Brasil	334	Brasil	375

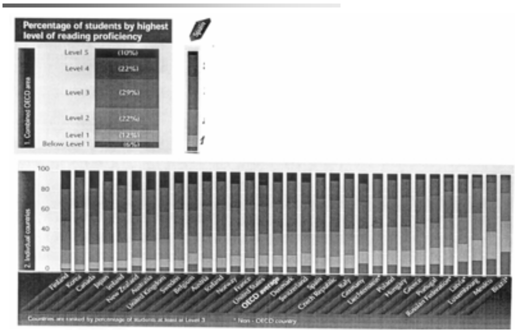
LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
PISA - 2000

La evaluación en el marco del Proyecto PISA (Proyecto Internacional para la Producción de Indicadores de Resultados Educativos de los Alumnos) de la OCDE (Organización para la Cooperación y el Desarrollo Económico).

Sitúa a España (476 puntos) en relación con la cultura matemática por debajo de la media que está en torno a los 500 puntos; en los primeros lugares figuran Japón y Corea con 557 y 547 puntos, respectivamente.

Nuevamente se constata las dificultades de los alumnos para el conocimiento matemático, de los que un 20% no alcanza el nivel mínimo y sólo un 9% alcanza el nivel de excelencia (NCES, 2001).

LA ENSEÑANZA DE LA MATEMÁTICA
EDUCACIÓN SECUNDARIA OBLIGATORIA
PISA - 2000



Estudio: Habilidades básicas en Matemáticas de alumnos que inician los estudios de Magisterio (Hernández, Noda, Palarea y Socas, 2003).

Participan: Universidades de Extremadura, Granada, Huelva, La Laguna, Las Palmas de Gran Canaria, Murcia y Zaragoza.

Duración del Estudio: 2000-2002

Edad: 18-19 años

Muestra: 883

Organismo: Área de Didáctica de la Matemática de las Universidades de La Laguna y Las Palmas de Gran Canaria.

El estudio muestra diferentes resultados en relación con los bloques de contenidos tratados: Números y operaciones, Medida, Geometría, Análisis de datos, Estadística y Probabilidad y Resolución de problemas,

En el bloque de Números y Operaciones, los alumnos presentan grandes dificultades en los problemas relacionados con la proporcionalidad, así como en la realización de cálculos básicos en los distintos campos numéricos (fracciones, decimales, ...).

En el bloque de Medida se observa que los alumnos presentan confusión en los conceptos de longitud de un segmento, perímetro, área y volumen, con especiales dificultades en la asignación de las unidades correctas.

La confusión entre área y perímetro ha sido puesta de manifiesto en numerosas investigaciones anteriores. En esta investigación se ha detectado mayor dificultad en el cálculo del perímetro que en el del área; por ejemplo, le confieren el mismo perímetro a figuras que tienen la misma área.

En bloque de Álgebra los alumnos obtienen mejores resultados que en los otros bloques, aunque siguen siendo deficitarios; en esta situación siguen apareciendo errores relacionados con la interpretación de las letras, uso de paréntesis,..., y con procedimientos: sacar factor común, uso de la propiedad distributiva, reglas de los signos, operaciones entre monomios, etc.

En Análisis de datos, Estadística y Probabilidad, el estudio muestra como los alumnos tienen dificultades para interpretar y resolver problemas sencillos sobre probabilidad, así como para interpretar y representar gráficas y códigos.

La resolución de problemas nos muestra que alumnos de 18 años tienen todavía la tendencia a operar con los datos del problema, sin mostrar una clara comprensión del mismo y sin identificar las relaciones conceptuales que se dan entre los datos, dando muchas veces soluciones que no pueden ser válidas para las condiciones del problema, lo que evidencia también una falta de pensamiento crítico.

En este estudio aparecen los bloques de "Números y operaciones" y "Medida", como los que plantean mayores dificultades a los alumnos.

Este estudio, al igual que los anteriores, sigue mostrando las enormes deficiencias que presentan los alumnos en conocimientos básicos en Matemáticas, pero no se encuentran diferencias significativas respecto a sus conocimientos básicos y los errores que cometen, según su procedencia curricular Bachillerato LOGSE o COU, e incluso según la modalidad: letras o ciencias.

LA ENSEÑANZA DE LA MATEMÁTICA. UN EJEMPLO: EL BACHILLERATO

Estas breves referencias a diferentes evaluaciones nacionales e internacionales, muestran que los resultados obtenidos en Matemáticas son ciertamente modestos, y, todos ellos son similares en las diferentes evaluaciones realizadas en Matemáticas en los últimos veinte años. Hemos de señalar que estos preocupantes niveles de dificultad que presentan los alumnos son un grave problema de nuestro Sistema Educativo que se ha ido arrastrando a lo largo de las diferentes reformas educativas.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

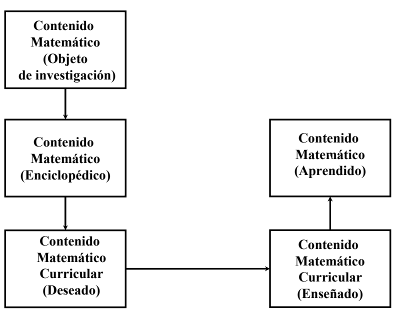
Formación del profesorado de matemáticas de Secundaria

Las reformas educativas plantean modificaciones profundas en el modo usual de enseñar Matemáticas. Los cambios curriculares afectan a las múltiples dimensiones del currículo. Organizar un currículo de Matemáticas para los estudiantes, se puede describir desde diferentes puntos de vista:

-La tradición alemana llama "Elementarización", a la transformación activa de un contenido matemático a formas más elementales con un doble sentido: ser fundamental y accesible para los grupos de estudiantes que lo reciban (Biehler et al., 1994).

-La tradición francesa describe este proceso con la teoría de la "Transposición Didáctica", Chevallard (1985), poniendo en evidencia las diferentes variables que intervienen en el paso del conocimiento matemático científico al conocimiento matemático deseado, susceptible de ser enseñado en una etapa educativa.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA



NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

El contenido matemático curricular es definible en el dominio del contenido matemático enciclopédico, aunque él no es enseñado ni organizado bajo esa forma. Son mecanismos y organizaciones precisas las que deben asegurar su extracción del contenido enciclopédico y su inserción en el discurso didáctico.

El currículo de Matemáticas que el profesor debe implementar ha sido determinado por diversos agentes del macrosistema educativo, mediante un proceso que generalmente resulta desconocido al futuro profesor. El currículo está organizado por una lista de contenidos que están relacionados con las capacidades que se pueden desarrollar e inmerso en una concepción determinada de entender la enseñanza y el aprendizaje, así como el proceso de evaluación. El futuro profesor debe reflexionar sobre este currículo, asimilarlo en su globalidad, en su coherencia, en su finalidad, y hacer sobre el mismo, una interpretación personal.

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

Los problemas de las reformas educativas están relacionados con los nuevos objetivos de la Educación Matemática:

- **Preparar a los estudiantes para una función social: análisis de procesos e interpretación de resultados**
- **Preparar a los estudiantes para una educación futura: necesidad de unas Matemáticas aplicadas a las distintas áreas (biología, economía ...)**
- **Desarrollar una nueva visión de la enseñanza y aprendizaje: los estudiantes aprenden por sí mismos**
- **Tratar nuevos conceptos matemáticos: grafos, matrices, problemas de optimización, análisis exploratorio de datos ...**
- **Usar nuevas tecnologías que han dado un impulso diferente al Área: programas de ordenador, calculadoras gráficas ... (De Lange, 1993).**

NATURALEZA DEL CONOCIMIENTO MATEMÁTICO Y SUS IMPLICACIONES EN LA ENSEÑANZA DE LAS MATEMÁTICAS EN LA EDUCACIÓN SECUNDARIA

Ahora bien, ¿qué conocimientos ayudan al profesor de Matemáticas a desarrollar con garantías estas propuestas curriculares?

Es cierto que pocos profesores de Matemáticas tienen una formación adecuada respecto a lo que están enseñando en términos de un conocimiento matemático como proceso, es decir, como un conocimiento que debe ser contextualizado y que tiene relaciones con las sociedades y culturas donde nace y se arraiga.

La tendencia más común es considerar el conocimiento matemático como un producto acabado, que implica abordar el conocimiento en su fase actual, descontextualizado, basado en el análisis lógico, donde las relaciones se establecen sólo a nivel de conceptos matemáticos.

Esta concepción es insuficiente para desarrollar las propuestas curriculares de Matemáticas en la Educación Secundaria.

Las tres reformas educativas: LGE, LOGSE y LOCE, plantean la exigencias de un título profesional para el profesorado de Educación Secundaria.

- La LGE determina para el profesorado de Secundaria (artículo 102): ... "Una formación Pedagógica adecuada a cargo de los Institutos de Ciencias de la Educación" ...

Curso de Aptitud Pedagógica (CAP)

-La LOGSE determina para el profesorado de Secundaria (artículos 24.2, 28, y 33.1): ... "La exigencia de un título profesional de especialización didáctica, obtenido mediante la realización de un Curso de Cualificación Pedagógica (CCP)" ...

-- La LOCE determina igualmente para el profesorado de Secundaria (artículo 58): ... "La exigencia de un título profesional de especialización didáctica" ...

Consideraciones finales

Las diferentes reformas están condicionadas especialmente por la extensión de la Educación Obligatoria hasta los 16 años y esto supone un cambio profundo en el funcionamiento del Microsistema Educativo, que afecta a todos los elementos que lo conforman, particularmente a los elementos sociocultural e institucional, que contextualizan este microsistema (cambios operados en la familia y en la vida de los niños y adolescentes,...), y a los referentes de este microsistema (alumnos, profesores y contenidos matemáticos).

Todo ello nos lleva a una reflexión necesaria sobre el papel cultural y social de la Matemática en la Educación Secundaria.

Las breves referencias a los resultados de diferentes evaluaciones nacionales e internacionales mostradas son suficientemente significativas y ponen de manifiesto que el problema del aprendizaje de las Matemáticas es un problema de "más calado" que la simple comparación entre un modelo educativo y otro.

Es necesario abordar en profundidad toda la problemática asociada a todas y cada una de las componentes del microsistema educativo, en lugar de plantearnos la discusión sobre la calidad de la Educación Matemática sólo en términos de comparación de modelos educativos. Una de estas componentes es la formación del profesorado de la Educación Secundaria, que no ha tenido un tratamiento adecuado en ningún momento en la historia de este país.

Es importante propiciar, organizar y coordinar la formación inicial y continua del profesorado. El modelo de formación debería ser objeto de un debate abierto con todos los implicados: Departamentos de Didáctica de las Matemáticas; de Matemáticas; Profesores de Matemáticas de los niveles no universitarios, a través de las diversas organizaciones existentes; Investigadores en Educación Matemática; Sociedades profesionales de Matemáticas, responsables educativos de las Administraciones Públicas, etc.

Aplicaciones estadísticas en las Ciencias Sociales

Juan Camacho Rosales

Profesor Titular de Metodología de las Ciencias del Comportamiento
Departamento de Didáctica e Investigación Educativa y del Comportamiento
Universidad de La Laguna

La Estadística Matemática

A medida que la Ciencia progresa, sus teorías se van haciendo más y más matemáticas en la forma. Hay una relación positiva entre el progreso de una Ciencia y el grado de su desarrollo matemático.

No es necesario que el investigador en Ciencias Sociales sea un especialista en áreas matemáticas concretas, lo verdaderamente importante es que sepa acercarse con mentalidad matemática a los problemas que se le plantean. La mentalidad matemática se define como la comprensión del proceso lógico subyacente al razonamiento matemático: entender la estructura formal del modelo matemático y las condiciones que lo hacen posible.

Tiene que haber un compromiso de manera que se simplifique la realidad concreta lo menos posible, pero, a la vez, lo suficiente para que el modelo creado a partir de la realidad sea fácilmente manejable desde el punto de vista instrumental matemático.

Es necesario una buena cierta matemática para conocer la potencia y debilidad de las técnicas estadísticas y consiguientemente para saber usarlas con eficacia y la vez con prudencia.

Para estudiar Estadística Matemática se necesita cálculo avanzado y álgebra de matrices, sin embargo tal madurez no es indispensable para comprender las bases de la Estadística Aplicada.

El sentimiento de satisfacción y tranquilidad que resulta de dominar un lenguaje lógico y no ambiguo compensa la ocasional ansiedad que se desencadena al descubrir que se ha expresado un absurdo explícitamente y a todas luces.

Desde un punto de vista matemático muchas de nuestras afirmaciones están incompletas, mal encuadradas o son imprecisas. Pero, por otro lado, muchas de estas ideas pueden ser entendidas intuitivamente, y es mejor una comprensión intuitiva que ninguna comprensión en absoluto.

Yo prefiero que los ejemplos que se usen en la enseñanza sean hipotéticos, porque es más importante tener un problema simple y plausible que el estudiante pueda comprender y que ilustre el método claramente que otro que simplemente asombre al estudiante con nuestra sabiduría.

Las principales aplicaciones estadísticas en cualquier campo, no sólo el de las Ciencias Sociales, descansan sobre el hecho de poder hacer observaciones o experimentos repetidos, esencialmente, bajo las mismas condiciones. En algunas áreas de la investigación, los objetos o fenómenos observados bajo las mismas condiciones variarán sólo en pequeña medida (en las ciencias físicas, donde las observaciones controladas dan prácticamente los mismos resultados). Pero, por otro lado, especialmente en las Ciencias Sociales, aunque el experimentador haga un esfuerzo sobrehumano para observar repetidamente bajo las mismas condiciones, se encontrarán diferencias entre las observaciones y las diferencias, ordinariamente, no serán despreciables.

La Estadística Matemática es una teoría acerca de la incertidumbre, la tendencia de los resultados a variar cuando observaciones repetidas se hacen bajo condiciones idénticas.

La Estadística es el estudio de fenómenos donde, bajo un mismo conjunto de condiciones, las medidas obtenidas presentan variabilidad, y por tanto resultados impredecibles a priori; es decir, existe incertidumbre asociada al conocimiento del objeto de estudio. Aceptado que la Estadística trata sobre la incertidumbre, cabe preguntarse si la naturaleza está determinada o, en realidad, la incertidumbre es inherente a la misma, y por tanto está indeterminada. Y si está indeterminada entonces la Estadística tratará sobre la misma esencia de la realidad empírica.

Definamos entonces la Estadística como aquella manera de pensar de la cual se deriva una forma de representar los sistemas y razonar sobre ellos, sobre una naturaleza que se muestra indeterminada. La Estadística puede considerarse una Ciencia que guía la extracción de conocimiento, e implica una manera de conceptuar cualquier problema donde la incertidumbre es inherente a la comprensión del objeto de estudio y, por lo tanto, nuestro discernimiento sólo puede ser probabilístico y expresado mediante leyes estadísticas.

Aunque la organización de la información, las transformaciones y la depuración de los datos no sean características esenciales de la Estadística, eso no implica que no puedan ser incluidas en una definición de la disciplina.

El objetivo de la Estadística como Ciencia es mejorar el nivel de vida de la sociedad. Estadística deriva de la palabra Estado, y etimológicamente significa recoger información para tomar decisiones de cómo repartir comida o trabajo.

La Estadística moderna se ocupa de la recolección, análisis e interpretación de información, tanto cuantitativa como cualitativa. Y los métodos estadísticos son particularmente útiles cuando hay variabilidad en la medición.

Utilidad de la Estadística en las Ciencias Sociales

Un estadístico trabajando en el campo de las Ciencias Sociales se ocupa de las siguientes cuestiones:

- ¿qué datos se necesita recoger?
- ¿cómo se pueden usar los recursos disponibles más eficientemente para recolectar los datos?
- ¿cómo especificar un modelo matemático que describa el proceso que ha generado los datos?
- depuración y transformación de los datos
- ¿cómo presentar los datos de manera que transmitan sus rasgos más esenciales de una manera clara?
- ¿qué conclusiones se pueden extraer de los datos y cuál es el grado de incertidumbre de estas conclusiones?
- ¿qué acciones se deben tomar en base a las conclusiones extraídas de los datos?

En la actualidad la Estadística es probablemente una de las disciplinas científicas más utilizada y estudiada en todos los campos del conocimiento humano. Por ejemplo:

- en la Administración de Empresas se utiliza para evaluar la aceptación de un producto antes de comercializarlo,

- en Economía para medir la evolución de los precios mediante números índice o para estudiar los hábitos de consumo mediante encuestas,
- en Ciencias Políticas para conocer las preferencias de los electores antes de la votación mediante sondeos y así orientar las estrategias de los candidatos,
- en Sociología para estudiar las opiniones de los colectivos sociales sobre temas de actualidad,
- en Psicología para elaborar las escalas de los tests y cuantificar aspectos del comportamiento humano,
- en general en las Ciencias Sociales para medir la relación entre variables y hacer predicciones sobre ellas.

En las Ciencias Sociales la Estadística se estudia en tres secciones: la Estadística Descriptiva, la Estadística Inferencial y el Diseño Experimental. La Estadística Descriptiva sirve de herramienta para describir, resumir o reducir las propiedades de un conglomerado de datos al objeto de que se pueda manejar. La Estadística Inferencial se utiliza para estimar las propiedades de una población a partir del conocimiento de las propiedades de una muestra de ella. Y en tercer lugar, el diseño y análisis de experimentos se desarrolla para determinar y confirmar relaciones causales entre variables.

En la investigación la Estadística es importante porque:

- permite el tipo más exacto de descripción,
- fuerza a ser exactos y definidos en nuestros procedimientos y pensamiento,
- permite resumir nuestros resultados de una forma conveniente,
- permite extraer conclusiones generales,
- permite predecir, y
- permite analizar algunos de los factores causales que subyacen a eventos complejos.

Dentro del campo de la Psicología hay tres vertientes metodológicas (lo cualitativo, lo no experimental y lo longitudinal) que son el autentico punto de partida de las actuales líneas de desarrollo. Las ecuaciones estructurales permiten la modelación de la causalidad. La regresión logística, los modelos log-linear y el análisis de correspondencias se utilizan para el análisis de datos cualitativos. Las series temporales investigan el aspecto longitudinal.

La mayoría de las técnicas utilizadas en los cuasi-experimentos se deriva del modelo de la regresión múltiple, de modo que las hipótesis rivales son probadas una a una. Por el contrario, en los estudios aleatorizados, se estima exactamente un efecto y se eliminan otros, dado que existe garantía de que influyen por igual en el grupo experimental y en el grupo control.

Estadística Descriptiva

La estadística puede estudiar tanto las características de las muestras en sí como hacer inferencias acerca de las características de las poblaciones. Población es el conjunto de valores que tienen una propiedad común y muestra es un subconjunto (aleatorio o no) de la población. Si se estudia la variable "edad de los estudiantes que estudian cuarto de Psicología en España" la población es el conjunto de edades de todos esos estudiantes, y muestra es el conjunto de las edades de los que estudian en una de sus universidades.

Las características de las poblaciones se estudian mediante indicadores de uno o más aspectos particulares. A estos indicadores se les llama parámetros cuando se refieren a la población, y se les llama estadísticos cuando se refieren a una muestra. Aunque en general se habla de estadísticos para referirse tanto a éstos como a los parámetros.

En la Estadística las variables se dividen en cuantitativas y cualitativas. Las variables cualitativas (o nominales) son variables de cuyos valores sólo se puede decir que son distintos. En las variables cuantitativas sus valores, además de ser distintos, se pueden ordenar (de mayor a menor). Las variables cuantitativas se dividen en tres tipos: ordinales, de intervalo y de razón. En las variables ordinales los valores son distintos y se pueden ordenar. En las variables de intervalo además de esos dos rasgos (distintos, ordenados) existe una unidad común. Y en las de razón además de esos tres rasgos existe un cero real. La afiliación política es una variable nominal. La altura o el peso son variables de razón.

En la estadística descriptiva el interés del estudio puede estar en una variable, en dos variables o en tres o más variables. En cuanto al estudio de una sola variable las características que se pueden estudiar de cada variable son:

1. Distribuciones de frecuencias a través de histogramas, gráficas de barras o polígonos de frecuencias.
2. La tendencia central, es decir, el valor más representativo de la muestra, se indica mediante la media, la mediana o la moda dependiendo del tipo de variable estudiada.
3. La dispersión de los datos, si están o no muy agrupados los números, se estudia mediante la desviación típica, la varianza, la amplitud semiintercuartílica o el rango dependiendo del tipo de variable.
4. La forma de las distribuciones, a través de los coeficientes de asimetría y apuntamiento.

La forma de la distribución toma como patrón la distribución normal o campana de Gauss. En la distribución normal la mayoría de las puntuaciones se agrupan en torno a la media, y en la que cuanto más lejos se encuentra la puntuación de la media más rara es. Además, existen igual número de puntuaciones a ambos lados de la media. Y la asimetría y el apuntamiento valen cero para una distribución normal.

Los estadísticos anteriores son aplicados a muestras, pero cuando el interés está en el individuo cada sujeto se puede estudiar con respecto a la muestra que pertenece. Se puede estudiar su puntuación directa, la puntuación diferencial (la diferencia con respecto a la media), y la puntuación típica (la distancia en desviaciones típicas a la media). La puntuación típica permite comparar verazmente las puntuaciones en distintas variables con medias y desviaciones típicas diferentes. Otro índice de la posición del sujeto en su grupo es el percentil, que indica el porcentaje de casos que tienen puntuaciones inferiores a la suya.

En el estudio de dos variables lo que interesa usualmente es hallar la relación que hay entre las variables. Se utiliza el coeficiente de correlación de Pearson cuando las variables son cuantitativas. Este coeficiente varía entre uno y menos uno; cuando vale cero indica que no hay relación entre las variables, y cuando vale uno o menos uno que la relación es perfecta. El estudio de la relación se puede complementar con su representación gráfica mediante la nube de puntos o gráfico de dispersión.

El estudio de la relación entre variables cualitativas se realiza mediante tablas de contingencia y estadísticos basados en el estadístico chi cuadrado. Hay otra serie de estadísticos cuando se intenta estudiar la relación entre variables cuantitativas y cualitativas.

Cuando se estudia la relación entre varias variables la técnica por excelencia es la regresión múltiple. Ésta da un índice, que varía entre cero y uno, de la relación entre una variable y un conjunto de variables. Otros índices estadísticos son: la correlación parcial, que estudia la relación entre dos variables cuantitativas eliminando el influjo sobre ellas de otras variables, y el coeficiente de correlación canónica, que analiza la relación entre dos conjuntos de variables, cada uno de ellos con dos o más variables.

Estadística Inferencial

El siguiente gran capítulo de la Estadística aplicada a las Ciencias Sociales es la Estadística Inferencial. Cuando se quieren hacer inferencias, es decir, suposiciones acerca de lo que ocurre en la población global de la que se han tomado las muestras que se están estudiando, se utilizan los contrastes de hipótesis. Estos contrastes permiten contestar a preguntas como: ¿Pertenece una cierta muestra a una determinada población? ¿Pertenecen a la misma población dos muestras?

Para contestar a las anteriores preguntas se adopta un nivel de significación (el máximo nivel de error que se asume en la decisión), y se compara con la probabilidad del estadístico de contraste. Entonces, si es menor la probabilidad del estadístico de contraste que el nivel de significación, se habla de que existen diferencias, o de que se han producido resultados significativos.

Este proceso se formaliza de la siguiente manera:

1. Fijar el nivel de significación o error alfa.
2. Establecer una hipótesis nula y una hipótesis alternativa.
3. Recoger los datos adecuados.
4. Hallar el estadístico de contraste.
5. Comparar la probabilidad del estadístico de contraste con el nivel de significación.

El estadístico de contraste es una cierta fórmula que da como resultado un valor. Este valor sigue una cierta distribución. Las distribuciones que siguen estos estadísticos son, en la mayoría de los casos: la distribución normal, la distribución t de Student, la distribución chi cuadrado y la distribución F.

Hay diferentes tipos de contrastes dependiendo del objetivo de estudio: diferencia entre medias, diferencia entre desviaciones típicas, diferencia entre proporciones, diferencia entre correlaciones, y así sucesivamente.

Diseño Experimental

Los principales tipos de problemas con que se enfrenta la Estadística en el campo de las Ciencias Sociales son los de: diferencia entre grupos, relación entre variables, estructura de los datos y separación entre grupos.

En el estudio de la diferencia entre grupos lo que usualmente se trata de analizar es la diferencia entre medias. Y para ello se utiliza la prueba t de diferencia entre medias o el análisis de varianza. En ambas técnicas lo que se investiga en realidad es si la diferencia entre las medias de los grupos (la llamada variabilidad intergrupos) es significativamente mayor que la diferencia dentro de los grupos (la denominada variabilidad intragrupos). El análisis de varianza ocupa una gran parte del contenido de la Estadística Inferencial en las Ciencias Sociales ya que el número y características de las variables experimentales puede ser muy variado. Y además es la base del diseño experimental. Diseño experimental que trata de establecer relaciones de causa a efecto entre variables.

En el estudio de las relaciones entre variables hay dos objetivos principales, uno de índole teórica y otro de naturaleza práctica. El primero halla índices analíticos que miden la relación entre variables mediante las técnicas ya mencionadas: correlación de Pearson, regresión múltiple o correlación canónica. En la regresión múltiple se halla la relación entre una variable (la variable predicha o variable dependiente) y una o más variables (las variables predictoras o variables independientes). La regresión múltiple ofrece no sólo un índice numérico global de la relación, sino un estudio pormenorizado de la importancia de cada una de las variables predictoras. La cuestión práctica en el estudio de la relación entre variables consiste en poder realizar predicciones del valor de la variable predicha. Es decir, una vez establecido que existe una relación significativa y substantiva entre la variable predicha y las variables predictoras, basta con aplicar una fórmula con los valores de las variables predictoras para obtener un valor en la variable predicha sin necesidad de medirla de antemano.

El problema de la estructura de los datos se estudia básicamente con una técnica estadística multivariada: el análisis de componentes principales. Esta técnica permite reducir un conjunto de variables relacionadas a un número de componentes (variables también) independientes entre sí. Con ello se consigue agrupar las variables originales en subconjuntos de variables que están relacionadas entre sí y no están relacionadas con las variables de los otros subconjuntos. Por ejemplo, en Psicología está técnica ha servido para constatar la existencia de distintos tipos de inteligencia: espacial, verbal, manual, etcétera.

En el problema de la separación entre grupos se aplican dos tipos de técnicas. Con la primera se trata de establecer la existencia de grupos que no son evidentes a priori; para ello se utiliza el análisis de conglomerados. Con esta técnica no sólo es posible descubrir estos grupos, sino fijar de antemano el número de ellos. Con la segunda técnica, el análisis discriminante, se estudia la estructura de la diferencia entre los grupos; puede ocurrir que la diferencia entre los grupos se explique por subconjuntos de distintas variables. Y tal como ocurre en la regresión múltiple, también se puede utilizar para predecir la pertenencia a grupos.

Bibliografía

J. Amón: *Estadística para Psicólogos: 1 Estadística Descriptiva*. Pirámide, Madrid, 1980.

J. Arnau: Técnicas de Análisis Avanzadas y Diseño de Investigación: Tendencias Actuales y Líneas Futuras de Desarrollo. En J. Arnau (editor): *Métodos y Técnicas Avanzadas de Análisis de Datos en Ciencias del Comportamiento*. EUB, Barcelona, 1996.

G. Glass, J. Stanley: *Métodos Estadísticos aplicados a las Ciencias Sociales.* Prentice Hall, Madrid, 1980.

W. Hays: *Statistics for the Social Sciences.* Holt, Rinehart & Winston, London, 1973.

D. Peña, J. Romo: *Introducción a la Estadística para las Ciencias Sociales.* McGraw-Hill, Madrid, 1997.

A. Solanas et al.: La Enseñanza de la Estadística en las Ciencias del Comportamiento a Inicios del Siglo XXI. *Metodología de las Ciencias del Comportamiento* **4**, no. 2 (2002), 157-183.

Modelos de Aproximación Racional en Economía

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Introducción

Como sabemos y experimentamos en nuestra vida cotidiana, las sociedades modernas exigen de manera creciente ciertas habilidades como pueden ser la eficiencia en la organización, clasificación, análisis y asimilación de la información, valoración de diferentes opciones, toma de decisiones, predicciones y finalmente actuaciones... habilidades que podríamos adquirir con el estudio y uso de las matemáticas.

Paralelamente, al menos en los países denominados *desarrollados* presumimos de disponer de muchos medios, en particular tecnológicos, y de mucha información. Sin embargo, se está comprobando que es más limitada la capacidad para captarla, interpretarla, analizarla y relacionar coherentemente todas las variables que influyen en los procesos de toma de decisiones. En este momento, las enormes posibilidades computacionales y la propia tecnología de la información no hacen más que demandar nuevas ideas matemáticas que permitan avanzar económica y tecnológicamente pero en equilibrio con la vida humana y, en general, con la conservación de nuestro planeta en condiciones favorables para el medio ambiente.

Dividimos el trabajo en dos secciones. En la primera, a modo de preámbulo, tratamos de exponer la **Economía como Ciencia**, para reconocer que desde el punto de vista histórico esta característica de Ciencia es realmente algo "reciente". Citaremos algunos ejemplos sobre el cómo, el porqué y para qué los economistas construyen modelos usando las Matemáticas tanto teórica como empíricamente. En la segunda mencionaremos un tipo particular de modelos útiles en Economía, a saber, los modelos para datos cronológicos. Lo ilustraremos mostrando modelos que se construyen utilizando **aproximación racional** y métodos numéricos de optimización. Veremos el uso de los mismos desde el punto de vista computacional y **experimental** en un caso específico de nuestro entorno económico – datos agrosociales del sector platanero de Canarias –. Así mismo mostraremos la importancia que en estos modelos tienen no sólo los datos pasados conocidos y acumulados a lo largo del tiempo en las bases de datos sino las expectativas que los agentes implicados se forman sobre un proceso económico concreto.

1. Ejemplos aislados y aspectos históricos

Ejemplos aislados como la llegada del euro o el uso básico de programas numéricos como las Hojas de Cálculo nos sirven para ilustrar el interés de las Matemáticas en la toma de decisiones en Economía y en Ciencias Sociales en general.

En el primer caso, por ejemplo, la propia normativa europea CE 1103/97 [7] reconoce la importancia del concepto de *cifra significativa* y establece que los tipos de conversión y los precios unitarios deben fijarse con 6 cifras significativas, realizando el redondeo a 2 decimales sólo en los saldos finales de las facturas. Este hecho de importancia no es observado directamente por el consumidor en el mercado que en su realidad cotidiana se encuentra con precios en moneda real, esto es, con 2 decimales. Sin embargo, podría observarse al menos parcialmente en las facturas de bienes de servicios (electricidad, agua, teléfono...). Profundizando un poco más en el proceso hacia la Moneda Única necesitaríamos conocimientos básicos sobre Matemática Numérica para entender las demostraciones que aparecen en el Informe *Análisis Aritmético del Redondeo en el Proceso de Moneda Única* elaborado por el Comité de Tecnología de Moneda Única respecto a los problemas que produce el propio proceso de redondeo, entre otros, la pérdida de reversibilidad (la conversión de pesetas en euros y la vuelta atrás de euros en pesetas no ofrece siempre el resultado inicial), la pérdida de neutralidad (se producen errores que estadísticamente no se compensan) y la pérdida de homogeneidad (la suma de las conversiones puede no ser igual a la conversión de la suma). A su vez todo ello plantea nuevas necesidades informáticas y matemáticas para afrontar con éxito las actualizaciones precisas en las bases de datos monetarios.

El otro ejemplo tiene que ver con el uso de las Hojas de Cálculo (Excel de Microsoft o similar) para generar y simular series de datos, por ejemplo, relacionados con el control, en función de ciertos umbrales establecidos, de precios que oscilan en el mercado. En este proceso intervienen expresiones complejas, incluyendo esquemas como

$$P_{t+1} = \begin{cases} 2P_t & 0 \le X_t \le 0{,}5 \\ 2(1-P_t) & 0{,}5 \le X_t \le 1 \end{cases},$$

que permiten simular la subida del precio si éste está por debajo de cierto umbral y la bajada del mismo si se encuentra por encima de dicho umbral. Pues bien, este ejemplo, muy simple desde el punto de vista simbólico, nos daría (comenzando en un valor de 0,1) la evolución oscilante 0,1;0,2;0,4;0,8;0,4;0,8... que repite indefinidamente estos dos últimos valores. En la Hoja de Cálculo se convertirá en 0,1;0,2;0,4;0,8;0,4;...;0,5;1;0;0... lo que rompe completamente y para siempre el modelo teórico planteado. Esto ocurre debido a los errores de máquina y a las características de la aritmética finita con la que trabajan numerosos programas de ordenador utilizados por los usuarios informáticos.

Podrían citarse numerosos ejemplos aislados como estos. Sin embargo, el uso de las Matemáticas en Economía va mucho más lejos ya que como en cualquier ciencia el verdadero interés subyace en la **construcción de modelos** lo más amplios posible que permitan comprender las regularidades en el comportamiento económico y elaborar decisiones. En este sentido, sería interesante tomar conciencia de la Economía como Ciencia para lo cual un pequeño resumen histórico nos hará caer en la cuenta de que esta visión de la Economía es algo bastante reciente en comparación con otras ciencias, tanto que hubo que esperar al siglo XIX para vislumbrar resultados esperanzadores [1,2,3].

Los **orígenes comunes** de la Economía y la Matemática en cuanto a las necesidades de contar y medir no fueron suficientes para impulsar desde un principio a la Economía como Ciencia. De hecho, suele establecerse como fecha de introducción sistemática de las Matemáticas en la Economía el año 1838 cuando Cournot publicó su libro *Investigación acerca de los Principios Matemáticos de la Teoría de las Riquezas*. Hasta ese momento aritmética básica, aplicaciones aisladas de la teoría de la probabilidad y tendencias es lo que encontramos en los escritos económicos. A partir del siglo XIX casi todas las áreas matemáticas (álgebra, teoría de conjuntos, análisis de funciones, optimización, convexidad, topología, programación matemática, teoría de juegos, no linealidad, aleatoriedad, incertidumbre, dinamicidad, asimetría, experimentación, análisis numérico, simulación, análisis cualitativo…) han ayudado a resolver diferentes problemas económicos y muchas han sido las personas que han contribuido a ello, normalmente con formación interdisciplinar en Economía y otras ciencias. Un repaso de las biografías de los Premios Nobel en Economía, por ejemplo, así lo manifiesta [6].

Además, una perspectiva histórica nos permite analizar de una forma más realista los verdaderos avances de la Economía como Ciencia. En efecto, como sabemos una teoría científica puede partir de ideas o de observaciones: lo mismo ocurre en Economía. Además, como en cualquier ciencia, **la realización de experimentos** es de vital importancia. No obstante, en cualquier ciencia los experimentos, especialmente los de laboratorio, deben poder repetirse bajo las mismas hipótesis, garantizando así que los resultados deben ser prácticamente los mismos. Esto no ocurre de manera formal en muchos ámbitos de la Economía donde cualquier experimento es irrepetible ya que las hipótesis incluyen comportamiento humano y, por tanto, varían inevitablemente con el tiempo. Esto puede ser cubierto, en parte, con estudios bajo incertidumbre, pero ni siquiera los avances tecnológicos pueden evitar este inconveniente. De la misma manera que en Medicina no existen las enfermedades sino los enfermos, en Economía el mercado es la abstracción de un conjunto de comportamientos económicos individuales. Pero no por eso dejan de tener valor los avances que la tecnología pueda imprimir en esas áreas, sino todo lo contrario. Incluso Newton dudó de las posibilidades que tendría la aplicación de la Teoría de la Probabilidad a los problemas sociales; por suerte el pesimismo de Newton fue precisamente la clave que animó a Daniel Bernoulli a estudiar ciertas aplicaciones al mundo real de tal teoría, ya desarrollada en ese momento para los juegos de azar. Fueron momentos decisivos puesto que los descubrimientos de este matemático y de otros miembros de su familia permitieron posteriormente a De Moivre descubrir y a Gauss formalizar la curva de campana que permite distribuir los errores al tomar medidas. Si bien ellos no lo aplicaron a datos sociales, sí lo hizo posteriormente Quetelec, siendo Kolmogorov, dos siglos más tarde, el que tuvo la capacidad de abstracción suficiente para ver la relación entre la teoría de la probabilidad y la teoría de la medida, que junto con el cálculo de Newton y Leibniz permite desde hace algunos años construir modelos dinámicos estocásticos de vital importancia para la toma de decisiones en muchas áreas de la Economía y que volveremos a retomar en la segunda sección de este trabajo. En concreto, el Premio Nobel en Economía en 1997 se concedió por una única fórmula, una ecuación diferencial estocástica debida a Black y Scholes y desarrollada más tarde por Merton, que permite calcular valores de opciones financieras. Esta fórmula fue publicada en 1973, después de algunos años de intentos infructuosos en varias revistas científicas, pero puesta en práctica por Wall Street apenas unos meses después de su publicación. Era una de esas ocasiones en las que una fórmula matemática tuvo más interés práctico que teórico.

A modo de resumen, podríamos decir que la Economía se convierte por el uso de las Matemáticas en el nexo de unión entre la cultura humanística (representada por la propia sociedad) y la cultura científica (representada por la ciencia, la tecnología y las matemáticas): Definitivamente, la Matemática es la herramienta que permite a los economistas construir sus propios modelos para comprender mejor, planificar y tomar decisiones. Como dijo Dorfman, matemático y estadístico, impulsor de la programación lineal e investigador en temas de medio ambiente y uso de recursos naturales, *Si quieres realismo mira al mundo que te rodea, si quieres comprender mira a las teorías*. También Allais, físico y economista, siempre dispuesto a investigar lo que podía haber de falso en las ideas recibidas y Premio Nobel en Economía en 1988 por sus estudios de mercado y asignación eficiente de recursos, aporta reflexiones interesantes sobre su visión de la Economía como Ciencia [5]...

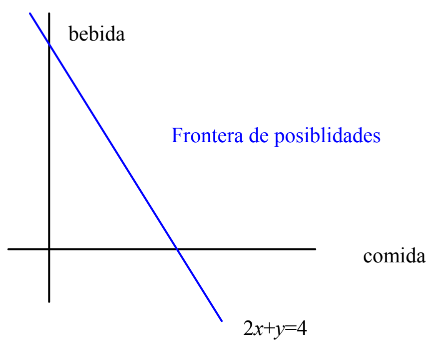
2. Construcción de modelos en Economía

Hay que partir de la base de que un modelo en Economía, como en cualquier ciencia, supone una **abstracción o simplificación de la realidad** y puede construirse partiendo de ideas teóricas o de experimentos, bajo unas hipótesis bien especificadas. Diremos, respectivamente, que construimos un modelo teórico o un modelo experimental. Esquemáticamente, los pasos que seguimos podrían ser, en general:

- Planteamiento del problema económico y concreción de hipótesis.
- Traducción al lenguaje matemático: Hipótesis matemáticas.
- Estudio del problema.
- Elección de la técnica de resolución.
- Resolución matemática.
- Obtención de las conclusiones matemáticas.
- Interpretación económica y contrastación con el problema real.
- Si fuera necesario, corrección de hipótesis tanto matemáticas como económicas.
- Planteamiento y resolución de modelos más realistas.

Modelos teóricos. Estos parten de ideas abstractas que el investigador concibe normalmente a partir de su visión de la realidad económica o sugeridas por otras teorías anteriores. Por tanto, no se construyen utilizando las bases de datos, que se reservan para el momento de la contrastación del modelo. Vamos a ilustrar este tipo de modelos a través de un modelo básico en la teoría de la elección que intenta explicar de forma simbólica la conducta económica de los individuos.

Pensemos que disponemos de 4 euros para merendar, pudiendo comprar algo para comer (que vale 2 euros por unidad) y algo para beber (que vale 1 euro por unidad). Nuestras posibilidades de elección estarían en el triángulo que forma la siguiente figura:



Supongamos que observamos que muchas personas eligen una unidad de comida y dos de bebida frente a otras elecciones posibles. ¿Es esa la "mejor" elección? Para intentar responder de alguna forma a esta cuestión construimos un modelo siguiendo los pasos indicados anteriormente:

- <u>Fenómeno económico</u>: La conducta económica de los individuos.

- Hipótesis: Los individuos se comportan como si utilizaran su poder adquisitivo para obtener la máxima *satisfacción* posible.
- Traducción al lenguaje matemático: Los individuos adquieren un número limitado de bienes, cada uno a un precio dado. La *satisfacción* se mide usando una función matemática que llamamos Utilidad y que simbolizamos por *U*.
- Elección de la técnica de resolución: Optimización matemática con restricción.
- Resolución matemática: Mostramos un caso sencillo en el que el individuo adquiere sólo dos bienes en cantidades x e y a precios respectivos p y q y U es continuamente diferenciable, de manera que se trata de resolver $\begin{array}{l} MAX\ U = U(x,y) \\ s.a.\ \ px+qy \le R \end{array}$.
- Conclusiones matemáticas:
  ⇒ La solución del problema está en la frontera de la región factible $px + qy = R$.

  ⇒ Derivando respecto a x la restricción obtenemos $p + q\,y' = 0$ y, por tanto, $y' = -\frac{p}{q}$.

  ⇒ Utilizando el método de Lagrange, al derivar

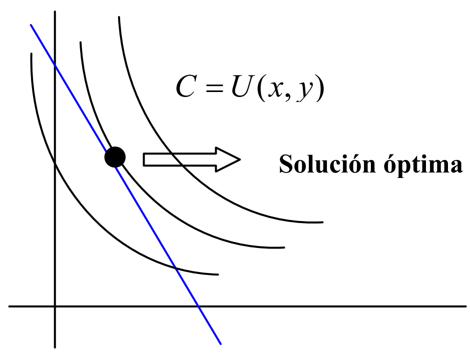
$$L(x,y)=U(x,y)-\lambda(px+qy-R)$$

  obtenemos

$$L_x(x,y)=U_x(x,y)-\lambda p; \qquad L_y(x,y)=U_y(x,y)-\lambda q.$$

  ⇒ En el óptimo se verifica: $\lambda = \frac{U_x}{p} = \frac{U_y}{q}$ y $\frac{U_x}{U_y} = \frac{p}{q}$

  ⇒ Haciéndonos una idea gráfica:

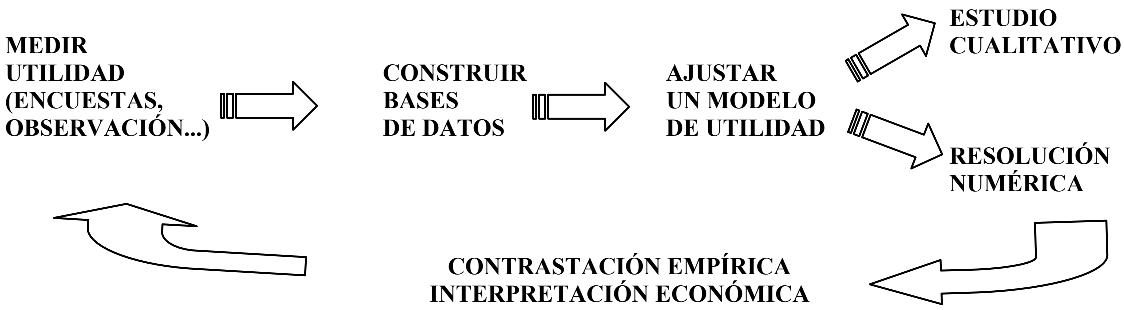


  donde se ve la conveniencia de que la función de utilidad sea convexa, si queremos asegurar solución única en el punto de tangencia.
- Interpretación económica:
  ⇒ El individuo debe invertir toda su renta para maximizar la "satisfacción".
  ⇒ Si la "satisfacción" es convexa, el individuo debe aumentar el consumo de uno de los bienes al disminuir el del otro si quiere conservar su satisfacción.

⇒ La relación marginal de sustitución entre los bienes coincide con la que existe entre los precios respectivos en el mercado.
⇒ Si el individuo maximiza su satisfacción, el valor de λ mide aproximadamente la satisfacción adicional que podría obtener por cada unidad monetaria adicional que gane.
⇒ La relación entre las tasas de cambio de la satisfacción al modificar el consumo de cada bien es la misma que entre los precios respectivos en el mercado.

- <u>Contrastación del modelo</u>: Se contrasta que este modelo tiene capacidad para representar de forma simbólica y abstracta el comportamiento de una mayoría amplia de consumidores.
- <u>Hacia modelos más realistas</u>: Se observa que el modelo estudiado ha permitido conclusiones interesantes pero algunas de ellas son muy restrictivas, lo que nos invita a plantear una serie de cuestiones, como: ¿qué ocurre si añadimos más de dos bienes?, ¿y si *U* no es diferenciable?, ¿y si *U* no es convexa?, ¿y si cambia el comportamiento de los individuos?… encaminadas a considerar nuevos modelos más realistas.

Modelos experimentales. Una posibilidad de ampliación se encuentra en la construcción de modelos experimentales siguiendo el siguiente esquema:



que completa los modelos teóricos y abre nuevas cuestiones relacionadas con la elaboración y el tratamiento de las bases de datos y con el uso de tecnología científica e informática.

Respecto a la elaboración, consulta y uso de las bases de datos, es necesario tener en cuenta que:

a) Los datos en Economía son de tipo histórico-sociológico, en general, fuera del control del investigador.
b) Es habitual obtener mediciones sustancialmente diferentes para una misma variable dependiendo de la fuente consultada.
c) Los datos socio-económicos están sujetos a "errores de medición y escasa precisión" por lo que requieren un tratamiento estadístico adecuado.
d) Los datos económicos exigen, en general, modelos con cierto grado de complejidad (no lineales, dinámicos, multivariantes, estocásticos...).
e) Se generan necesidades informáticas propias del área y usuarios cualificados.

El uso de la computación y de análisis numérico parte del reconocimiento de que lo exacto no existe y que, por tanto, la aproximación es una necesidad. De modo que los modelos experimentales descansan en métodos estadísticos y numéricos evaluados a través de algoritmos programados por el propio investigador o a través de programas comerciales ya elaborados. Este tipo de modelos tiene la ventaja de lograr mayor eficiencia y mayor acercamiento al mundo real, de permitir contrastación, experimentación y simulación de casos complejos así como de servir de complemento sustancial a los estudios cualitativos y de base para elaborar predicciones. No obstante, por un lado, hay que pensar que este proceso de computación se lleva a cabo en muchas ocasiones por usuarios no especializados para los cuales el programa de ordenador que utilizan es una especie de "caja negra" y, por tanto, los resultados computacionales que obtienen les resultan poco intuitivos e incomprensibles; por otro, muchos modelos se construyen con programas que no son lo suficientemente sofisticados para los fines que se persiguen porque algunas empresas consideran los "grandes" programas como herramientas infrautilizadas y, por ende, un derroche de recursos.

Vamos a ilustrar este apartado a través de modelos racionales en el campo de la economía agrícola en Canarias [4].

Intentamos esquematizar los pasos, de manera similar a como lo hicimos en el modelo de la elección:

- Fenómeno económico: La evolución del sector platanero en las islas y sus repercusiones económicas y paisajísticas.
- Hipótesis:
  - ⇒ El pasado predice el futuro.
  - ⇒ La producción se planifica en función del ingreso percibido por el agricultor en años anteriores.
  - ⇒ El año 1993 es una fecha clave para el sector platanero. Se liberaliza el mercado y los efectos negativos se intentan compensar con el establecimiento de una ayuda compensatoria por parte de la Unión Europea.
- Características de los datos disponibles:
  - ⇒ Datos anuales históricos oficiales y de fincas particulares sobre producción y precios de mercado a partir de 1938.
  - ⇒ Datos anuales oficiales de ayuda compensatoria a partir de 1993.
- Traducción al lenguaje matemático:
  - ⇒ Llamamos P_t a la producción (en millones de toneladas) e I_t al ingreso percibido por el agricultor en el año t (en pesetas constantes del año 1996).
  - ⇒ Contrastamos estadísticamente la relación $I_t \rightarrow P_t$
- Elección de la técnica de resolución:
  - ⇒ Preparación de los datos y de las variables: Adecuación del tamaño y naturaleza de las series usando interpolación numérica, función logaritmo (*ln*) y operador diferencia B.
  - ⇒ Definición de las nuevas variables y_t=(1-B)lnP_t y x_t=(1-B)lnI_t.
  - ⇒ Elección de modelos dinámicos causales de aproximación racional basados en aproximantes de Padé y teoría Box-Jenkins para series temporales, esquematizados como sigue:

$$\underset{\substack{\downarrow\\ \text{Output}\\ \text{actual}}}{y_t} = \underbrace{v_0 x_t + v_1 x_{t-1} + \ldots}_{\substack{\text{Input actual}\\ \text{y pasado}\\ \text{Parte}\\ \text{Determinista}}} + \underset{\substack{\downarrow\\ \text{Parte}\\ \text{Estocástica}}}{N_t} = \sum_0^{\infty} v_i B^i x_t + N_t \cong \underbrace{\frac{P(B)}{Q(B)} x_t + \frac{\theta_q(B)\vartheta_Q(B^s)}{\Theta_p(B)\Phi_P(B^s)}}_{\substack{\text{Modelos}\\ \text{Racionales}}} \underset{\substack{\downarrow\\ \text{Proceso}\\ \text{Ruido}\\ \text{Blanco}}}{a_t}$$

- Resolución matemática:
  Construcción del modelo para el periodo 1938-1993 usando programas informáticos científicos (MATHEMATICA y SCA) con técnicas basadas en aproximación racional de Padé, optimización por mínimos cuadrados y mínimo-máximo.

$$y_t=0.1707_{(5.39)}\,B^4 x_t+(1+0.9318_{(27.78)}B)a_t$$

- Conclusiones matemáticas:
  ⇒ Hemos separado la estructura determinista de la estocástica en los datos.
  ⇒ El término en $x_{t\text{-}4}$ es el más significativo de la serie.
  ⇒ Ambos modelos se ajustan bien a los datos y se obtiene una predicción que luego contrastamos empíricamente.
- Interpretación económica y contrastación del modelo:
  ⇒ El ingreso percibido influye en la planificación de la producción básicamente a 2 años vista.
  ⇒ El ajuste a los datos es "bueno" pero no tanto las predicciones contrastadas con los datos a partir de 1993. Lo vemos en gráfica incluida al final de esta sección.
- Hacia modelos más realistas:
  ¿Podemos modificar el modelo para mejorar las predicciones?

A continuación intentamos dar una primera respuesta parcial a esta cuestión construyendo un modelo semejante al anterior pero que tome en cuenta de alguna forma las expectativas que sobre su ingreso futuro tienen los propios agricultores.

- Hipótesis:
  ⇒ La producción se planifica en función de los ingresos percibidos en años anteriores y de las expectativas del ingreso que se espera percibir en años venideros.
  ⇒ Respecto a las expectativas hacemos 4 hipótesis diferentes:

♦ ESCENARIO 1: El agricultor supone que sus ingresos se mantendrán la próxima década.
♦ ESCENARIO 2: El agricultor supone que la ayuda compensatoria no va a considerar la subida anual del IPC para reducir los gastos por parte de la Unión Europea por lo que sus ingresos van a descender paulatinamente.
♦ ESCENARIO 3: El agricultor supone que se avecina una época de crisis pero que los ingresos se recuperan más tarde volviendo a niveles aceptables.
♦ ESCENARIO 4: El agricultor supone que se avecina una crisis en la que los ingresos se sitúan por debajo de un umbral permitido y no vislumbra el final de dicha crisis.

- Traducción al lenguaje matemático:
  Se generaliza el modelo anterior, introduciendo una modificación para incluir las expectativas de la variable I_t.
- Elección de la técnica de resolución:
  ⇒ Preparación de los datos completando las series con las expectativas, generadas de acuerdo a las hipótesis supuestas con respecto al comportamiento del agricultor.

$\Rightarrow$ Modelo semejante al anterior pero sustituyendo los aproximantes de Padé por aproximantes de Padé-Laurent y generalizando la teoría Box-Jenkins para series temporales que incluyan expectativas del input.

$$\underbrace{y_t}_{\text{Output actual}} = \underbrace{\underbrace{\ldots + v_{-1}x_{t+1}}_{\text{Expectativas Input}} + \underbrace{v_0 x_t + v_1 x_{t-1} + \ldots}_{\text{Input actual y pasado}}}_{\text{Parte determinista}} + \underbrace{N_t}_{\text{Parte Estocástica}} \cong \underbrace{\frac{P_{-p_1,p_2}(B)}{Q_{-q_1,q_2}(B)} x_t + \frac{\theta_q(B)\vartheta_Q(B^s)}{\Theta_p(B)\Phi_P(B^s)}}_{\text{Modelos Racionales}} \underbrace{a_t}_{\text{Proceso Ruido Blanco}}$$

- Resolución matemática:

$$y_t = (0.2001_{(4.07)} + 0.1296_{(2.69)} B)B^{-7} x_t + (1 + 0.8776_{(11.32)} B - 0.1153_{(1.48)} B^2) a_t$$

(ESCENARIO 1)

$$y_t = \frac{0.1661_{(3.00)}}{1 - 0.5417_{(3.62)} B^2} B^{-7} x_t + (1 + 0.7589_{(8.81)} B - 0.2469_{(2.89)} B^2) a_t$$

(ESCENARIO 2)

$$y_t = \frac{0.1645_{(3.06)}}{1 - 0.5543_{(3.86)} B^2} B^{-7} x_t + (1 + 0.7342_{(8.94)} B - 0.2770_{(3.41)} B^2) a_t$$

(ESCENARIO 3)

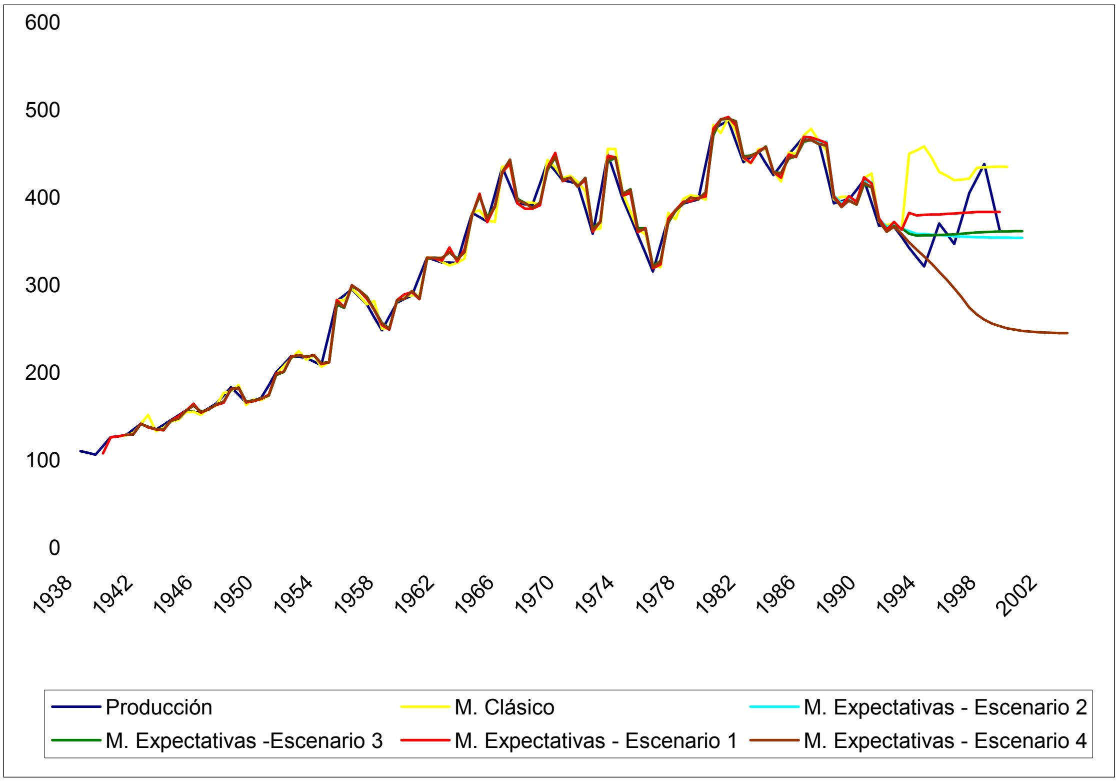
$$y_t = \frac{0.1649_{(3.03)}}{1 - 0.5480_{(3.60)} B^2} B^{-7} x_t + (1 + 0.7959_{(10.29)} B - 0.2129_{(2.82)} B^2) a_t$$

(ESCENARIO 4)

- Conclusiones matemáticas:
  - $\Rightarrow$ Aparece la dependencia racional de forma explícita.
  - $\Rightarrow$ Se ve la influencia de x_t+7 sobre y_t, aspecto que no podía observarse con el modelo anterior.
  - $\Rightarrow$ Todos los modelos se ajustan bien a los datos y se obtiene una predicción mejor que la ofrecida por el modelo clásico.
- Interpretación económica:
  - $\Rightarrow$ La vemos sobre la gráfica.
  - $\Rightarrow$ Las expectativas sobre el ingreso influyen en la planificación de la producción a 3 ó 4 años vista.
  - $\Rightarrow$ Se recoge la importancia de la ayuda compensatoria.
  - $\Rightarrow$ El Escenario 1 es el más deseable. Compensa la sobreproducción.

⇒ Los Escenarios 2 y 3 confirman el optimismo vigente en el año 97 por parte del agricultor de recuperación posterior.

⇒ El Escenario 4 manifiesta la importancia de la ayuda compensatoria o vías alternativas de compensación por pérdida de renta para conservar la producción platanera y el paisaje.



- <u>Hacia modelos más realistas:</u>

⇒ Consideración de otras variables, por ejemplo, costes de producción y comercialización.

a) Modelos Multivariantes.
b) Modelos estacionales.
c) …

⇒ Búsqueda de alternativas para conservar expectativas positivas por parte del agricultor.

a) Vía precios. Competencia en un mercado globalizado.
b) Vía costes. Competencia con otros países donde la producción es más barata.
c) Explotación de cambios tecnológicos apropiados.
d) Vía potenciación de variedades diferenciadas.
e) Vía nuevos mercados, nuevos consumos.
f) Vía compensación por conservar paisaje.
g) Vía influencia indirecta en el sector turístico.
h) ...

⇒ Sustitución del plátano por otros cultivos.

3. Conclusiones

La modelización en Economía, como en otras ciencias, es una actividad cíclica e interdisciplinaria en la que siempre quedan muchos aspectos que mejorar, hipótesis que estudiar y una infinidad de posibilidades para investigar. Los modelos racionales dinámicos que hemos estudiado, como sustitutos de modelos polinómicos, constituyen un ejemplo que ilustra esta idea.

Bibliografía

[1] H.J. Arrow, M.D. Intriligator, W. Hildebrand (editores): *Handbook of Mathematical Economics.* North-Holland, Amsterdam, 1984-1991.
[2] K. Devlin: *El lenguaje de las matemáticas.* Ediciones Robinbook, Barcelona, 2002.
[3] C. González-Concepción, M.C. Gil-Fariña: *El lenguaje de la ciencia económica: ¿Por qué la economía no prescinde de las matemáticas?* Editorial RA-MA, Madrid, 2000.
[4] M.C. Gil-Fariña, C. González-Concepción: La producción del plátano de Canarias y las expectativas del agricultor sobre la ayuda compensatoria. *Revista Española de Estudios Agrosociales y Pesqueros*, Ministerio de Agricultura, Pesca y Alimentación, no. 194 (2002), 127-146.
[5] M. Szenberg (editor): *Grandes economistas de hoy.* Editorial Debate, Madrid, 1994.
[6] http://almaz.com/nobel/economics
Nobel Prize in Economic Sciences Winners 2002-1969
The Nobel Prize Internet Archive.
[7] http://www.euro.mineco.es
EuroWeb
Ministerio de Economía de España.

La Matemática y la sabiduría popular de los canarios

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Introducción

Quizá sea en las Matemáticas, en la Física y en otras Ciencias de las denominadas exactas donde la visualización práctica de la teoría en la vida cotidiana pueda parecer menos evidente y necesaria. Estas materias, entendidas como paradigma de lo abstracto y, en apariencia, alejadas del contraste han venido enseñándose desde un punto de vista enteramente formal, sobre todo a partir de la revolución Bourbakiana que extralimitó el rigor en la experiencia investigadora. Aunque históricamente, y coincidiendo con la Revolución que acaeciera en torno al siglo XVI, el Método Científico se convirtió en el único Modelo Conceptual aceptado como herramienta imprescindible para interpretar la realidad, en el albor de la Ciencia Moderna, las disciplinas aplicadas: Metrología, Astronomía, Meteorología y otras, conformaban el corpus de las enseñanzas propias de los currículos medievales, y el desarrollo de la nueva Ciencia se dio en conjunción con tales aplicaciones. Conviene recordar que los avances del Álgebra Renacentista no se pueden entender sin los trabajos de Lucca Pacioli, Viéte o Fibonacci, que, en su origen, estuvieron interesados en la aplicación de las nuevas técnicas en la formalización rigurosa de la Contabilidad Analítica.

En la perspectiva actual, son escasos los ejemplos de teorías que expliquen la naturaleza tal como la entiende la gente ajena a la instrucción en niveles superiores. En contadas ocasiones podemos encontrar alguna interpretación mítica de los fenómenos asociados con el movimiento de la Tierra y de los Astros, de la composición y estructura de nuestro Planeta o de las causas y efectos de la transmisión del calor o de la propagación de la luz. Estas interpretaciones, que los etnólogos vascos y catalanes han recopilado en el Norte de la Península, no han sido contrastadas entre los campesinos y marineros canarios; por cuanto éstos, además, se muestran reacios a exponer sus ideas.

En todo caso, podemos centrar nuestra atención sólo en la descripción de las prácticas y técnicas que, cotidianamente, ejecuta nuestro pueblo. Son éstas estrategias productivas o formas de actuación sobre el medio, carentes de cuestiones especulativas, pero de fácil aplicación y comprensión. Dichos procedimientos atañen fundamentalmente a las Ciencias Aplicadas del estudio de la Naturaleza: Meteorología, Cronología, Geofísica, Metrología, y otras; y en cada una de ellas cabe extraer un buen número de aplicaciones pedagógicas.

Conocemos al menos dos formas de transmisión y recopilación del saber popular. Una, más universal y generalmente extendida, se expresa con la ayuda de refranes, proverbios, “aberruntos” y calendarios, que son conocidos por la mayoría de la población y que se aplican indistintamente para valorar la incidencia de las condiciones naturales en las tareas cotidianas. Otra, de carácter particular de cada comarca o región, se recoge en conocimientos específicos, más elaborados y contrastados científicamente, y es patrimonio singular de ciertas personas, conocidos y reconocidos como sabios, zahoríes o adivinadores.

El origen de la primera de las formas de conocimiento popular y la génesis de sus estructuras se pierden en la tradición grecolatina y mediterránea. Refranes que usan nuestros hombres del campo y de la mar ya fueron recogidos por Rodrigo Zamorano en su "Cronología de la razón de los tiempos", 1594; prácticas adivinatorias y procedimientos para ejecutar las labores agrícolas de acuerdo con los movimientos de los astros se reconocen iguales a los actuales en los textos de Columella, Vitruvio y Paladio, y prácticas de predicción meteorológica, enteramente similares a las que han sido recopiladas en Canarias, se encuentran en los textos clásicos de Alonso de Herrera y Vitruvio. Podemos argumentar, por tanto, que este conjunto de saberes, transmitidos de forma oral de generación en generación, forma parte de un corpus general, reconocible en todos los ámbitos geográficos iberoamericanos, y sujetos a escasas variaciones en su temática, estructura y aplicación práctica.

Por otra parte, los saberes populares atesorados por magos, curanderos o zahoríes se nos muestran específicos de cada zona. Sus fórmulas adivinatorias o de predicción son poco o nada conocidas por el resto de la población. Se ocultan celosamente al investigador etnográfico, y sólo admiten una aplicabilidad local. Éstos gozan de gran reconocimiento entre sus vecinos, sus aseveraciones admiten un alto predicamento general y son destacados como "sabios". Difícilmente sabremos descubrir el origen de sus conocimientos. En opinión de algunos investigadores, son sucesores de los adivinos o zahoríes que Fray Alonso de Espinosa denominara *Guanameñes* entre los aborígenes tinerfeños. Mas, habremos de coincidir con D. José Padrón Machín que estos sabios del pueblo han adquirido su sabiduría desentrañando todas las recetas, prácticas de procedencias varias y remedios con los que se ha enriquecido el acervo tradicional isleño, ya sean portugueses, peninsulares o americanos. La sabiduría de estos hombres se alimenta en muchas ocasiones de los conocimientos extraídos de manuales y libros de Aritmética y Geometría. Estos textos, sencillos y prácticos, han sido difundidos en la mayor parte de las escuelas rurales del Archipiélago, y sus fórmulas y técnicas operativas han perdurado en la memoria de aquellos hombres más aptos para el estudio y la curiosidad.

Como ya comentamos, los conocimientos populares atañen a las ramas aplicadas de las ciencias exactas; y, en particular, a la Metrología, o ciencia de las Medidas; a la Aritmética, aplicada en contabilidades mercantiles; a la Meteorología y a las aplicaciones geofísicas, cartográficas y de Agrimensura de la Geometría plana y de los cuerpos sólidos. Detengámonos en describir los contenidos de la Sabiduría Tradicional Canaria en cada una de estas materias.

Medida, Estimación y Cálculo de Magnitudes: la Metrología Tradicional

Las medidas de uso común en Canarias coincidían, casi por completo, con aquellas que conquistadores y tratantes peninsulares introdujeron en épocas de Conquista. A partir de entonces, los pesos y medidas premétricas se regularon a través de Ordenanzas y Disposiciones de los Cabildos; y, así, lo que en principio constituía un apretado amasijo de patrones y formas de medición de orígenes diversos, confluyó en un nuevo Modelo Metrológico Tradicional, de carácter eminentemente ergométrico. Su uso quedó regulado por leyes promulgadas desde la capital del Reino (Ley de 7 de Enero de 1496, sancionada por los Reyes Católicos; Pragmática de 24 de Junio de 1568, firmada por Felipe II; Real Orden de 26 de Enero de 1801, de Carlos IV), que imponían una unificación metrológica en todas las posesiones de la Corona española.

Con todo, siendo las Islas tradicional "tierra de promisión", frecuentemente visitadas, pobladas, y, ante todo y sobre todo, gobernadas por tratantes y comerciantes de orígenes y procedencia variopintas, a lo largo de la evolución metrológica isleña aparecen innumerables unidades procedentes de sistemas diversos que, en algunos casos, fueron recogidas por nuestros antepasados, quienes las incorporaron así a nuestro rico acervo.

En todo caso, las medidas de nuestros mayores no se entienden como restos primitivos de un proceder arcaico y obsoleto, sino, más bien, como herramientas precisas, aplicadas como estrategias métricas inteligentes, que posibilitan resolver los problemas de cálculo exigibles en todas y cada una de las tareas propias del trabajo cotidiano. La Metrología Tradicional supo responder con éxito a las necesidades matemáticas de todas aquellas personas, que, carentes de instrucción, hubieron de afrontar la valoración de sus producciones agrícolas o marineras, el reparto de las cosechas y zafras o la cuantificación de sus propiedades y pertenencias.

Patrones de capacidad para áridos

Las causas del éxito de estas prácticas metrológicas, que aún hoy en día permanecen fuertemente arraigadas, desafiando el uso generalizado de los patrones decimales, radican en que nuestro Sistema Métrico Decimal es un modelo de medidas fuertemente jerarquizado, de reciente invención (fue instaurado por primera vez en la Francia Revolucionaria, en 1795) y de aún más cercana popularización (hasta mediados del siglo pasado no se generalizó su uso en Canarias, cuando ya desde 1849 fue impuesto como sistema legal de medidas en toda España). En él, los patrones se hallan ligados entre sí por factores de conversión convencionales, con divisores y múltiplos en escala decimal; proponiendo un entramado de patrones intangibles, universales, invariables e invariantes. Y la práctica cotidiana con las unidades métricas no resulta sencilla y accesible. Al contrario, su estructura decimal imposibilita los repartos y los cálculos, por cuanto sólo permite divisiones exactas entre múltiplos de cinco y diez; sus patrones, intangibles y abstractos, carentes de significado ergonómico concreto, imposibilitan su uso reiterado, y su estructura matemática, compleja y convencional, provoca la incomprensión de todos los que no han sido instruidos en los principios básicos de dicha Ciencia.

Por contra, los patrones metrológicos tradicionales se estructuran en sistemas individualizados: de capacidad, longitud, peso y superficie; en los cuales las unidades se materializan en moldes tangibles: almudes, raposas, arrobas, quintales, etc.; los factores de conversión responden a las exigencias propias de cada proceso productivo, y los múltiplos y divisores, o bien están en relación dicotómica (son divisibles por dos), o en escala duodecimal (esto es, divisibles por doce, y, por lo tanto, también por dos, tres, cuatro y seis). Como consecuencia, el uso de los patrones tradicionales simplifica y no dificulta la práctica metrológica y encarece, por tanto, su éxito.

Números y Operaciones: Aplicaciones de la Aritmética

La aplicación más apreciada de la matemática en general, y de la Aritmética en particular, se identifica con la capacidad de ejecución de "cuentas". La facilidad para realizar cálculos numéricos siempre fue apreciada con gran reconocimiento entre las capas populares, y todas estas habilidades se encuentran enteramente imbricadas con la tradición oral del Archipiélago. Así podemos comprobarlo retomando el inicio de la historia de nuestra Cultura Popular.

Entre las actividades aborígenes que siguieron desarrollándose tras la Conquista destacan las asociadas con el pastoreo. Los pobladores prehispánicos de Canarias continuaron en gran medida su oficio de pastores y se adaptaron en cierta forma a las nuevas condiciones del modelo económico impuesto por los conquistadores. De procedencia aborigen, y heredada por los actuales "cabreros" de las Islas, perdura su habilidad a la hora de reconocer, contar o valorar el número total de cabezas de ganado que pastorean a su cuidado. Esta práctica matemática primitiva representa una habilidad notable, propia del estado intermedio en el desarrollo mental del hombre primitivo; se reconoce de igual forma en otras sociedades de tradición pastoril, y ya fue descrita por los primeros cronistas de nuestra historia reciente.

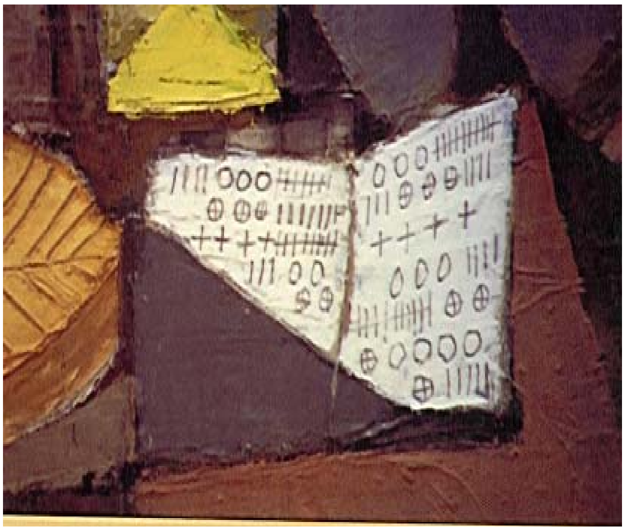
En concreto, los primeros colonizadores europeos quedaron notablemente sorprendidos por la facilidad de los pastores guanches para reconocer con exactitud el número de cabezas de ganado que poseían, y que computaban enteramente "de memoria". Los primitivos habitantes de Tenerife, y sus descendientes actuales, distribuyen el ganado en subgrupos (a modo de complejos lógicos), que quedan delimitados de acuerdo al color de su pelaje, al nombre de las cabras o por sus características en el comportamiento diario. La disposición en el terreno (cabras delanteras o traseras), su estado de salud (enfermas, preñadas, etc.), y las relaciones de consanguinidad, les aporta, asimismo, una clasificación topológica en clases diferenciadas.

Comparando estas técnicas de asociación y clasificación con los rudimentos de la teoría del conocimiento, podemos entender el recuento de nuestros pastores como un modelo elemental de clasificación y seriación, realizado con herramientas de contaminación y limitación, propias de las etapas preoperatorias en el desarrollo de los conceptos numéricos.

También entre los primeros sistemas de registro conocidos por la humanidad encontramos la notación con ayuda de marcas o muescas, reconocible en todas las sociedades de tradición pastoril. En Canarias se discute sobre el uso de tales "tájaras" o "tablas de contar" entre los aborígenes, y, aunque no existe evidencia material de su utilidad entre guanches, canarios o majos, su presencia actual en numerosas contabilidades agrícolas (en el cómputo de cosechas de cereales en Fuerteventura, en el registro de las cargas de bestia en el cultivo de la papa y de la vid en Tenerife, etc.) nos habla de un modelo de contabilidad y registro elementales, que se reconoce en toda la tradición comercial isleña.

De naturaleza similar son los recuentos que pescadoras y venteras de todas las Islas han venido ejecutando con ayuda de sus peculiares signos. En concreto, nuestras abuelas analfabetas se han apoyado en un complejo sistema de grafos, que utilizaron para representar el dinero y realizar el cómputo de las operaciones elementales en sus comercios. Tales signos presentan una gran uniformidad en cada sector comercial (en la venta al por menor y a domicilio del pescado, el pan o la leche) y recopilan un patrimonio ancestral, de clara procedencia pastoril. Conocemos con precisión el origen de tal simbología, que se muestra enteramente diferenciada según el tipo de actividad donde se ejecuta, diferenciándose las grafías de pescadoras y sus áreas de influencia de las reconocibles entre las "venteras" de las medianías.

Precisada la notación estándar de cada moneda en uso, los cálculos con tales signos permiten efectuar operaciones elementales: sumas; productos sencillos por adición reiterada; sustracciones, que se ejecutan al devolver el cambio; y repartos proporcionales, sus toscas divisiones. Tales cálculos comportan un avance técnico y teórico respecto del cómputo realizado con ayuda de "tarjas" y tablas de contar, y pueden servir como ejemplificación práctica de los rudimentos teóricos que subyacen en la manipulación de las operaciones.



Los signos de nuestras venteras

En todo caso, en ambos modelos de cómputo primitivo se obvia el cálculo mental, muy valorado entre campesinos y tratantes. Éste aparece en la dilatada práctica de conversión de cuentas valoradas en reales, onzas o duros a pesetas. La reducción de tales contabilidades en distintas unidades monetarias posibilita entender el producto y la división como estrategias de cálculo diversificadas, abundando en el significado numérico de los conceptos de múltiplo y divisor.

Geometría Práctica: Técnicas de Medición y Representación y Organización del Espacio

De la Agrimensura se pueden extraer diversos métodos para la valoración de distancias inaccesibles y prácticas de cómputo de áreas de terrenos limitados por contornos irregulares. Se encuentran detallados con todo lujo de aplicaciones cotidianas en el librito de D. José Estevez Méndez, donde se explica, en particular, el conocido método del "fraguero", propio y particular de Canarias, que ya fuera descrito por el farmacéutico D. Cipriano de Arribas y Sánchez a comienzos de este siglo. Este método permite medir la altura de árboles o edificios contando tan sólo con la ayuda del cuerpo del observador. D. Cipriano lo comenta como sigue:

> *El pino de la madre del agua tiene 66 metros de altura con 7'80 metros de circunferencia.*
> *Como carecía de medios a propósito para medirlos, una persona que me acompañaba sacóme del apuro, diciéndome: - Dé V. la espalda al pino, vaya marchando de frente y mirando por entre las piernas lo más posible, con la cabeza cercana a la tierra hasta*

que vea la copa del árbol; verificando así, mida la distancia que exista desde donde V. está hasta el tronco del árbol y ésta será la medida absoluta de su altura.

El método del fraguero, que aparece anotado en numerosos tratados de Geometría, se reconoce como uno de los procedimientos más antiguos, pues ya fue recogido por Oroncio Fineo en 1553 en la traducción del texto de Jerónimo Girava, y su fundamento científico se basa en la aplicación correcta del teorema de Tales. Ha perdurado en la memoria de esos hombres sabios, destacados entre sus pares por su sabiduría, aunque su práctica y uso no exigen ni determinan conocimiento matemático avanzado.

En Geometría esférica y de los cuerpos sólidos el saber tradicional recoge prácticas relativas al cálculo de volúmenes y a las imbricaciones planetarias del cómputo horario. En el primero de los tópicos, la valoración de los volúmenes de los cuerpos sólidos hace un uso generalizado del principio de Cavalieri. En particular, sabemos de un sencillo procedimiento que posibilita obtener la distancia que se da entre un observador y el horizonte y que se fundamenta también en la esfericidad de nuestro planeta. Dicho procedimiento se encuentra detallado en algunos manuales de geometría antigua, y, en esencia, consiste en:

Medirle la altura sobre el nivel del mar a la que se encuentra el observador, sumarle a esta altura la estatura del observador. Se obtiene entonces la raíz cuadrada del resultado y se multiplica por el factor 3,5. El valor que aparezca al final de las operaciones supone la distancia en kilómetros que lo separa de la línea del horizonte.

Este procedimiento se fundamenta en el hecho de que todo observador puede divisar desde cualquier punto de la superficie terrestre un horizonte equivalente a la extensión de cualquier círculo máximo. En este caso, podemos suponer que nos encontramos en un plano donde observador y horizonte abarcan la visión máxima de una circunferencia comprendida entre el punto de observación P y el punto (a,b) donde intersecta la tangente a dicha curva trazada desde P. La ecuación de dicha recta tangente viene dada por:

$$\mathbf{y \cdot b + x \cdot a = R^2}$$

donde R es el radio de la Tierra. Entonces la intersección (a,b) se dará en:

$$\mathbf{a = R^2/(R + h),}$$
$$\mathbf{b = \sqrt{\;}\,[R^2 - R^4/(R + h)^2],}$$

siendo h la suma de la altura del observador y de la distancia que lo separa verticalmente del mar. Calculando la distancia euclídea entre los puntos (a,b) y (R+h,0) encontramos que la visual hasta el horizonte se extiende hasta el valor:

$$\mathbf{V} = \sqrt{h} \cdot \sqrt{2R}$$

y, como quiera que el valor aproximado de raíz de 2R es equivalente a 3,5683 · 100, la fórmula anterior coincide casi por completo con la que apuntara el método práctico que hemos comentado.

La aplicación de la Geometría práctica se extiende también al cómputo de las capacidades de barriles, barricas y toneles, estímulo de la pericia de "cubicadores" de barriles, toneleros y bodegueros.

El "Aforo Diagonal" es el procedimiento práctico para aforar toneles y barriles de menor dificultad en su implementación práctica, pero que atesora el mayor contenido matemático. Es técnica antigua, que interesó notablemente a los matemáticos medievales y renacentistas, atentos siempre a la pericia de los "gauger", esto es, de los "cubicadores".

"Cubicadores" ejecutando el Aforo Diagonal

El procedimiento de Aforo Diagonal consiste en medir la distancia L que se extiende entre la boca del barril horadada en su vientre hasta el extremo más alejado de uno de los fondos; se eleva el valor calculado al cubo y se multiplica el resultado por el factor corrector 0,625, obteniéndose de esta forma el volumen:

$$\mathbf{V_8 = 0,625 \cdot L^3}$$

Esta fórmula, recogida por numerosos textos de geometría elemental (J. Estévez, Morroyo y Gago, Bruño, etc.), es la que se utilizaba en Canarias, en ejercicio de maestros de tonelería y viticultores expertos. La explicación de la exactitud de este método nos la propone P. Gianni en su obra "Práctica de Geometría y Trigonometría", de 1784, quien identifica el prototipo de tonel con un esferoide, del cual se conoce su volumen ("solides" según la terminología dieciochesca) igual a 2/3 del área de la máxima latitud circular por la longitud total de la figura.

Así, en un esferoide formado por dos esferas tangentes de radio R y una que las envuelve, tangente a ambas, el volumen se calcula por la fórmula:

$$\mathbf{V = 2 \cdot 3,141516 \cdot (VV')^2 \cdot AB / 12}$$

La distancia VV' coincide con la longitud L que se usa en la medición diagonal, pues la circunferencia envolvente es tangente a las dos en S y S'. Entonces quedará:

$$\mathbf{V = 2 \cdot 3{,}141516 \cdot L^2 \cdot AB / 12}$$

La circunferencia superior, de ecuación:

$$\mathbf{x^2 + (y - R)^2 = R^2}$$

es tangente a la envolvente:

$$\mathbf{(x - L/2)^2 + y^2 = L^2}$$

con lo cual deberá existir un único punto de intersección entre ambas. Éste será:

$$\mathbf{y_0 = 64 \cdot R / 40}$$

Además, por ser tangentes, existe una relación entre los valores de L y R, dada por:

$$\mathbf{R = 3 \cdot L / 8;}$$

entonces, el volumen del tonel, que tan sólo se extiende hasta el valor de y_0, vendrá dado por:

$$\mathbf{V = 2 \cdot 3{,}141516 \cdot L^2 \cdot 2y_0}$$
$$\mathbf{V = 3{,}141516 \cdot L^3 / 5}$$

que de forma aproximada equivale a:

$$\mathbf{V = 0{,}625 \cdot L^3}$$

Como vemos, existe un compendio de procedimientos prácticos, de uso generalizado entre nuestros campesinos, que esconden una explicación científica nada trivial. Y habremos de concluir corroborando que estos ejemplos de aplicaciones de conocimientos y saberes, sin ser simples, han otorgado carácter de sabiduría a las prácticas de nuestros agricultores y campesinos.

Bibliografía

C. de Arribas Sánchez: *A través de las Islas Canarias*. Museo Arqueológico, Cabildo de Tenerife, S/C de Tenerife, 1993.

A. de Chaves: *Espejo de navegantes.* Museo Naval, Madrid, 1983.

J. Estévez Méndez: *Problemas de Matemáticas.* S/C de Tenerife, 1954.

J.M. González Rodríguez: *Medidas y contabilidades populares: Las cuentas de las pescadoras y venteras del Valle de La Orotava.* Centro de la Cultura Popular Canaria, La Laguna, 1992.

J.M. González Rodríguez: *La sabiduría popular: Técnicas y conocimientos científicos tradicionales en Canarias.* Centro de la Cultura Popular Canaria, La Laguna, 1993.

A. de Herrera: *Agricultura general.* Ministerio de Agricultura, Madrid, 1996.

N. de Hoyos Sainz: *Manual de Folklore: La vida popular tradicional en España*. Ediciones Istmo, Madrid, 1985.

B. Morroyo y Gago: *Tratado elemental de Geometría*. Imprenta y Librería Moderna, Logroño, 1916.

J. Padrón Machín: *El Hierro: Séptima Isla.* Aula de Cultura del Cabildo Insular de Tenerife y Centro de la Cultura Popular Canaria, S/C de Tenerife, 1989.

Vitruvio: *Los diez libros de arquitectura.* Alianza Forma no. 133, Madrid, 1995.

En Internet

http://nti.educa.rcanaria.es/culturacanaria/matematicas/mattrad.htm

Las matemáticas y las ciencias tradicionales en Canarias

Curso de Cultura Canaria de la Consejería de Educación, Cultura y Deportes de Canarias.

Optimización Matemática: Ejemplos y aplicaciones

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Resumen

Se introducen aquí diversas aplicaciones de la Optimización Matemática, tanto de la Lineal Continua como de la Entera. El objetivo es mostrar algunas de las tantas situaciones reales que demandan métodos de optimización. En cada situación se propone un *Modelo Matemático*, punto de partida fundamental para intentar afrontar su resolución mediante Programación Matemática. En la charla veremos además algunas herramientas que permiten encontrar soluciones para los problemas a partir de los modelos.

1. Introducción

La Programación Lineal Continua trata sobre la resolución de problemas de optimización que pueden modelizarse en la forma

$$\min \sum_{j=1}^{n} c_j x_j$$

sujeto a

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{para todo } i = 1,\ldots,m,$$

donde n representa el número de variables, m el número de restricciones, x_j $(j = 1,\ldots,n)$ las variables del problema, y a_{ij}, c_j, b_i $(i = 1,\ldots,m; j = 1,\ldots,n)$ números reales dados. Al número c_j se le llama *costo* asociado a la variable j-ésima $(j = 1,\ldots,n)$, mientras que al número b_i se le llama *recurso* asociado a la restricción i-ésima $(i = 1,\ldots,m)$. Utilizando notación vectorial, el problema anterior puede ser representado en la *forma compacta* siguiente:

$$\min \mathbf{c}^T \mathbf{x}$$
$$\mathbf{A}\mathbf{x} \le \mathbf{b},$$

siendo $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ y $\mathbf{A}$ una matriz con m filas y n columnas. Observemos que la región factible es un poliedro.

Cualquier problema de Programación Lineal puede reescribirse de la siguiente forma, llamada *forma estándar*:

$$\begin{aligned} \min \mathbf{c}^T \mathbf{x} \\ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0, \end{aligned}$$

aunque la transformación desde otra forma puede alterar el número de variables y/o restricciones del problema.

En general, la conversión puede llevarse a cabo como sigue:

1. Una función objetivo de tipo "max" se convierte en "min" cambiando de signo los valores c_j (y de signo la nueva función).
2. Una restricción de tipo $\sum_{j=1}^{n} a_j x_j \leq b_i$ se convierte en ecuación añadiendo una variable $x^h \geq 0$, creando $\sum_{j=1}^{n} a_j x_j + x^h = b_i$. Esta variable x^h se llama *variable de holgura*. Una restricción de tipo $\sum_{j=1}^{n} a_j x_j \geq b_i$ se convierte en ecuación añadiendo una variable $x^h \geq 0$, creando $\sum_{j=1}^{n} a_j x_j - x^h = b_i$.
3. Una variable x_j no limitada en signo se convierte en dos variables $x_j^+ \geq 0, x_j^- \geq 0$ mediante la sustitución $x_j = x_j^+ - x_j^-$.

Mediante pautas similares a las anteriores puede demostrarse que las siguientes formas son equivalentes:

1. $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$
2. $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$
3. $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$
4. $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$
5. $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\}$

También hay otras formas similares. Sin embargo, ninguna es equivalente a la forma $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$, ya que la región factible de ésta es un subespacio afín o el conjunto vacío, y no hay siempre alternativa para hacer desaparecer todas las desigualdades de un sistema lineal de ecuaciones e inecuaciones. Precisamente ésta es una de las grandes dificultades que afronta la Programación Matemática, y que proceden de las ineludibles limitaciones de recursos existentes en muchos problemas del Mundo Real.

Dado que las variables x_j son números reales, no necesariamente enteros, se dice que los modelos anteriores pertenecen a la Programación Lineal *Continua.* Cuando las variables x_j deben ser necesariamente números enteros, entonces se dice que los modelos son de Programación Lineal *Entera.*

En lo que sigue veremos diversos ejemplos concretos que demuestran las numerosas aplicaciones de la Programación Matemática, tanto Lineal Continua como Entera. Es conveniente observar que dado un problema real puede haber varios modelos matemáticos asociados, pudiendo resultar algunos menos útiles que otros para la resolución práctica del problema. En este sentido avisamos al lector de que en los ejemplos hemos optado sólo por modelos simples de presentar.

2. Problema de optimizar mezclas

Problema real

En la refinería de Santa Cruz de Tenerife (*C.E.P.S.A.*) se producen 3 tipos de gasolinas que describimos a continuación:

Tipo	Variedad	Octanaje
A	STAR-98	98 Oct.
B	Sin Plomo	95 Oct.
C	Súper	97 Oct.

Para ello se mezclan cuatro productos base, que representaremos con un número, y cuyo costo y disponibilidad son:

Producto	Disponibilidad	Costo/unidad
1	3000	3
2	2000	6
3	4000	4
4	1000	5

Para la clasificación de la mezcla en uno de los tres tipos de gasolina se atiende a la proporción de los productos que la componen según la siguiente tabla:

Producto	Prod. 1	Prod. 2	Prod. 3	Prod. 4	Beneficio/Unidad
A	$\leq 30\%$	$\geq 40\%$	$\leq 50\%$	s.l.	5,5
B	$\leq 50\%$	$\geq 10\%$	s.l.	s.l.	4,5
C	$\geq 70\%$	s.l.	s.l.	s.l.	3,5

donde “s.l.” significa que no importa la proporción de ese producto.

Modelo matemático

Consideremos las siguientes variables:

- $y_A \equiv$ Cantidad de gasolina de tipo A (STAR-98).
- $y_B \equiv$ Cantidad de gasolina de tipo B (Sin Plomo).
- $y_C \equiv$ Cantidad de gasolina de tipo C (Súper).
- $z_1 \equiv$ Cantidad de producto 1.
- $z_2 \equiv$ Cantidad de producto 2.
- $z_3 \equiv$ Cantidad de producto 3.
- $z_4 \equiv$ Cantidad de producto 4.
- $x_{ij} \equiv$ Cantidad de producto $i \in \{1,2,3,4\}$ invertido en $j \in \{A,B,C\}$.

Entonces, un modelo matemático es:

$$\max 5{,}5\, y_A + 4{,}5\, y_B + 3{,}5\, y_C - 3 z_1 - 6 z_2 - 4 z_3 - 5 z_4$$

sujeto a:

$$\begin{aligned} y_A &= x_{1A} + x_{2A} + x_{3A} + x_{4A} \\ y_B &= x_{1B} + x_{2B} + x_{3B} + x_{4B} \\ y_C &= x_{1C} + x_{2C} + x_{3C} + x_{4C} \end{aligned}$$

$$\begin{aligned} z_1 &= x_{1A} + x_{1B} + x_{1C} \\ z_2 &= x_{2A} + x_{2B} + x_{2C} \\ z_3 &= x_{3A} + x_{3B} + x_{3C} \\ z_4 &= x_{4A} + x_{4B} + x_{4C} \end{aligned}$$

$$\begin{aligned} z_1 &\le 3000 \\ z_2 &\le 2000 \\ z_3 &\le 4000 \\ z_4 &\le 1000 \end{aligned}$$

$$\begin{aligned} x_{1A} &\le 0{,}3\, y_A \\ x_{2A} &\ge 0{,}4\, y_A \\ x_{3A} &\le 0{,}5\, y_A \\ x_{1B} &\le 0{,}5\, y_B \\ x_{2B} &\ge 0{,}1\, y_B \\ x_{1C} &\ge 0{,}7\, y_C \end{aligned}$$

$$x_{ij} \ge 0 \quad \text{para todo } i \in \{1,2,3,4\} \text{ y para todo } j \in \{A,B,C\}.$$

Notemos que un simple análisis del modelo nos permite eliminar las variables z_i y las variables y_j haciendo uso de las ecuaciones. Es siempre muy importante realizar este proceso de simplificación (generalmente llamado *preproceso*).

3. Problema de planificar una cosecha

Problema real

Un agricultor tiene 500 hectáreas de terreno para cultivar próximamente y desea planificar tal cultivo. Sabe que necesitará disponer de 200 toneladas de trigo y 240 toneladas de maíz para alimentar a su ganado, lo que puede obtener mediante su propia cosecha o mediante compra en el mercado. Lo que produzca, y que no dedique a su ganado, lo puede vender. Los precios de venta son de 170 euros y 150 euros por cada tonelada de trigo y de maíz, respectivamente. Los precios de compra son un 40% superior debido a las ganancias de intermediarios y a los costos de transporte.

Otro cultivo posible es el de caña de azúcar, que se vende a 36 euros cada tonelada producida. Sin embargo, normas de la Comisión Europea imponen una cuota máxima para la producción de azúcar, lo que conlleva que cada tonelada de caña de azúcar producida sobre tal cuota tendrá un precio de venta de 10 euros. Para el próximo cultivo se espera que tal cuota sea 6000 toneladas.

Basado en experiencias anteriores, el agricultor conoce que la producción media es 2,5, 3 y 20 toneladas por hectárea de trigo, maíz y caña de azúcar, respectivamente. El costo de plantar una hectárea de trigo, maíz y caña de azúcar es de 150, 230 y 260, respectivamente.
Plantear un modelo matemático cuya solución pueda ayudar al agricultor en su deseo de maximizar sus beneficios.

Modelo matemático

Consideremos las siguientes variables:

- $x_1 \equiv$ hectáreas que dedicará a trigo,
- $x_2 \equiv$ hectáreas que dedicará a maíz,
- $x_3 \equiv$ hectáreas que dedicará a azúcar,
- $y_1 \equiv$ toneladas que comprará de trigo,
- $y_2 \equiv$ toneladas que comprará de maíz,
- $w_1 \equiv$ toneladas que venderá de trigo,
- $w_2 \equiv$ toneladas que venderá de maíz,
- $w_3 \equiv$ toneladas que venderá de azúcar a 36 euros,
- $w_4 \equiv$ toneladas que venderá de azúcar a 10 euros.

Un modelo matemático es:

$$\max -150x_1 - 230x_2 - 260x_3 - 238y_1 - 210y_2 + 170w_1 + 150w_2 + 36w_3 + 10w_4$$

sujeto a:

$$x_1 + x_2 + x_3 \leq 500$$

$$2{,}5x_1 + y_1 - w_1 \geq 200$$

$$3x_2 + y_2 - w_2 \leq 240$$

$$w_3 + w_4 \leq 20x_3$$

$$w_3 \leq 6000$$

$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0.$$

Este modelo pertenece a la Programación Lineal, y mediante algún método de resolución es posible concluir que una solución óptima es:

$$x_1 = 120,\ x_2 = 80,\ x_3 = 300,\ y_1 = 0,\ y_2 = 0,$$

$$w_1 = 100,\ w_2 = 0,\ w_3 = 6000,\ w_4 = 0,$$

con beneficio óptimo 118600. Esto significa que el agricultor deberá dedicar 120 hectáreas a trigo, 80 a maíz y 300 a caña de azúcar, y con ello se espera que venderá 100 toneladas de trigo y la cuota máxima de azúcar (es decir, al precio más favorable), obteniendo un beneficio total de 118600 euros.

Dada la sencillez del ejemplo, resulta evidente que también se puede obtener esta misma decisión (óptima) mediante una simple regla lógica: "dedicar cada hectárea de terreno a la producción que más beneficio conlleve". Así, tras producir lo que el ganado necesita, el resto conviene considerarlo en el siguiente orden: azúcar al precio favorable, trigo, maíz, azúcar al precio reducido. Este tipo de técnicas de resolución (que se denominan *heurísticas*) no siempre proporcionan una solución óptima de un problema si éste tiene otras restricciones adicionales más complejas.

4. Problema de optimizar músicos

Problema real

Una banda de músicos consta de 9 músicos (A,B,C,D,E,F,G,H,I) y debe repetir cada tarde un repertorio de 7 sinfonías (1,2,3,4,5,6,7). No todas las sinfonías necesitan de todos los músicos y cada músico recibe un salario proporcional al número de sinfonías en las que está presente (tocando o no su instrumento) desde la primera hasta la última en la que interviene. Ningún músico recibirá sueldo por sinfonías en las que esté presente antes de la primera en la que sea necesario ni después de la última en la que sea necesario, pero sí por todas las demás. La tabla 1 muestra las sinfonías en las que cada músico debe tocar su instrumento, así como el sueldo que recibe por cada sinfonía en la que tenga necesariamente que estar presente. ¿Cómo debe ordenar el director las sinfonías para minimizar el coste total de los salarios?

Músico	A	B	C	D	E	F	G	H	I
Sinfonías	1,7	2,4,7	1,2,5,7	1,3,5	2,3,5,6	1,2,4,6,7	3,5,7	4,6	1,2,3
Coste/sinfonía	2	3	3	2	1	2	2	1	2

Tabla 1. Sinfonías y coste por músico.

Modelo matemático

Sea c_k el coste por sinfonía del músico k. Notemos que para cada músico el coste de las sinfonías en las que toca su instrumento es un valor constante. Por ejemplo, para el músico A su coste concerniente con las dos sinfonías en las que interviene es 4, y luego habrá que añadir dos unidades de coste adicional por cada sinfonía que el director decida colocar entre las sinfonías 1 y 7. Por tanto, el problema equivale a minimizar el costo por las sinfonías en las que el músico no toca y que están entre dos en las que sí toca.

Consideremos la matriz $\mathbf{A}=[a_{ik}]$ donde cada fila i corresponde con una sinfonía y cada columna k con un músico. Mediante la tabla 1 es fácil notar que:

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Entonces, el problema puede ser entendido matemáticamente como el problema de reordenar las filas de $\mathbf{A}$ de manera que sea mínima la suma del número de ceros entre unos en una misma columna k por el costo de dicha columna c_k.

Un posible modelo consiste en definir, para cada sinfonía i y cada posible posición j en la ordenación final, la variable decisional

$$x_{ij} := \begin{cases} 1 & \text{si la sinfonía } i \text{ se toca en la posición } j \\ 0 & \text{en otro caso,} \end{cases}$$

donde $i,j \in \{1,2,3,4,5,6,7\}$. Para cada músico k también consideramos la variable u_k representando la posición en la ordenación final de la primera sinfonía donde es necesario, y la variable w_k representando la posición en la ordenación final de la última sinfonía donde es necesario, siendo $k \in \{A,B,C,D,E,F,G,H,I\}$. Entonces, un modelo matemático es:

$$\min \sum_{k\in\{A,\ldots,I\}} (w_k + 1 - u_k)c_k = 18 + \sum_{k\in\{A,\ldots,I\}} (w_k - u_k)c_k$$

sujeto a:

$$\begin{aligned}
\sum_{j\in\{1,\dots,7\}} x_{ij} &= 1 \quad \text{para todo } i \in \{1,\dots,7\}\\
\sum_{i\in\{1,\dots,7\}} x_{ij} &= 1 \quad \text{para todo } j \in \{1,\dots,7\}\\
x_{ij} &\in \{0,1\} \quad \text{para todo } i,j \in \{1,\dots,7\}\\
\sum_{j\in\{1,\dots,7\}} j\, x_{ij} &\geq u_k \quad \text{para todo } i \in \{1,\dots,7\},\, k \in \{A,\dots,I\} \,:\, a_{ik} = 1\\
\sum_{j\in\{1,\dots,7\}} j\, x_{ij} &\leq w_k \quad \text{para todo } i \in \{1,\dots,7\},\, k \in \{A,\dots,I\} \,:\, a_{ik} = 1.
\end{aligned}$$

Nótese que $v_i := \sum_{j\in\{1,\dots,7\}} j\, x_{ij}$ es la posición donde conviene tocar la sinfonía i. Dado que cuando $x_{ij} \in \{0,1\}$, v_i será automáticamente entera y, por el carácter de la función objetivo, también sucede que

$$\begin{aligned}
u_k &= \min_{i\in\{1,\dots,7\}:a_{ik}=1} v_i\\
w_k &= \max_{i\in\{1,\dots,7\}:a_{ik}=1} v_i
\end{aligned}$$

para todo músico $k \in \{A,\dots,I\}$.

Conviene observar que este modelo se basa fuertemente en la condición $x_{ij} \in \{0,1\}$, es decir, en lo que distingue a este modelo de Programación Lineal Entera del modelo de Programación Lineal Continua que resulta de prescindir de tal condición. En algunas aplicaciones ambos modelos producen la misma solución; en otras suelen estar próximos (permitiendo que muchas técnicas para la Programación Lineal Entera se apoyen en técnicas de Programación Lineal); sin embargo, en este caso no es así. Es fácil notar que olvidando la integrabilidad de las variables x_{ij} (es decir, manteniendo sólo que deben asumir valores reales no negativos) se obtiene una solución óptima del problema lineal con valor objetivo cero (todas las u_k y w_k son iguales). Por tanto, el modelo matemático presentado es malo si se pretende abordar la resolución de este problema mediante técnicas que usen el modelo lineal.

Otra alternativa de modelo matemático es la siguiente. Añadamos una sinfonía ficticia 0 que hará el papel de apertura y cierre pero que no necesita de ningún músico (¡es música de cassette!). Para cada par de sinfonías $i,j \in \{0,1,2,3,4,5,6,7\}$ distintas ($i \neq j$) consideremos la variable decisional

$$y_{ij} := \begin{cases} 1 & \text{si cuando acaba } i \text{ comienza inmediatamente } j\\ 0 & \text{otro caso,} \end{cases}$$

y la variable t_i que representa el instante en el que debe ser interpretada la sinfonía i ($t_0 := 0$ y $t_i \in \{1,\ldots,7\}$). Además usamos las variables u_k y w_k antes definidas. Entonces, otro modelo matemático (también de Programación Lineal Entera) es:

$$\min \sum_{k\in\{A,\ldots,I\}} (w_k - u_k)c_k$$

sujeto a:

$$\begin{aligned}
\sum_{j\in\{0,1,\ldots,7\}\setminus\{i\}} y_{ij} &= 1 && \text{para todo } i \in \{0,1,\ldots,7\} \\
\sum_{i\in\{0,1,\ldots,7\}\setminus\{j\}} y_{ij} &= 1 && \text{para todo } j \in \{0,1,\ldots,7\} \\
y_{ij} &\in \{0,1\} && \text{para todo } i,j \in \{0,1,\ldots,7\} \quad (i \neq j) \\
t_0 &:= 0 && \\
t_j &\geq t_i + 7y_{ij} - 6 && \text{para todo } i,j \in \{0,1,\ldots,7\} \quad (i \neq j, j \neq 0) \\
t_i &\geq u_k && i \in \{1,\ldots,7\}, k \in \{A,\ldots,I\} \ : \ a_{ik} = 1 \\
t_i &\leq w_k && i \in \{1,\ldots,7\}, k \in \{A,\ldots,I\} \ : \ a_{ik} = 1
\end{aligned}$$

Aunque innecesaria para el modelo de Programación Lineal Entera, la siguiente restricción debe también verificarse:

$$\sum_{i\in S, j\notin S} y_{ij} \geq 1 \qquad \text{para todo } S \subset \{0,1,\ldots,7\},$$

ya que las variables y_{ij} deben definir una secuencia de las sinfonías.

Otro modelo diferente aparece al considerar también las variables y_{ij} con el sentido antes indicado, y para cada par de sinfonías i,j sea la variable z_{ij} que representa el número de las sinfonías que faltan por tocar cuando acaba i y empieza j. Además usamos las variables u_k y w_k antes definidas. Entonces, otro modelo matemático (también de Programación Lineal Entera) es:

$$\min \sum_{k\in\{A,\ldots,I\}} (w_k - u_k)c_k$$

sujeto a:

$$\sum_{j\in\{0,1,\ldots,7\}\setminus\{i\}} y_{ij} = 1 \quad \text{para todo } i \in \{0,1,\ldots,7\}$$

$$\sum_{i\in\{0,1,\ldots,7\}\setminus\{j\}} y_{ij} = 1 \quad \text{para todo } j \in \{0,1,\ldots,7\}$$

$$\sum_{i\in S, j\notin S} y_{ij} \geq 1 \quad \text{para todo } S \subset \{0,1,\ldots,7\}$$

$$y_{ij} \in \{0,1\} \quad \text{para todo } i,j \in \{0,1,\ldots,7\} \quad (i \neq j)$$

$$\sum_{j\in\{0,1,\ldots,7\}} z_{0j} = 7$$

$$\sum_{j\in\{0,1,\ldots,7\}\setminus\{i\}} (z_{ji} - z_{ij}) = 1 \quad \text{para todo } i \in \{1,\ldots,7\}$$

$$\sum_{j\in\{1,\ldots,7\}} z_{ij} \geq u_k \quad i \in \{1,\ldots,7\}, k \in \{A,\ldots,I\} : a_{ik} = 1$$

$$\sum_{j\in\{1,\ldots,7\}} z_{ij} \leq w_k \quad i \in \{1,\ldots,7\}, k \in \{A,\ldots,I\} : a_{ik} = 1$$

$$0 \leq z_{ij} \leq 7y_{ij} \quad \text{para todo } i,j \in \{1,\ldots,7\} \quad (i \neq j)$$

$$z_{i0} = 0 \quad \text{y} \quad z_{0i} = 7y_{0i} \quad \text{para todo } i \in \{1,\ldots,7\}$$

Al igual que el modelo anterior, este modelo puede ser "fortalecido" mediante restricciones adicionales notando que las variables y_{ij} deben definir una secuencia ininterrumpida cíclica de todas las sinfonías.

Dejamos propuesto al lector comparar la resolución de este problema mediante el uso de cada uno de los tres modelos. Una solución óptima de este problema viene dada por la secuencia 5,7,2,8,1,4,6,3, lo que produce 6 huecos.

5. Problema de optimizar la banda de una matriz

Problema real

Consideremos cualquier matriz cuadrada simétrica de dimensión $n \times n$. Las diagonales de una matriz se numeran atendiendo a su distancia con respecto a la diagonal principal. Así la diagonal principal tiene la etiqueta 0 mientras que las esquinas de la matriz que no están en la diagonal constituyen diagonales con etiqueta $n-1$. Se llama *ancho de banda* de $\mathbf{A}$ a la mayor etiqueta de alguna diagonal que contenga algún coeficiente no nulo. Por ejemplo, la matriz

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

tiene ancho de banda igual a 3. Encontrar una reordenación que, aplicada tanto a las filas como a las columnas, produzcan una nueva matriz con menor ancho de banda.

Modelo matemático

Consideremos la variable decisional x_{ij} que asume el valor 1 si la fila i (y la columna i) es colocada en la j-ésima posición y 0 en otro caso. Entonces, un modelo matemático es:

$$\min z$$

sujeto a:

$$\begin{aligned}
\sum_{j\in\{1,..,5\}} \sum_{l\in\{1,..,5\}} |j-l|\, x_{ij}\, x_{kl} \le z \quad & \text{para todo } i,k \in \{1,...,5\}: \quad a_{ik} \neq 0 \\
\sum_{j\in\{1,..,5\}} x_{ij} = 1 \quad & \text{para todo } i \in \{1,...,5\} \\
\sum_{i\in\{1,..,5\}} x_{ij} = 1 \quad & \text{para todo } j \in \{1,...,5\} \\
x_{ij} \in \{0,1\} \quad & \text{para todo } i,j \in \{1,...,5\}.
\end{aligned}$$

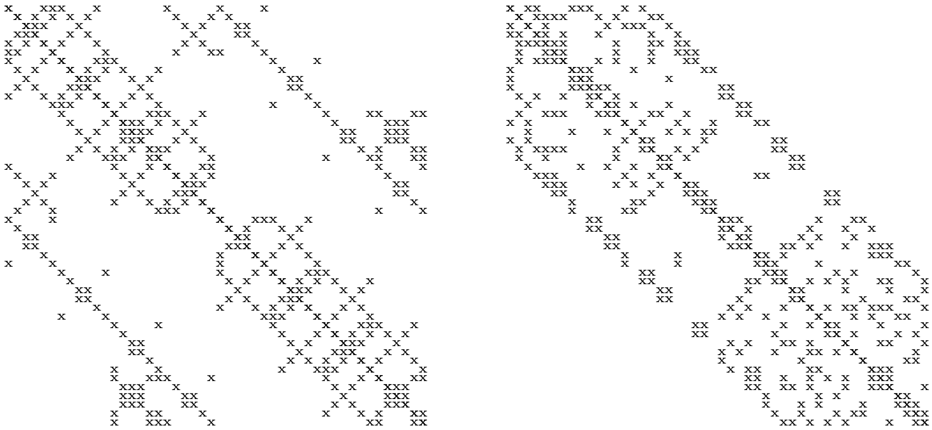
Éste es un ejemplo de un modelo matemático que *no* pertenece a la Programación Lineal Entera. Para presentar un modelo matemático de Programación Lineal Entera basta notar que la posición final de un elemento i cuando se conocen las asignaciones x_{ij} viene definida por $\sum_j jx_{ij}$. En consecuencia, otro modelo es:

$$\min z$$

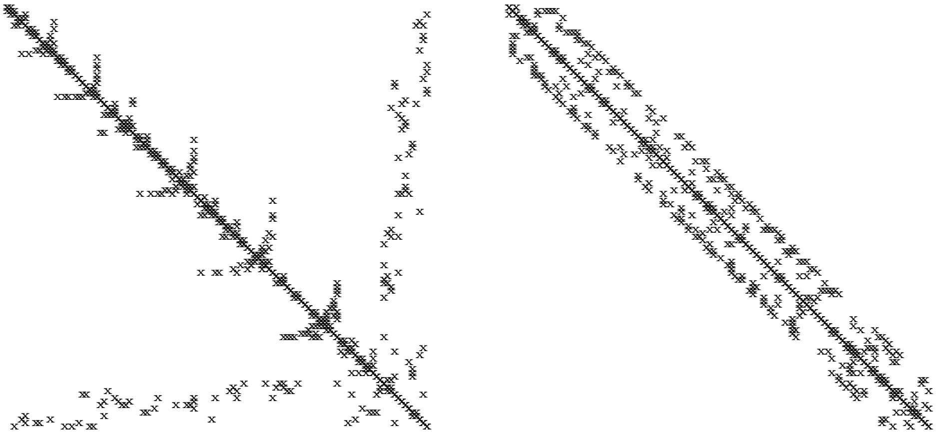
sujeto a:

$$\begin{aligned}
\sum_{j\in\{1,..,5\}} j(x_{ij} - x_{kj}) \le z \quad & \text{para todo } i,k \in \{1,...,5\} : a_{ik} \neq 0 \\
\sum_{j\in\{1,..,5\}} x_{ij} = 1 \quad & \text{para todo } i \in \{1,...,5\} \\
\sum_{i\in\{1,..,5\}} x_{ij} = 1 \quad & \text{para todo } j \in \{1,...,5\} \\
x_{ij} \in \{0,1\} \quad & \text{para todo } i,j \in \{1,...,5\}.
\end{aligned}$$

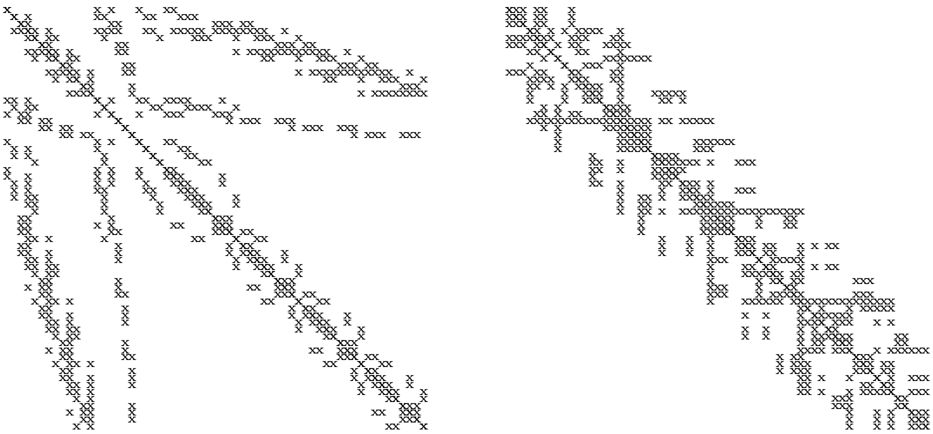
Usando este modelo es posible permutar las filas y columnas de algunas matrices dando lugar a otras matrices análogas con menor banda. La figura 1 muestra tres matrices de la colección *MatrixMarket* en su formato original y en el formato de banda mínima obtenido mediante el modelo matemático anterior.



Matriz “bcsstk01” de la colección *MatrixMarket*: en la izquierda tiene banda 35, en la derecha 16.



Matriz “bcspwr03” de la colección *MatrixMarket*: en la izquierda tiene banda 115, en la derecha 12.



Matriz “can___61” de la colección *MatrixMarket*: en la izquierda tiene banda 50, en la derecha 13.

Figura 1. Tres matrices en formatos original (izquierda) y con banda mínima (derecha).

6. Problema de optimizar una inversión

Problema real

El *C.D. Tenerife* ha puesto sus acciones en bolsa y un inversor ha descubierto la clave para sacar el beneficio que en el terreno de juego el resto de los accionistas no han obtenido en toda la temporada. El funcionamiento es el siguiente: al inicio de la temporada se puede invertir en ella una cantidad cualquiera de x euros, al comenzar la siguiente temporada se debe invertir adicionalmente $x/2$ euros, y luego pasada otra temporada se obtienen $2x$ euros. Lo obtenido en esas acciones al final de una temporada puede ser reinvertido de nuevo en dichas acciones al principio de la siguiente, si se desea. Si en el momento actual el inversor dispone de 100000 euros, ¿cuál debe ser su plan de inversión en tales acciones para disponer de un máximo capital dentro de 6 años?

Modelo matemático

Consideremos la variable x_i asociada a la temporada i-ésima ($i = 1,...,6$), representando el dinero invertido al inicio de dicha temporada. Es claro que, por las condiciones de las acciones, conviene que $x_5 := x_6 := 0$, ya que dichas inversiones no producen beneficios dentro de las 6 temporadas. Entonces, un esquema de inversión es:

Temporadas	**Nueva Inversión**	**Inversión adicional**	**Beneficios**
0	x_0	——	——
1	x_1	$x_0/2$	——
2	x_2	$x_1/2$	$2x_0$
3	x_3	$x_2/2$	$2x_1$
4	x_4	$x_3/2$	$2x_2$
5	——	$x_4/2$	$2x_3$
6	——	——	$2x_4$

Por tanto, un posible modelo matemático es el definido por las restricciones:

1^{er} año: $x_1 + x_0/2 \le (100000 - x_0)$.

2^{o} año: $x_2 + x_1/2 \le (100000 + x_0/2 - x_1)$.

3^{er} año: $x_3 + x_2/2 \le (100000 + x_0/2 + x_1/2 - x_2)$.

4^{o} año: $x_4 + x_3/2 \le (100000 + x_0/2 + x_1/2 + x_2/2 - x_3)$.

5^{o} año: $x_4/2 \le (100000 + x_0/2 + x_1/2 + x_2/2 + x_3/2 - x_4)$.

$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$x_3 \ge 0$$
$$x_4 \ge 0,$$

y su función objetivo viene dada por la maximización de la función lineal:

$$6^{\text{o}}\ \text{año}: \qquad 100000 + x_0/2 + x_1/2 + x_2/2 + x_3/2 + x_4/2,$$

equivalente (como función para maximizar) a $2x_0 + 2x_1 + 2x_2 + 2x_3 + 2x_4$.

6. Problema de optimizar un divisor de tensión

Problema real

A la hora de diseñar un circuito electrónico hay que tener presente que sus componentes están caracterizadas por diversos parámetros, cada uno de los cuales tiene asociada una cierta *tolerancia*. Por ejemplo, una resistencia de 48 ohmios puede tener una tolerancia de $\varepsilon = \pm 10\%$, debido a temperatura, tiempo en uso, etc. Al valor dado por el fabricante del componente (en el ejemplo, el número 48) lo llamaremos valor *centrado*. Las tolerancias también suelen venir dadas por el fabricante. Es importante tener presente estas tolerancias al tiempo de diseñar un circuito, y no sólo los valores centrados, ya que ellas nos permitirán controlar mejor los límites bajo los cuales el circuito se mantiene operativo, y en consecuencia optimizar el rendimiento del mismo.

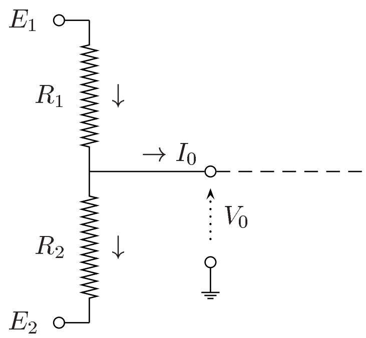


Figura 2. Divisor de tensión.

Consideremos el divisor de tensión de la figura 2. En él se asumen dos generadores de tensión, cada uno de los cuales produce una fuerza electromotriz de E_1 y E_2 voltios, respectivamente. Asumiremos que $E_1 > E_2$ para que el trozo de circuito representado en la figura mueva corriente en el sentido indicado por las flechas. También se asumen conocidas las tolerancias asociadas a dichos potenciales, es decir, se asumen dados valores mínimos E_1^-, E_2^- y máximos E_1^+, E_2^+ para E_1, E_2, respectivamente. Se desea determinar los valores centrados de las resistencias R_1 y R_2 de manera que la impedancia resistiva del divisor de tensión sea mínima y el potencial de salida V_0 se mantenga siempre dentro de un intervalo predeterminado $[V_0^{\min}, V_0^{\max}]$ cuando la corriente I_0 que se desea sacar del divisor (al conectar algún componente adicional) está entre un mínimo igual a $I_0^{\min}$ y un máximo igual a $I_0^{\max}$. Se asumen conocidas las tolerancias que tendrán las resistencias y que $E_1^- \geq V_0^{\max} \geq V_0^{\min} \geq E_2^+$.

Modelo matemático

La impedancia resistiva total del divisor viene dada por el valor:

$$R_0 := \frac{R_1 R_2}{R_1 + R_2},$$

ya que las dos resistencias están colocadas en paralelo. A fin de obtener un modelo lineal en las variables (las resistencias), conviene trabajar con las admitancias asociadas, es decir, con los valores

$$G_i := 1/R_i \text{ para todo } i \in \{0,1,2\}.$$

Con esta notación la admitancia total del divisor es $G_0 := G_1 + G_2$. Representemos con $+$ y $-$ el mayor y menor valor, respectivamente, del valor de cada característica del circuito al considerar su tolerancia.

El objetivo del problema propuesto equivale a minimizar el mayor valor R_0^+ que puede alcanzar la impedancia, o alternativamente a maximizar el menor valor que puede alcanzar su admitancia $G_0^- = G_1^- + G_2^-$. Si denotamos por $\overline{G}_1$ y $\overline{G}_2$ los valores centrados de las admitancias G_1 y G_2, respectivamente, es decir, valores tales que $G_1^- := (1-\varepsilon_1)\overline{G}_1$ y $G_2^- := (1-\varepsilon_2)\overline{G}_2$ para ε_1 y ε_2 dos tolerancias conocidas, entonces el objetivo es:

$$\max \qquad (1-\varepsilon_1)\overline{G}_1 + (1-\varepsilon_2)\overline{G}_2 \qquad (1)$$

Para expresar las restricciones sobre las variables $\overline{G}_1$ y $\overline{G}_2$, observemos que

$$V_0 = \frac{E_1 G_1 + E_2 G_2 - I_0}{G_1 + G_2}.$$

Esto es fácilmente deducible mediante la figura 3, donde se representa el mismo circuito con fuentes de intensidad (la figura 2 dio una representación del circuito con fuentes de voltaje).

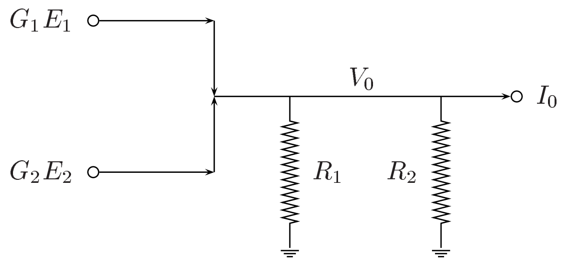


Figura 3. Circuito equivalente con fuentes de intensidad.

Nótese que de esta representación alternativa se deduce que, si mantenemos fija R_2, V_0 aumenta cuando R_1 disminuye, es decir, cuando G_1 aumenta. En efecto, analíticamente también se deduce esta misma conclusión observando que la derivada parcial de V_0 respecto de G_1 es:

$$\frac{\partial V_0}{\partial G_1} = \frac{G_2(E_1 - E_2) + I_0}{(G_1 + G_2)^2} > 0.$$

De igual forma, V_0 aumenta cuando G_1 se mantiene y G_2 disminuye. Consecuentemente, V_0 asumirá su menor valor posible (que se desea sea no inferior a $V_0^{\min}$) cuando G_1 asuma su menor valor G_1^-, G_2 asuma su mayor valor G_2^+, y la corriente que se extraiga I_0 sea $I_0^{\max}$. De este modo la restricción $V_0 \geq V_0^{\min}$ equivale a imponer

$$\frac{E_1^- G_1^- + E_2^- G_2^+ - I_0^{\max}}{G_1^- + G_2^+} \geq V_0^{\min},$$

o, alternativamente,

$$(1-\varepsilon_1)(E_1^- - V_0^{\min})\overline{G}_1 + (1+\varepsilon_2)(E_2^- - V_0^{\min})\overline{G}_2 \geq I_0^{\max}. \qquad (2)$$

Análogamente la restricción $V_0 \leq V_0^{\max}$ equivale a imponer

$$(1+\varepsilon_1)(E_1^+ - V_0^{\max})\overline{G}_1 + (1-\varepsilon_2)(E_2^+ - V_0^{\max})\overline{G}_2 \leq I_0^{\min}. \qquad (3)$$

La unión de las expresiones (1), (2) y (3), junto con la no negatividad de las variables $\overline{G}_1$ y $\overline{G}_2$, configuran un modelo matemático de Programación Lineal.

8. Problema de descubrir datos ocultos

Problema real

Supongamos que el Instituto Nacional de Estadística publica la tabla de datos recogida como tabla 2 con distintos gastos medios de diversos colectivos de una región. Nótese que se dan tanto datos concretos como sumas marginales y totales. Sin embargo, algunos datos se consideran confidenciales, ya que su publicación revelaría información privada. Por ejemplo, se considera que el dato referente al gasto medio en "vicios" de los "obispos" de una región es información confidencial, porque en dicha región sólo hay 1 obispo y, por ello, su publicación estaría revelando información de un individuo con nombre y apellidos conocidos. No ocurre igual con otros colectivos porque tienen más miembros. No obstante, algunos deben igualmente ser suprimidos para proteger el caso anterior. Los datos suprimidos son los que aparecen marcados con asterisco. Si sólo sabemos que ninguno de los datos ocultos puede ser negativo, ¿cuál es el rango más estrecho de valores que esta tabla revelará a un posible *curioso* sobre el gasto del obispo en vicios?

	policía	**profesor**	**maestro**	**vigilante**	**obispo**	**estudiante**	**TOTAL**
lectura	5	345	130	15	212	105	812
vicios	52	*	212	234	*	234	953
gimnasia	432	*	45	*	7	32	726
ropa	34	90	85	*	*	52	271
TOTAL	523	576	472	447	321	423	2762

Tabla 2. Gastos de diversos colectivos según distintos conceptos.

Modelo matemático

Consideremos una variable x_{ij} asociada a cada celda (es decir, a cada fila i y a cada columna j) representando el verdadero valor en la tabla. Es claro que las variables no son independientes entre sí, sino que están atadas por ecuaciones: una por cada fila y por cada columna. Así cualquier tabla de valores posibles para esta tabla debe cumplir:

$$\begin{aligned}
x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16} &= x_{17}\\
x_{21}+x_{22}+x_{23}+x_{24}+x_{25}+x_{26} &= x_{27}\\
x_{31}+x_{32}+x_{33}+x_{34}+x_{35}+x_{36} &= x_{37}\\
x_{41}+x_{42}+x_{43}+x_{44}+x_{45}+x_{46} &= x_{47}\\
x_{51}+x_{52}+x_{53}+x_{54}+x_{55}+x_{56} &= x_{57}\\
x_{11}+x_{21}+x_{31}+x_{41} &= x_{51}\\
x_{12}+x_{22}+x_{32}+x_{42} &= x_{52}\\
x_{13}+x_{23}+x_{33}+x_{43} &= x_{53}\\
x_{14}+x_{24}+x_{34}+x_{44} &= x_{54}\\
x_{15}+x_{25}+x_{35}+x_{45} &= x_{55}\\
x_{16}+x_{26}+x_{36}+x_{46} &= x_{56}\\
x_{17}+x_{27}+x_{37}+x_{47} &= x_{57}\\
x_{11},x_{12},\ldots,x_{56},x_{57} &\geq 0.
\end{aligned}$$

Además hay variables que en realidad son parámetros (aquellas asociadas a celdas cuyos verdaderos valores serán publicados) y hay otras de las que sólo sabemos que son no negativas. Por ello, el conjunto de todas las posibilidades que además son compatibles con la tabla publicada cumplirán:

$$\begin{aligned}
x_{22}+x_{25} &= 221\\
x_{32}+x_{34} &= 210\\
x_{44}+x_{45} &= 10\\
x_{22}+x_{32} &= 141\\
x_{34}+x_{44} &= 198\\
x_{25}+x_{45} &= 102\\
x_{22},x_{25},x_{32},x_{34},x_{44},x_{45} &\geq 0.
\end{aligned}$$

Ahora el problema planteado consiste en resolver dos problemas lineales, minimizando y maximizando la variable x_{25}, respectivamente. Ambos valores óptimos definen el rango dentro del cual debe estar el valor no publicado de la celda en fila 2 y columna 5.
Este tipo de problemas se llama *problema del curioso*, y es particularmente relevante en el control de la privacidad durante la publicación de tablas estadísticas. En este contexto aparecen numerosos problemas diferentes de optimización, como el que se plantea un instituto de estadística cuando desea encontrar las celdas que no debe publicar en una tabla estadística para asegurar ciertos niveles de protección en algunas celdas consideradas “sensibles”.

9. Problema de redondeos en una tabla

Problema real

Supongamos que el anterior Instituto Nacional de Estadística dispone de la tabla con totales marginales que se muestra como tabla 3, y desea publicarla tras redondear cada valor numérico fraccionario a su entero por exceso o por defecto. Ahora bien, no se admiten cualesquiera redondeos, sino que deben ser tales que en la tabla redondeada se mantengan las mismas relaciones de suma entre las celdas internas y marginales que en la tabla original. Las tablas 4 y 5 son dos posibles tablas redondeadas para la tabla 3 original. En caso de haber varias posibles tablas redondeadas correctas, el Instituto Nacional de Estadística desearía una que minimice la suma de las diferencias entre los valores redondeados y los valores originales. Plantear un modelo matemático para resolver el problema.

	hombre	**mujer**	**TOTAL**
infantil	1,666666	2,666666	4,333332
adulto	2,000000	4,750000	6,750000
anciano	1,250000	4,250000	5,500000
TOTAL	4,916666	11,666666	16,583332

Tabla 3. Tabla a redondear.

	hombre	**mujer**	**TOTAL**
infantil	2	3	5
adulto	2	5	7
anciano	1	4	5
TOTAL	5	12	17

	hombre	**mujer**	**TOTAL**
infantil	2	2	4
adulto	2	4	6
anciano	1	5	6
TOTAL	5	11	16

Tablas 4 y 5. Posibles tablas redondeadas.

Modelo matemático

Consideremos una variable decisional x_{ij} asociada con la celda (interna o marginal) de la tabla en la fila i y columna j, representando

$$x_{ij} = \begin{cases} 1 & \text{si el redondeo en la celda } (i,j) \text{ es por exceso} \\ 0 & \text{si el redondeo en la celda } (i,j) \text{ es por defecto.} \end{cases}$$

Entonces, toda solución al problema puede identificarse con valores para las variables cumpliendo:

$$
\begin{array}{rrcl}
\text{primera fila}: & (1+x_{11})+(2+x_{12}) & = & (4+x_{13}) \\
\text{segunda fila}: & (2)+(4+x_{22}) & = & (6+x_{23}) \\
\text{tercera fila}: & (1+x_{31})+(4+x_{32}) & = & (5+x_{33}) \\
\text{cuarta fila}: & (4+x_{41})+(11+x_{42}) & = & (16+x_{43}) \\
\text{primera columna}: & (1+x_{11})+(2)+(1+x_{31}) & = & (4+x_{41}) \\
\text{segunda columna}: & (2+x_{12})+(4+x_{22})+(4+x_{32}) & = & (11+x_{42}) \\
\text{tercera columna}: & (4+x_{13})+(6+x_{23})+(5+x_{33}) & = & (16+x_{43}) \\
\text{integrabilidad}: & x_{11},x_{12},x_{13},x_{22},x_{23},x_{31},x_{33} & \in & \{0,1\},
\end{array}
$$

y la función objetivo consiste en minimizar, para cada celda, la contribución del redondeo considerado, es decir:

$$
\begin{gathered}
0,666666(1-x_{11})+0,333334x_{11}+0,666666(1-x_{12})+0,333334x_{12}+ \\
0,333332(1-x_{13})+0,666668x_{13}+0,75(1-x_{22})+0,25x_{22}+0,75(1-x_{23})+ \\
0,25x_{23}+0,25(1-x_{31})+0,75x_{31}+0,25(1-x_{32})+0,75x_{32}+0,5(1-x_{33})+0,5x_{33}+ \\
0,916666(1-x_{41})+0,083334x_{41}+0,666666(1-x_{42})+ \\
0,333334x_{42}+0,583332(1-x_{43})+0,4166668x_{43}.
\end{gathered}
$$

Puede demostrarse que cuando la tabla original es de tipo bidimensional la condición de integrabilidad

$$x_{ij}\in\{0,1\} \text{ para toda celda } (i,j)$$

puede sustituirse simplemente por

$$0\le x_{ij}\le 1 \text{ para toda celda } (i,j),$$

con lo que el modelo propuesto pertenece a la Programación Lineal y además tiene siempre una solución óptima entera.

Bibliografía

A.V. Aho, J.E. Hopcroft, J.D. Ullman: *Data Structures and Algorithms*. Addison-Wesley, Reading (Massachusetts), 1983.

M.S. Bazaraa, J.J. Jarvis, H.D. Sherali: *Linear Programming and Network Flows*. John Wiley & Sons, New York, 1990.

V. Chvátal: *Linear Programming*. Freedman, New York, 1983.

W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A. Schrijver: *Combinatorial Optimization*. John Wiley & Sons, New York, 1998.

G.L. Nemhauser, L.A. Wolsey: *Integer and Combinatorial Optimization.* John Wiley & Sons, New York, 1988.

C.H. Papadimitriou, K. Steiglitz: *Combinatorial Optimization: Algorithms and Complexity.* Prentice-Hall, New Jersey, 1982.

J.J. Salazar: *Programación Matemática.* Díaz de Santos, Madrid, 2001.

H.A. Taha: *Investigación de Operaciones, una introducción.* Prentice Hall, México, 1998.

H.P. Williams: *Model Building in Mathematical Programming.* John Wiley & Sons, Chichester, 1999.

L.A. Wolsey: *Integer Programming.* John Wiley & Sons, New York, 1998.

En Internet

http://neos.mcs.anl.gov
NEOS Server for Optimization
Servidor web sobre Optimización.

http://www.caam.rice.edu/~mathprog
MPS
Mathematical Programming Society.

http://www.euro-online.org
EURO
Asociación de las Sociedades Europeas de Investigación Operativa.

http://www.informs.org
INFORMS Online
Institute for Operations Research and the Management Sciences.

http://mat.gsia.cmu.edu
Michael Trick's Operations Research Page

http://www.dash.co.uk
Dash Optimization - Home of Xpress-MP optimisation software
Software comercial para Optimización.

http://www.lindo.com
LINDO Systems
Software comercial para Optimización.

http://www.ilog.com/products/cplex
ILOG CPLEX
Software comercial para Programación Matemática.

Ciencia Computacional y Finanzas

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Resumen

Los modelos matemáticos pretenden captar la estructura de las relaciones causales entre las características fundamentales de cierta realidad concreta. Su utilidad es funcional. Un buen modelo es aquel que permite analizar y diagnosticar esa realidad, pronosticar su evolución (sea ésta determinista, probabilista o caótica) y tomar decisiones sobre la interacción más conveniente o el curso de acción a seguir; una suerte de test de Turing: el modelo es bueno si funciona (y si nos ayuda a comprender una realidad).

La necesidad de ser capaces de computar con el modelo escogido ha limitado tradicionalmente la ambición de su diseño. Los modelos han sido lineales, continuos, aplicables sólo a cortos espacios de tiempo, o en condiciones ideales, etc., no porque se creyera que así era la realidad, sino porque con esa primera aproximación se podía calcular, y porque, a no dudarlo, han funcionado extraordinaria, e, incluso, sorprendentemente bien. No hay mejor ejemplo de esto que las leyes de gravitación de Newton: elegantes, profundas, sencillas, y de precisión asombrosa.

Pero, hoy en día, animados por la prodigiosa potencia computacional de que ahora disponemos, nos atrevemos a modelizar realidades cada vez más complejas, donde muchas ecuaciones prescriben el comportamiento simultáneo de muchas variables, donde se incorpora la retroalimentación de causas, con efectos no-lineales que combinan ingredientes aleatorios y caóticos, y donde observamos y analizamos evoluciones temporales de largo alcance. Una complejidad que rehuye formulación cerrada, y que sólo se puede abordar mediante la simulación del modelo en el ordenador. Sistemas biológicos, económicos o financieros, el clima, o la turbulencia, enmarcan el ámbito de estas cuestiones.

El ordenador constituye un verdadero laboratorio de realidades complejas: un instrumento que permite trasladar el conocimiento organizado, a través de ese software mental que son las matemáticas, y de los modelos que concibe, en una realidad virtual sobre la que podemos actuar inocuamente. Nos permite experimentar recetas de política económica, para luego escoger la más conveniente, sin un (inadmisible) proceso de prueba y error sobre economías reales. Permite diseñar completamente un avión como el Boeing 777 pasando directamente de su concepción en el ordenador a la fase de producción, sin túneles de viento, ni prototipos. O simular explosiones termonucleares de distantes estrellas, y también de bombas atómicas, sin agredir desiertos ni atolones. O analizar los efectos de políticas alternativas de gestión ambiental sobre un ecosistema sin alterarlo irremediablemente.

Esta potencia requiere control. La simulación de un modelo no puede ser una suerte de caja negra, porque queremos entender. La complejidad de las realidades, la ambición de los modelos y la repercusión de las decisiones que emanan de su análisis generan inestabilidad. Son muchos los ejemplos de fiascos derivados de una excesiva fe en la modelización y su

simulación, al fin y a la postre (pero esto es casi una tautología) por no haber profundizado en la comprensión que la computación aporta en cuanto a diagnóstico de relaciones causales en la realidad a estudio.

En la gestión financiera, y para la toma de decisiones que conlleva, se comenzaron a desarrollar modelos científicos hace tan sólo unas decenas de años. Se trataba de modelos, ¡cómo no!, estilizados. Pero los sistemas financieros son sistemas complejos, y ya se ha generalizado el uso de ambiciosos modelos estocásticos complejos que permiten simular la evolución aleatoria integrada del negocio, de la estructura financiera, de las condiciones macroeconómicas y de los resultados de estrategias de gestión alternativas. Se trata de modelos que facilitan un proceso de decisión que tiene en cuenta no sólo un escenario medio de referencia, sino la incertidumbre inherente y la ulterior gestión activa.

Bibliografía

D.N. Arnold: *Mathematics in Industry and Government*. Presentación, NSF VIGRE meeting, Reston (Virginia), 4 de mayo de 2002.
[Disponible en http://www.ima.umn.edu/~arnold/talks/industry.pdf].

N. Barberis, R. Thaler: *A survey of behavioral finance*. National Bureau of Economic Research, Documento de trabajo 9222, septiembre de 2002.
[Disponible en http://papers.nber.org/papers/w9222].

A.J.G. Cairns: *An Introduction to Stochastic Pension Plan Modelling*. Nota técnica 94/11. Workshop on Interest Rate Risk, Vancouver, 19-20 de agosto de 1994.
[Disponible en http://www.ma.hw.ac.uk/~andrewc/papers/ajgc6.pdf].

S.P. D'Arcy: Enterprise Risk Management. *Journal of Risk Management of Korea* **12**, no. 1 (por aparecer).
[Disponible en http://www.cba.uiuc.edu/~s-darcy/papers/erm.pdf].

R. Kaufmann: *Dynamic Financial Analysis*. Presentación, 4 de julio de 2000.
[Disponible en http://www.math.ethz.ch/~kaufmann/DFA/DFA.pdf].

A.J. McNeil: *Risk Management, An Overview*. Presentación, Swiss Banking School, 17-18 de septiembre de 2001.
[Disponible en http://www.math.ethz.ch/~mcneil/ftp/bankschool.pdf].

R.J. Shiller: Human behavior and the efficiency of the financial system. En J.B. Taylor and M. Woodford (editores): *Handbook of Macroeconomics, Vol. 1* (1999), pp. 1305-1340.
[Disponible en http://cowles.econ.yale.edu/P/cd/d11b/d1172.pdf].

Enterprise Risk Management: An Analytic Approach. Monografía de la Consultora Tillinghast-Towers Perrin, 2000.
[Disponible en http://www.tillinghast.com/tillinghast/publications/reports/Enterprise_Risk_Management_An_Analytic_Approach/erm2000.pdf].

En Internet

http://www.afi.es
InfoAnalistas
Portal del Grupo Analistas Financieros Internacionales.

http://www.grupoanalistas.com
Grupo Analistas
Información corporativa del Grupo Analistas Financieros Internacionales.

http://15aniversario.afi.es
XV aniversario Grupo Analistas
http://archives.math.utk.edu/topics/computationalScience.html
Computational Science
Archivo de artículos y enlaces en Ciencia Computacional.

Ciencia Computacional, Simulación, ... y Finanzas

José Luis Fernández Pérez

***La Laguna*, 20 de marzo de 2003**

Matemáticas

Las matemáticas $nacen$ buscando abstraer

simetrías, formas, estructuras

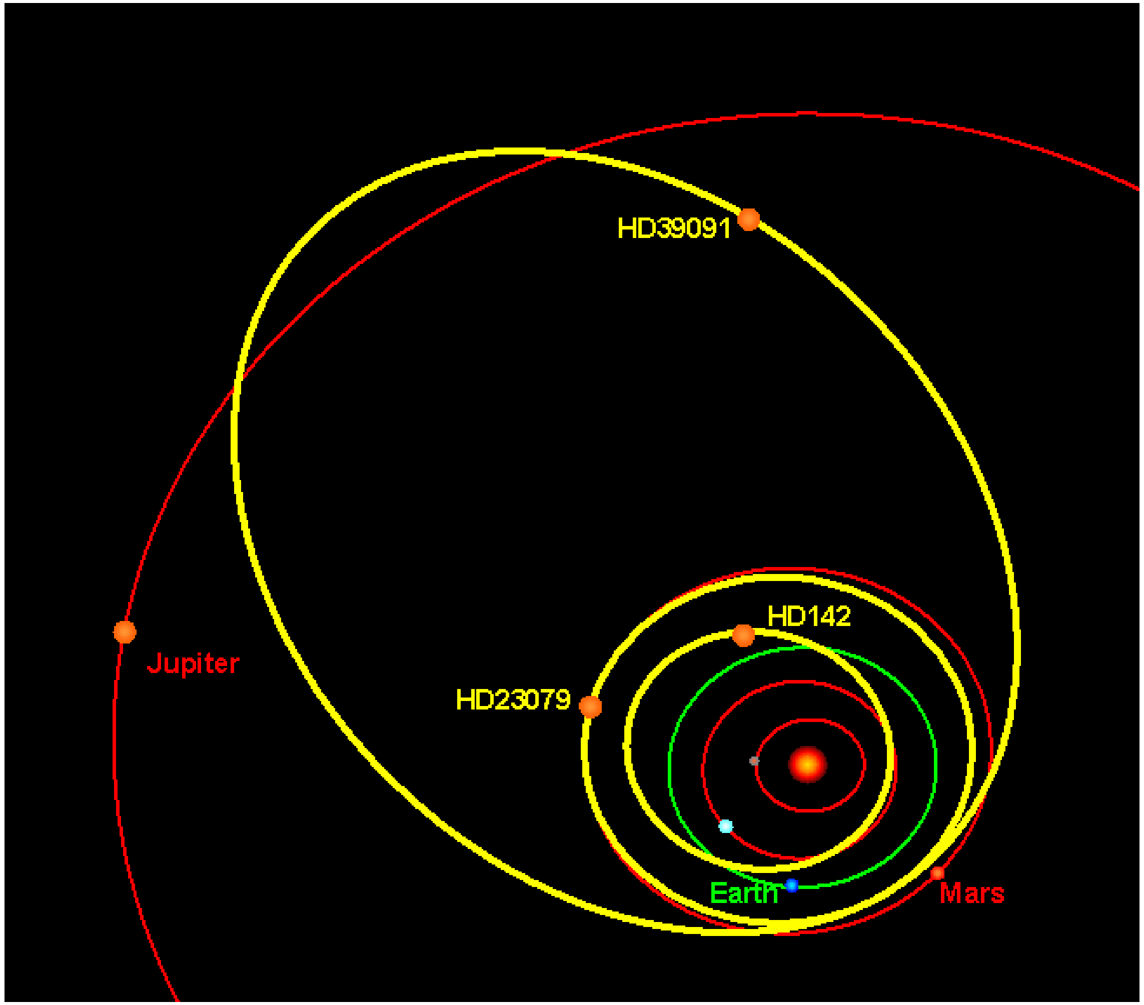
para entender el mundo.

Matemáticas: lenguaje. Matemáticas, software mental.

Simplificando . . . para entender

- Corto tiempo
- Lineal
- Pocos ingredientes
- Muy estilizados[1]
- Increíble que funcione. Un asombroso misterio.

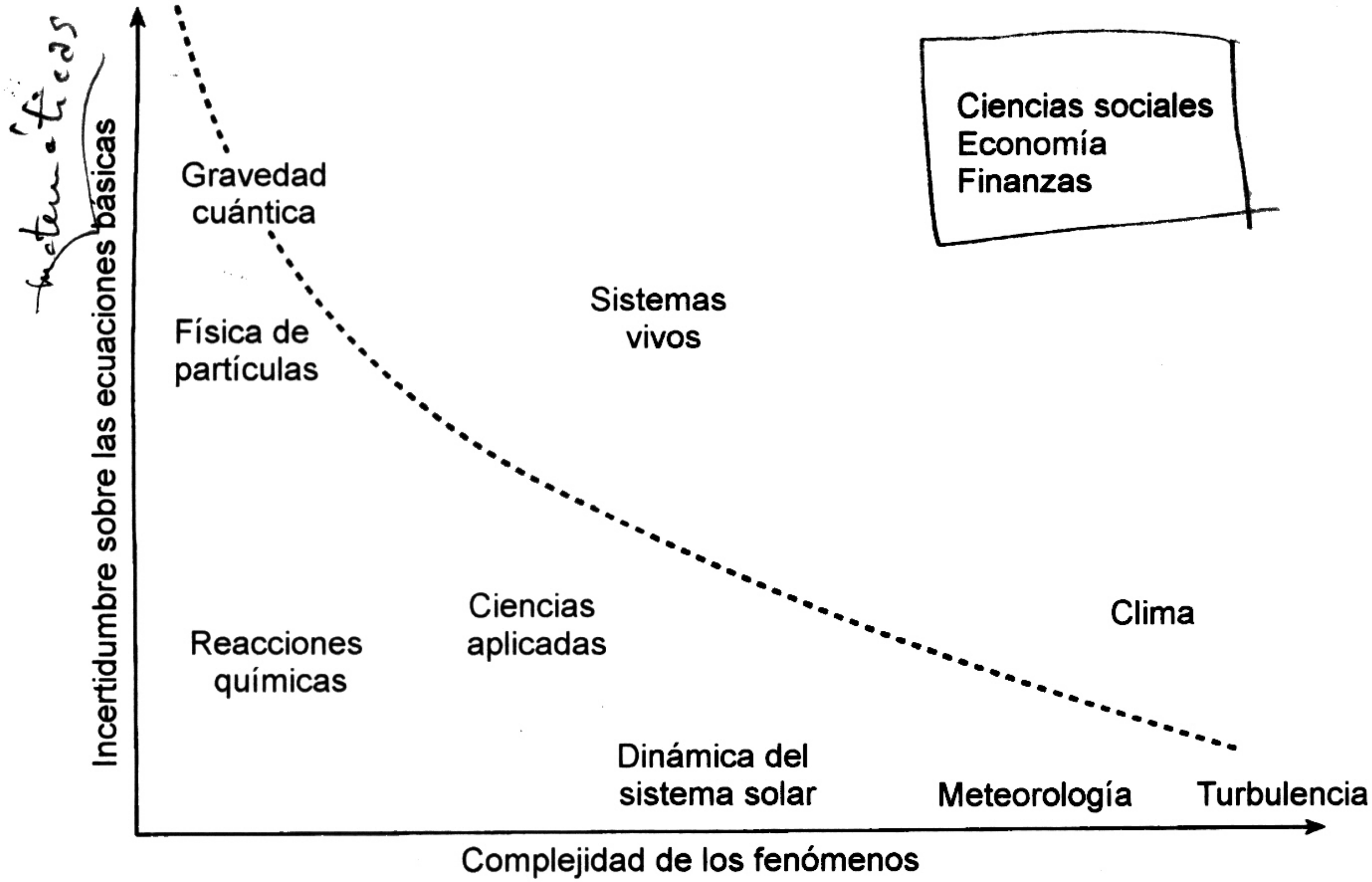
[1]Estilizar: [DRAE] Interpretar convencionalmente la forma de un objeto haciendo resaltar tan solo sus rasgos más característicos


HD39091
HD142
Jupiter
HD23079
Earth
Mars

La campiña inglesa

Sistemas Complejos

- Economía
- Biología
- Turbulencia
- Clima
- ...



Incertidumbre sobre las ecuaciones básicas
matemáticas
Ciencias sociales
Economía
Finanzas
Gravedad cuántica
Física de partículas
Sistemas vivos
Reacciones químicas
Ciencias aplicadas
Clima
Dinámica del sistema solar
Meteorología
Turbulencia
Complejidad de los fenómenos

D. Ruelle

- Retroalimentación.
- No-linealidad y Caos.
- Dimensión, capas y cascadas.
- Aleatoriedad.
- ...

- Regla de tres.
- Grandes Números.
- *Ceteris paribus*

Retroalimentación

La mayor deficiencia de la raza humana es su incapacidad para comprender la función exponencial.

A. A. Bartlett, *físico*.

- Botella de Coca-Cola y Ecología
- Interés continuo

Un niño que medía dieciocho meses en la escala de Richter.

. . . mínimos, gigantescos, qué más da:
después de todo, nadie sabe qué es lo pequeño y qué lo enorme . . .

José Hierro, *Libro de las alucinaciones*.

No-linealidad

Mecanismos perfectamente deterministas no lineales que actúan durante largo tiempo, suponen relación caótica (impredecible) ente el estado inicial del sistema y su estado futuro.

En lo caótico, lo determinista deviene en aleatorio.

Laplace. La mesa de Billar. Moneda al aire. Pascal, Cleopatra y las mariposas.

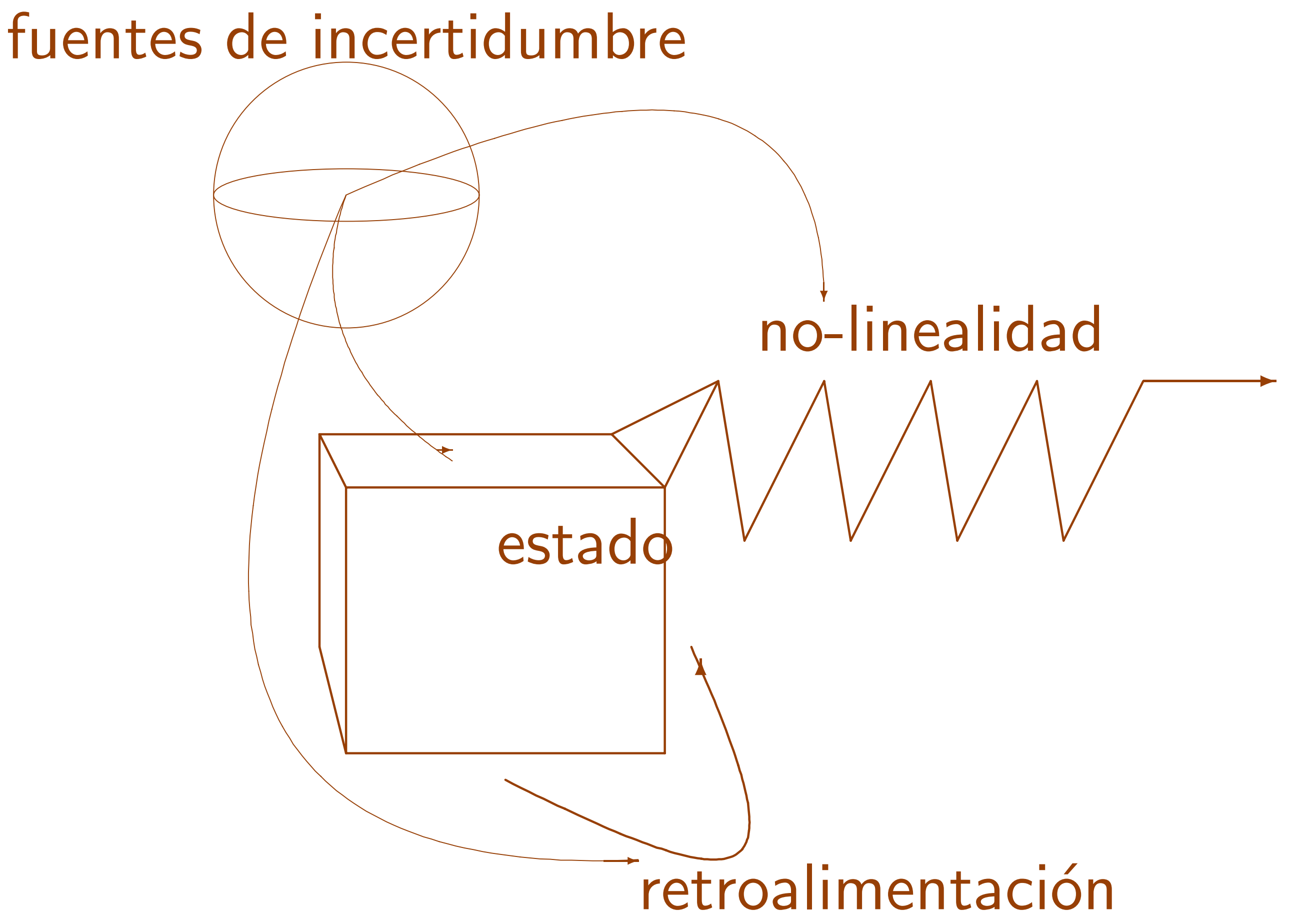
La dimensión

Sistemas grandes con un número enorme

- de variables,
- de sub-modelos,
- de capas intermedias de acción

que operan en sucesivas cascadas acumulando efectos.

Y, por supuesto: La aleatoriedad. La relación entre acción y efecto es aleatoria, intrínsecamente impredecible, a corto plazo y en acción directa.


fuentes de incertidumbre
no-linealidad
estado
retroalimentación

- En prosa: Una economía, un sistema financiero.

- En poesía: . . .

Dios, que es digital, creó la realidad analógica por la misma razón por la que nosotros hemos creado Internet: por enredar. De vez en cuando entraba en nuestro mundo como nosotros entramos ahora en la Red y disfrutaba viendo los días y las noches y el Sol y las tormentas. Y en cada una de esas incursiones, a la realidad atómica añadía alguna cosa nueva: los peces, las ranas, las serpientes, la polio, los instintos, la gripe . . . Todo ello sin calcular que la lógica de los átomos conduciría a la bomba atómica del mismo modo que la lógica digital conduce a la digitalina. Dios sólo es responsable de la puesta en marcha. Lo demás se dio por añadidura y ÉI fue el primero en extrañarse del modo singular que eligieron los mamíferos para reproducirse o las jirafas para llegar a la copa de los árboles. Cuando fabricas un calidoscopio, tampoco hay forma de predecir todas sus combinaciones posibles.

Con la misma extrañeza con que observaba Dios la realidad analógica, construida por Él mismo, nos asomamos ahora a la realidad virtual, hecha a nuestra imagen y semejanza. La hemos diseñado nosotros, sí, pero quién iba a imaginar que engendraría cosas tan curiosas por su cuenta. Y eso que aún estamos en el primer día de la creación como el que dice. Faltan los wap y los umts y la pantalla tridimensional, y los reptiles y las aves, y los Adanes y las Evas de ese mundo incipiente. Más que una realidad, hemos creado una lógica con capacidad para desarrollarse por sí misma, aunque la abandonáramos ahora mismo a su suerte. Dios tampoco necesitó crear los lunes ni los martes ni los miércoles... Desde el momento en que te inventas el domingo, el resto de la semana sale del huevo fecundado con cara de haberse confundido de estación.

Ahora bien, lo interesante de todo esto es el hecho de haber abierto en nuestra dimensión un agujero por el que podríamos ver el rostro de Dios, que quizá nos observa espantado por la misma abertura. No pierdan el tiempo buscándolo en dios punto com ni en satán punto es. Se trata de un hacker más experimentado que todo eso. Sepan en todo caso que, mientras navegamos, nos observa.

Juan José Millás, *Génesis*. EL PAIS

Matemáticas, *software mental*

- Simulación DRAE *Simular*. Representación de una cosa, fingiendo o imitando lo que no es. *Simulación*. Alteración aparente de la causa, la índole o el objeto verdadero de un acto o contrato.

- Modelización DRAE *Modelizar*. No'tá. *Modelo* . . . Esquema teórico, generalmente en forma matemática, de un sistema o de una realidad compleja, (por ejemplo, la evolución económica de un país), que se elabora para facilitar su comprensión y el estudio de su comportamiento.

En el *Seco*: *Simulador.* Aparato que permite reproducir artificialmente un fenómeno o un funcionamiento real. *Modelizar.* Establecer el modelo.

Los matemáticos no comprenden la realidad hasta que la encierran en una ecuación, pero los burócratas son incapaces de medir el tamaño de una catástrofe hasta que la transforman en un expediente.

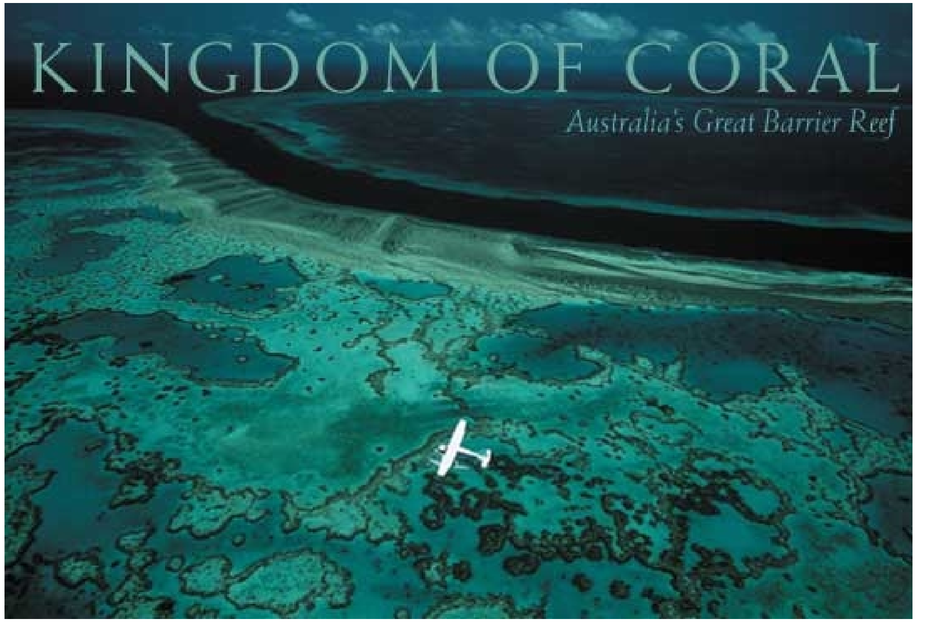
Juan José Millás, *La oficina*. EL PAIS

Simulación

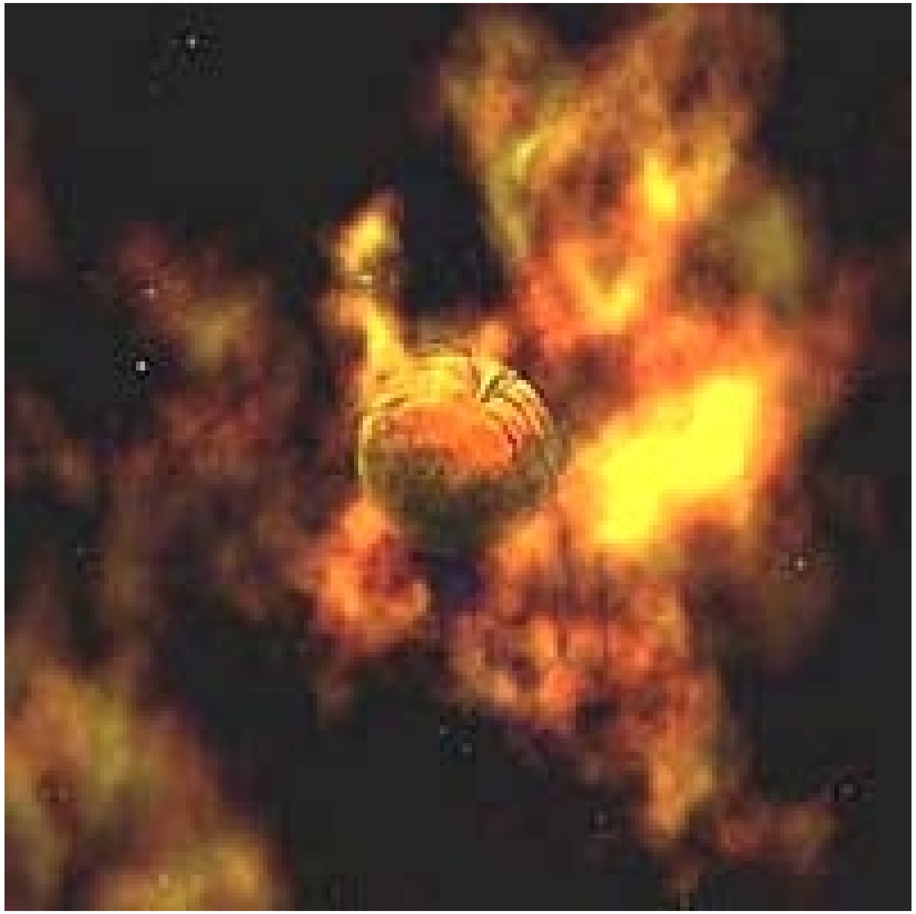
- Laboratorio de lo complejo
  - Diagnóstico y análisis
  - Control
  - Comparación de acciones y respuestas
- Potencia computacional actual (futura!) que permite
  - modelizar ambiciosamente, sistemas complejos
  - *whatif's*, escenarios, optimización, análisis, decisión

¡Un cambio de paradigma! (?)

- Arrecifes australianos

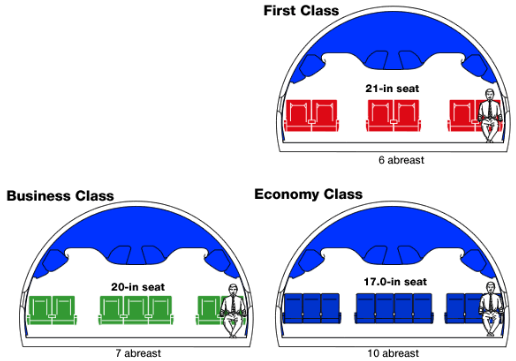


- Explosiones nucleares en estrellas



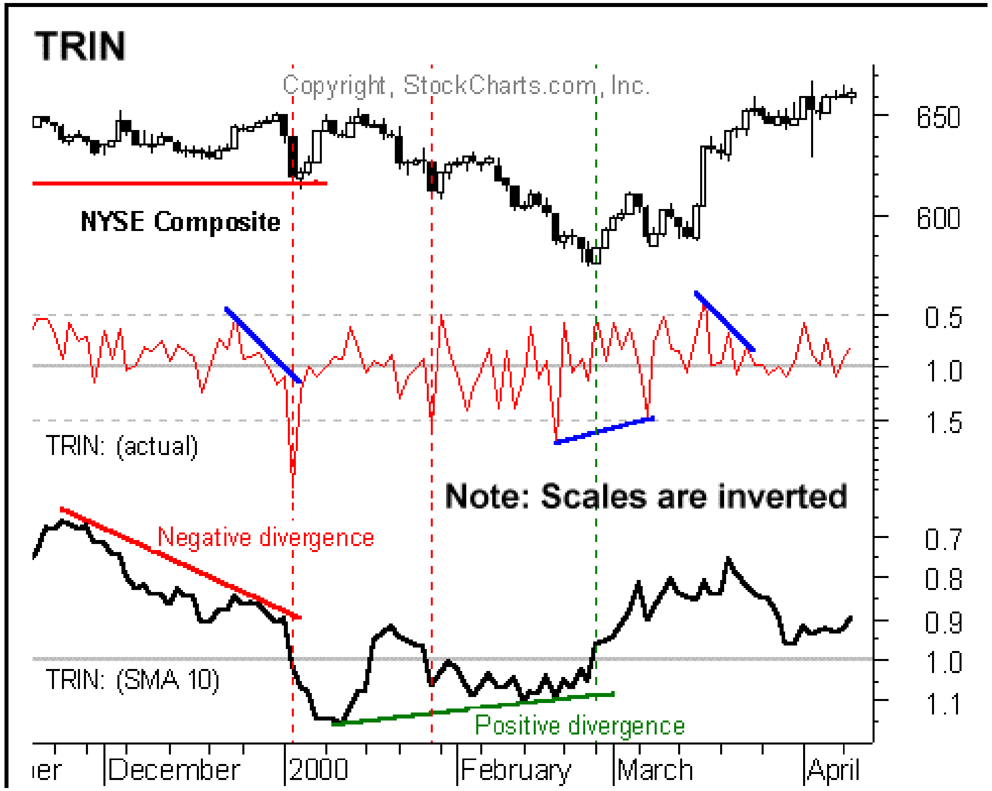
- Boeing777


BOEING 777-300
777

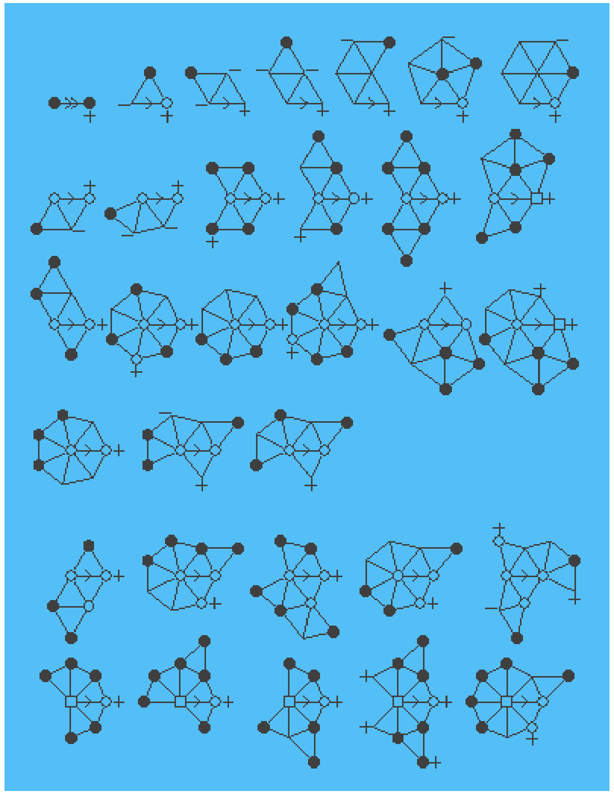

First Class
21-in seat
6 abreast
Business Class
20-in seat
7 abreast
Economy Class
17.0-in seat
10 abreast

- Plataformas petrolíferas en el Mar del Norte

- *Program trading*



- Cuatro colores



- La paradoja de Braess

- Regulación y supervisión de mercados

Acotando con declaraciones

"Although it is true that, in the 300 years since Newton, most of theoretical science has been done using the rigorous, analytical approach, the reason for that is simply that that is the only kind of science *could* be done ... The lack of computational power meant that researchers could only answer questions that had clean, elegant solutions ... It is only now that we have the ability to do complex calculations and simulations that we are discovering that a great many systems seem to have an inherent complexity that cannot be simplified ... After another 300 years, we will no doubt feel as comfortable using computer simulations to analyze nature as scientists today feel using Newton's laws of motion to describe the trajectory of a falling stone."

G. W. Rowe, *Theoretical models in Biology*

Most interesting problems presented by nature are likely to be formally undecidable or computationally irreducible, rendering proofs and predictions impossible. . . .

Mathematicians and scientists have managed to keep busy only by carefully choosing to work on the relatively small set of problems that have simple solutions.

S. Wolfram, *A new kind of science*.
Recensión de J. Gray en el *Notices* de la AMS

La urna de Polya

Una de las grandes de las ventajas de la Teoría de la Probabilidad es que nos enseña a desconfiar de nuestras primeras impresiones. *Laplace*

- VHS y Betamax; Neanderthal y Cro-Magnon
- La urna de Polya
- Democracia (?) animal y Mercados financieros

Peros . . .

Entender, entender, . . . [ciencia]

¿y, el rigor, y la certeza, . . . ?[matemáticas]

Exceso de confianza.[sociedad]

El modelo y las agentes

Efectos cuánticos cuando se modelizan comportamientos en teoría descriptiva (que no normativa).

- Encuestas / votaciones
- Evaluación científica
- *Stock options*
- Supervisión / Control de riesgos financieros: Los mercados son sistemas complejos en los que la observación altera los fundamentos. Rentabilidad, volatilidad, correlación, *todos a una*.
- Lo público y lo privado. Default cuando probabilidad de default es $\geq 10\%$.

Finanzas computacionales

- Escenario medio
- *ceteris paribus*
- Incertidumbre
- Dependencia
- ...

Los marcos metodológicos usados en finanzas: Markowitz, Black-Scholes, CAPM, suponen modelizaciones estilizadas

- Cálculo estocástico
  - Cálculo de Ito
  - Cálculo de Malliavin
- Hipótesis que permiten tratamiento analítico
- pero que no captan la inestabilidad e incertidumbre reales

- Premio Nobel
- El fiasco de LTCM. Finanzas forenses.

Gestión de la incertidumbre económica

Modelizar la actividad económica: del negocio y de la financiación

Modelización de los procesos generadoras de incertidumbre

Parametrización de los procesos alternativos de gestión

$\Longrightarrow$

Resultados de la gestión

Énfasis en

- la modelización de los procesos exógenos,
- parametrización de las gestiones,
- determinación del criterio de optimalidad.

Optimización de la gestión.

EJEMPLO: Compañía de seguros

Fuentes de incertidumbre	$\Rightarrow$	Tipos de interés Inflación Bolsa Siniestralidad, mortalidad
Parametrización de gestión	$\Rightarrow$	Selección de inversiones Tarificación de pólizas Liquidez Mezcla de sectores
Resultados	$\Rightarrow$	Rentabilidad/Riesgo Niveles de riesgo

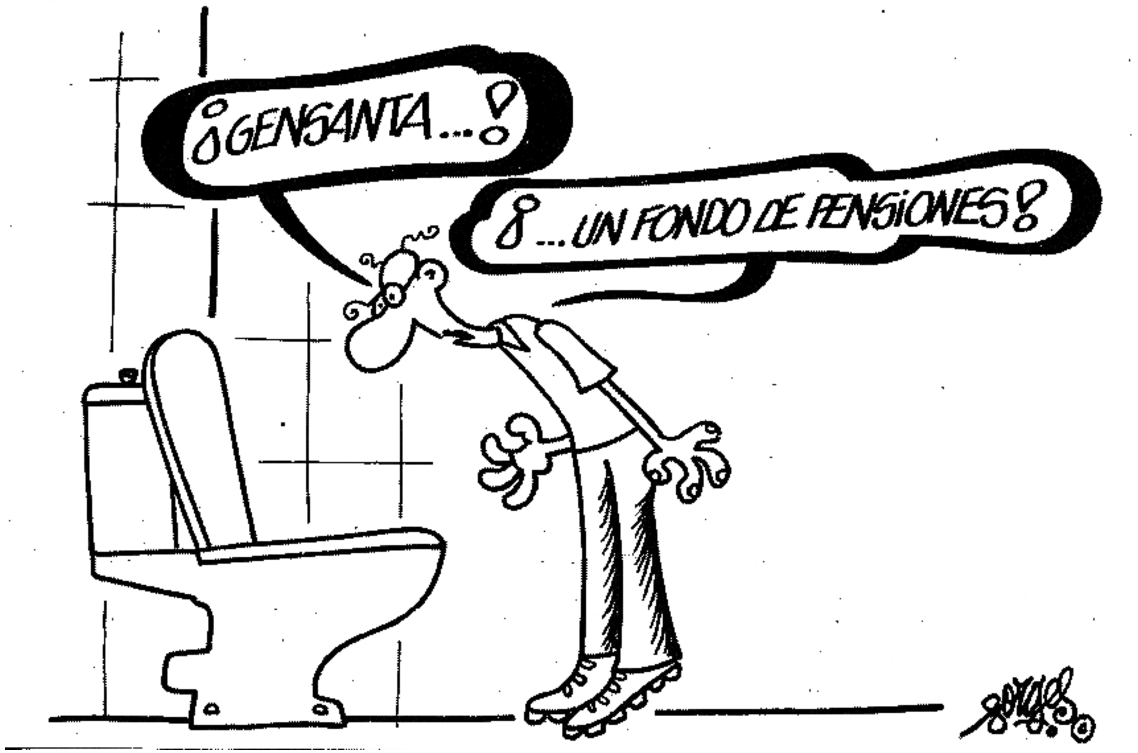
- Precios de seguros.
- La situación general de las aseguradoras.

El sistema de pensiones

- *New deal*. 65 años
- Deuda nacional

- Pasivo
  - Más pensionistas
  - Más longevos
  - Mayor tasa de reposición
- Activo
  - Menor población
  - Menor carrera laboral
- El pacto de Toledo
- 3 generaciones

Fondos de pensiones



An Introduction to Stochastic Pension Plan Modelling [1] [2]

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Abstract

In this paper we consider models for pension plans which contain a stochastic element. The emphasis will be on the use of stochastic interest models, although we will also consider stochastic salary growth and price inflation. The paper will concentrate primarily on defined benefit pension plans. In doing so we will look at how the size of the fund and the contribution rate vary through time and examine how these are influenced by factors which are within the control of a plan's managers and advisers. These factors include the term over which surplus is amortized; the period between valuations; the delay between the valuation date and the implementation of the new contribution rate; and the asset allocation strategy.

The paper will stress the importance of having a well defined objective for a pension plan: optimal decisions and strategies can only be made when a well defined objective is in place.

The paper will also consider, briefly, defined contribution pension plans. The primary decision here relates to the construction of suitable investment strategies for individual members. Again, a well defined objective must be formulated before a sensible strategy can be designed.

Finally, computer simulation methods will be discussed.

[1] Technical Note 94/11
[2] Presented to the workshop on Interest Rate Risk, Vancouver, 19-20 August, 1994

Contents

1 Introduction

In this paper we will consider stochastic pension plans. Pension plans generally fall into one of two categories: defined benefit plans; and defined contribution plans. Both of these are common in countries such as Canada, the USA, the UK and Australia. In all of these countries defined contribution plans are growing significantly in number at the cost of defined benefit plans as employers shift the burden of investment risk over to employees.

In this work we consider how the effects of investment risk can be reduced by making effective use of factors which are within the control of the scheme. These are

- Defined benefit: the method and period of amortization; the intervaluation period; the delay in implementing a recommended contribution rate; the funding method; the valuation basis; the asset allocation strategy.
- Defined contribution: the asset allocation strategy (age dependent); the contribution rate.

In the following sections we will look at each of these factors and consider the effects which each has on levels of uncertainty. In attempting to analyse such problems, a stochastic framework is the only sensible one to use. Within a deterministic framework there is no concept of uncertainty: the very thing we are attempting to quantify and control. For some factors the effect is the intuitive one, while in others the effect may not be known until some sort of exact or numerical analysis can be carried out.

2 Defined Benefit Pension Plans

Defined benefit pension plans provide benefits to members which are defined in terms of a member's final salary (according to some definition), and the length of membership in the plan. For example,

$$
\begin{aligned}
\text{Annual pension} &= \frac{N}{60} \times FPS \\
\text{where } N &= \text{number of years of plan membership} \\
FPS &= \text{final pensionable salary}
\end{aligned}
$$

In defined benefit pension plans pension and other benefits do not depend on past investment performance. Instead the risk associated with future returns on the funds assets is borne by the employer. This manifests itself through the contribution rate which must vary through time as the level of the fund fluctuates above and below its target level. If these fluctuations are not dealt with (that is, if the contribution rate remains fixed) then the fund will ultimately either run out of assets from which to pay the benefits or grow exponentially out of control.

2.1 A simple model

A number of the factors which we will look at can be first investigated by looking at a very simple stochastic model. By doing so we are able to focus quite quickly on the problem and to give ourselves a good feel for what might happen when we look at more realistic and complex models. This approach follows that of Dufresne (1988, 1989 a,b, 1990), Haberman (1992, 1993 a,b, 1994), Zimbidis and Haberman (1993), Cairns (1995) and Cairns and Parker (1995).

Suppose, then, that we have a fund which has a stable membership and a stable level of benefit outgo. Assuming that all benefits and contributions are paid at the start of each year we have the following relationship:

$$AL(t+1) = (1+i'_v)\,(AL(t)+NC(t)-B(t))$$

where

$$\begin{aligned} AL(t) &= \text{actuarial liability at time } t \\ B(t) &= \text{benefit outgo at time } t \\ NC(t) &= \text{normal contribution rate at time } t \\ \text{and } \; i'_v &= \text{valuation rate of interest} \end{aligned}$$

Suppose that salary inflation is at the rate s per annum and that benefit outgo increases in line with salaries each year. Then

$$\begin{aligned} B(t) &= B.(1+s)^t \\ AL(t) &= AL.(1+s)^t \\ NC(t) &= NC.(1+s)^t \end{aligned}$$

giving

$$\begin{aligned} AL(1+s) &= (1+i'_v)(AL+NC-B) \\ \text{or } \; AL &= (1+i_v)(AL+NC-B) \end{aligned}$$

where $i_v = (1+i'_v)/(1+s)-1 = (i'_v-s)/(1+s)$ is the real valuation rate of interest. Hence

$$NC = B-(1-v_v)L$$

where $v_v = 1/(1+i_v)$.

For convenience we will work in real terms relative to salary growth. In effect this means that we may assume that $s=0$, without losing any level of generality.

Now let $F(t)$ be the actual size of the fund at time t. Then

$$F(t+1) = (1+i(t+1))\,(F(t)+C(t)-B)$$

where $i(t+1)$ is the effective rate of interest earned on the fund during the period t up to $t+1$, and $C(t)$ is the contribution rate at time t.

$C(t)$ can be split into two parts: the normal contribution rate, NC; and an adjustment $ADJ(t)$ to allow for surplus or deficit in the fund relative to the actuarial liability. Thus

$$C(t) = NC + ADJ(t)$$

We will deal with the calculation of this adjustment in the next two sections.

The deficit or unfunded liability at time t is defined as the excess of the actuarial libility over the fund size at time t. Hence we define

$$\begin{aligned} UL(t) &= \text{unfunded liability at time } t \\ &= AL - F(t) \end{aligned}$$

In North America it is common also to look at the loss which arises over each individual year. This is defined as the difference between the expected fund size (based on the valuation assumptions) and the actual fund size at the end of the year given the history of the fund up to the start of the year. This gives us

$$\begin{aligned} L(t) &= \text{loss in year } t \\ &= E[F(t)] - F(t) \ \text{ given the fund history up to time } t-1 \\ &= UL(t) - E[UL(t)] \ \text{ given the fund history up to time } t-1 \end{aligned}$$

We will make use of $UL(t)$ and $L(t)$ in the next section.

No mention has been made so far of the interest rate process $i(t)$. Initially we will assume that $i(1), i(2), \ldots$ form an independent and identically distributed sequence of random variables with

$$\begin{aligned} i(t) &> -1 \ \text{ with probability 1} \\ E[i(t)] &= i \\ Var[i(t)] &= Var[1+i(t)] = \sigma^2 \\ \Rightarrow E[(1+i(t))^2] &= (1+i)^2 + \sigma^2 \end{aligned}$$

For notational convenience we will define

$$\begin{aligned} v_1 &= \frac{1}{E[1+i(t)]} = \frac{1}{1+i} \\ v_2 &= \frac{1}{E[(1+i(t))^2]} = \frac{1}{(1+i)^2+\sigma^2} \end{aligned}$$

These will be made use of in later sections.

2.2 Two methods of amortization

The Spread Method: This is in common use in the UK. The adjustment to the contribution rate is just a fixed proportion of the unfunded liability: that is,

$$
\begin{aligned}
ADJ(t) &= k.UL(t) \\
\text{where } k &= \frac{1}{\ddot{a}_{\overline{m}|}} \text{ at rate } i_v \\
\text{and } m &= \text{the period of amortization.}
\end{aligned}
$$

The period of amortization is chosen by the actuary, and commonly ranges from 5 years to over 20 years. For accounting purposes in the UK m must be set equal to the average future working lifetime of the membership.

The Amortization of Losses Method: This is in common use in the USA and Canada. The adjustment is calculated as the sum of the losses in the last m years divided by the present value of an annuity due with a term of m years calculated at the valuation rate of interest: that is,

$$
ADJ(t) = \frac{1}{\ddot{a}_{\overline{m}|}} \sum_{j=0}^{m-1} L(t-j)
$$

The interpretation of this is that the loss made in year s is recovered by paying m equal instalments of $L(s)/\ddot{a}_{\overline{m}|}$ over the next m years. These m instalments have the same present value as the loss made in year s.

Dufresne (1989b) showed that the unfunded liabilities and the losses are linked in the following way:

$$
\begin{aligned}
UL(t) &= \sum_{j=0}^{m-1} \lambda_j L(t-j) \\
\text{where } \lambda_j &= \frac{\ddot{a}_{\overline{m-j}|}}{\ddot{a}_{\overline{m}|}}
\end{aligned}
$$

Intuitively this makes sense, since $\lambda_j L(t-j)$ is just the present value of the future amortization instalments in respect of the loss made at time $t-j$. Hence $UL(t)$ is equal to the present value of the outstanding instalments in respect of all losses made up until time t.

The Spread Method can also be defined in terms of the loss function. Whereas the Amortization of Losses Method recovers the loss at time t by taking in m *equal* instalments of $L/\ddot{a}_{\overline{m}|}$, the Spread Method recovers this by making a geometrically decreasing, infinite sequence of instalments which starts at the same level.

We are now in a position to calculate the long term mean and variance of the fund size and of the contribution rate. Details of these are provided in Dufresne (1989) (in the case when the valuation and the true mean rate of interest are equal) and Cairns (1995) (covering the case when $i \neq i_v$). For the Spread method we find that

$$
\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2
\end{aligned}
$$

When $i = i_v$ these simplify to

$$
\begin{aligned}
E[F(t)] &= AL \\
E[C(t)] &= B-(1-v_1)AL \\
Var[F(t)] &= \frac{(v_1^2-v_2)}{(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(v_1^2-v_2)}{(v_2-(1-k)^2)}AL^2
\end{aligned}
$$

Now $v_1 > v_2$ and we must have $Var[F(t)]$ and $Var[C(t)]$ greater than 0. Hence we must have $(1-k)^2 < v_2 \;\Rightarrow\; k > 1-\sqrt{v_2}$. This then automatically implies that $k > 1-v_1$ and if this is combined with $k > 1-v_v$ it ensures that the mean fund size is also positive.

Looking at the Amortization of Losses Method we have, when $i = i_v$,

$$
\begin{aligned}
Var[L(t)] &= \frac{\sigma^2(1+i)^{-2}AL^2}{1-\sigma^2(1+i)^{-2}\sum_{j=1}^{m-1}\lambda_j^2} = V_\infty \text{ say} \\
Var[F(t)] &= V_\infty \sum_{j=0}^{m-1}\lambda_j^2 \\
Var[C(t)] &= \frac{m.V_\infty}{(\ddot{a}_{\overline{m|}})^2}
\end{aligned}
$$

2.3 The period of amortization

We now consider the first factor which we have within our control: the period of amortization, m.

For the time being, assume that $i = i_v$: we will look at the more general case in a later section. The following results can be shown to hold for the Spread Method (for example, see Dufresne, 1989b)

- $Var[F(t)]$ increases as m increases.

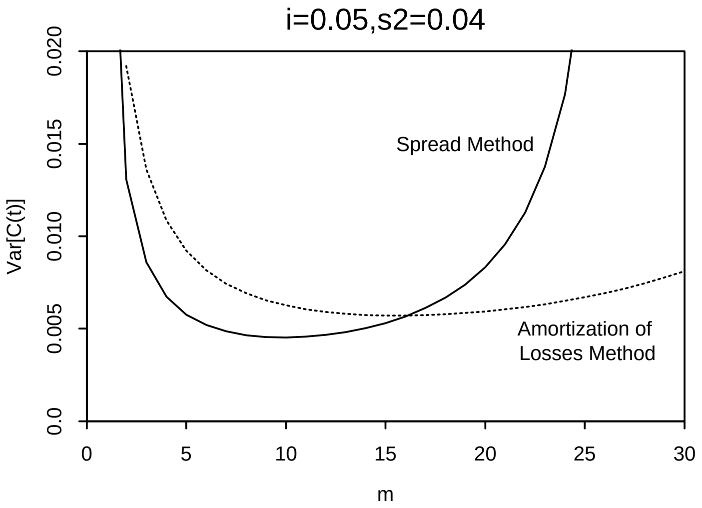

Figure 1: The effect of the period of amortization on the variance of the contribution rate with $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$.

- $Var[C(t)]$ decreases initially as m increases from 1 up to some value m^* and then increases as m increases beyond m^*. The optimal value, m^*, is such that
$k^* = 1/\ddot{a}_{\overline{m^*|}} = 1 - v_2$.

Looking at the Amortization of Losses Method no such analytical results have been proved but numerical examples show that the same qualitive behaviour holds, as illustrated in the following example.

Suppose $E[i(t)] = i = 0.05$ and $Var[i(t)] = \sigma^2 = 0.2^2$. Figure 1 illustrates how the variance of the contribution rate (with $AL = 1$) depends on m. The Spread Method has its minimum at about 10 while the Amortization of Losses Method has its minimum at about 16, and this minimum is higher.

In Figure 2 we compare the variance of the fund size against the variance of the contribution rate. We do this because we may be interested in controlling the variance of both the contribution rate *and* the fund size. As m increases each curve moves to the right, first decreasing and then increasing as m passes through m^*. Above m^* both the variance of the fund and the variance of the contribution rate are increasing. It is clear then that no value of m above m^* can be 'optimal' because the use of some lower value of m (say, m^*) can lower the variance of both the fund size and the contribution rate. The range $1 \leq m \leq m^*$ is the so-called *efficient* region: that is, given a value of m in this range there is no other value of m which can lower the variance of both the fund size and the contribution rate. There is therefore a trade-off between variability in the fund size and the contribution rate and settling on what we regard as an optimal spread period can only be done with reference to a more specific objective than 'minimize variance'.

It is significant that the Amortization of Losses Method curve always lies above the Spread Method curve. This means that the Spread Method is certainly more efficient than the Amortization of Losses Method: that is, for any value of m in combination with the Amortization of Losses Method there is a (different) value m' for which the variance of both the fund size and the contribution rate can be reduced by switching to the Spread Method.

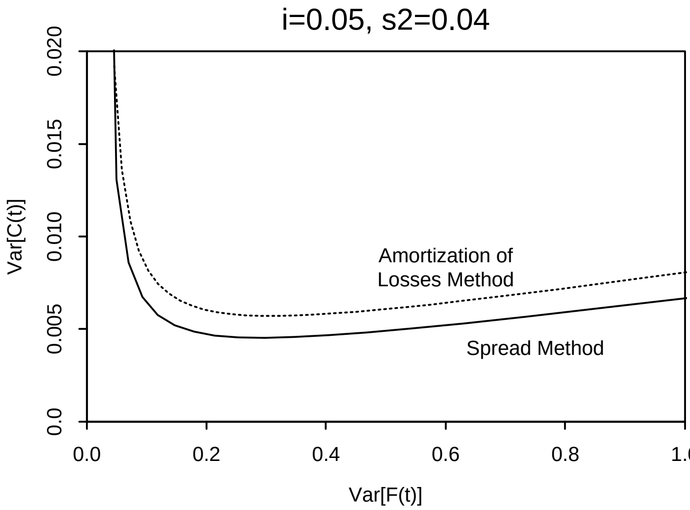

Figure 2: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Notes: $Var[F(t)]$ increases as m increases; the efficient frontier for the Spread Method is always more efficient than that for the Amortization of Losses Method.

2.4 The intervaluation period

The time between valuations is nominally a factor which is within the control of the scheme. We have so far considered the case where valuations are carried out on an annual basis. Such an approach is common amongst larger funds but this is often felt to be uneconomic for smaller funds to carry out such frequent valuations. Instead smaller funds often opt for a three year period between valuations ($3\frac{1}{2}$ years being the statutory maximum in the UK).

The effects of changing from annual to triennial valuations have been considered by Haberman (1993b). He finds that under the Spread Method of amortization

- the optimal spread period for $Var[C(t)]$, m^*, increases by about 1 year;
- the variances of both $F(t)$ and $C(t)$ are increased for most values of m below about m^*.

Continuing the example of the previous section we looked at 1 and 3 year intervaluation periods. Figure 3 plots $Var[C(t)]$ against m. For low values of m lengthening the intervaluation period has the effecct of increasing the variance of $C(t)$: the intuitive effect. For higher values of m, however, the reverse is true. This perhaps reflects the fact that over each three year period $C(t)$ is being held fixed thereby reducing the overall variance.

Comparing the variances of $F(t)$ and $C(t)$ (Figure 4) we see that, in this example at least, the efficient range for annual valuations lies below that for triennial valuations. We conclude that annual valuations are preferrable, although for values of m close to m^* there is little difference in the variances, so the benefit of annual valuations is marginal.

2.5 The delay period

The original analysis asumes that the new contribution rate can be implemented at the valuation date. In reality the results of a valuation are often not known until 6 or even 12 months after the

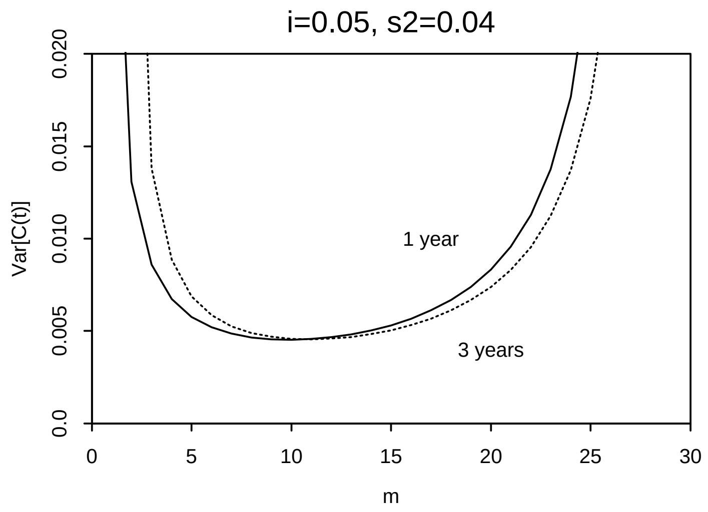

Figure 3: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for annual and triennial valuations.

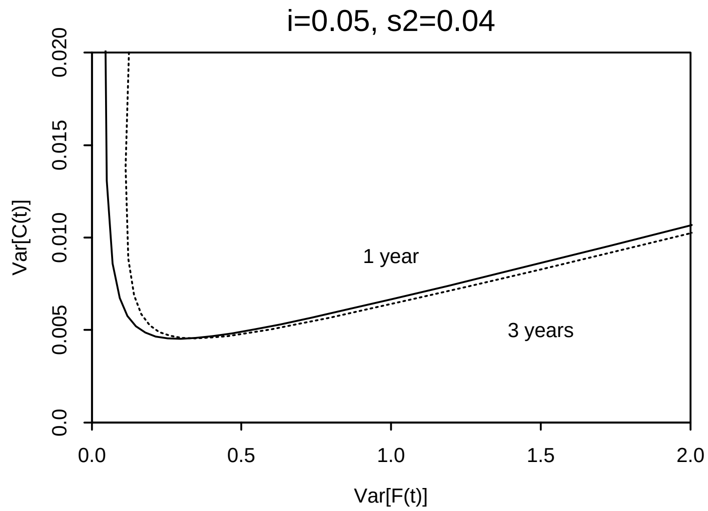

Figure 4: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Note: the efficient frontier for the annual valuation case is, for most values of m less than m^*, below that for the triennial valuation case.

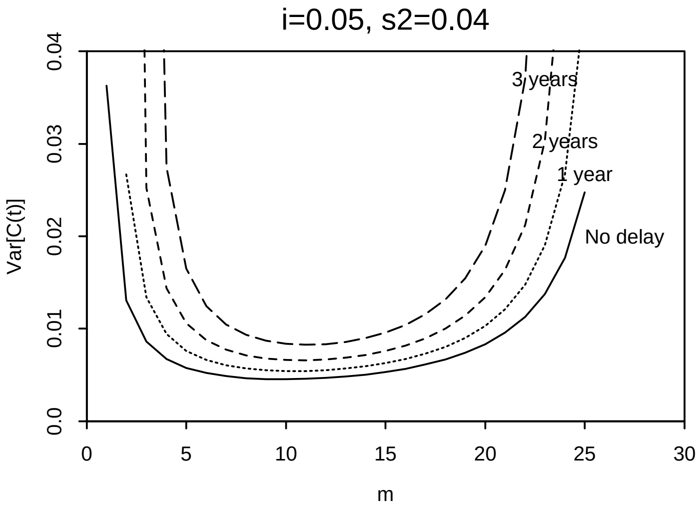

Figure 5: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for delay periods of 0, 1, 2 and 3 years.

valuation date. The new recommended contribution rate is therefore typically not implemented until one year later. There is a delay period of 1 year.

This problem has been investigated by Zimbidis and Haberman (1993). In the example under consideration each extra year's delay increases the variance of $F(t)$ and $C(t)$ by at least 20% and by much more substantial amounts for small values of m. Figures 5 and 6 illustrate the results for this example. One point to note is that where there is a delay period then $Var[F(t)]$ initially decreases with m before increasing as in the no-delay case. This has the effect of reducing the efficient range for m. For example, with a delay of 3 years the efficient range is $5 \leq m \leq 11$ as compared with $1 \leq m \leq 10$ when there is no delay.

In view of the substantial increases in variance caused by a delay it is felt that the delay should be kept as short as possible and perhaps that allowance should be made in the current rate even if the final results of a valuation are not known.

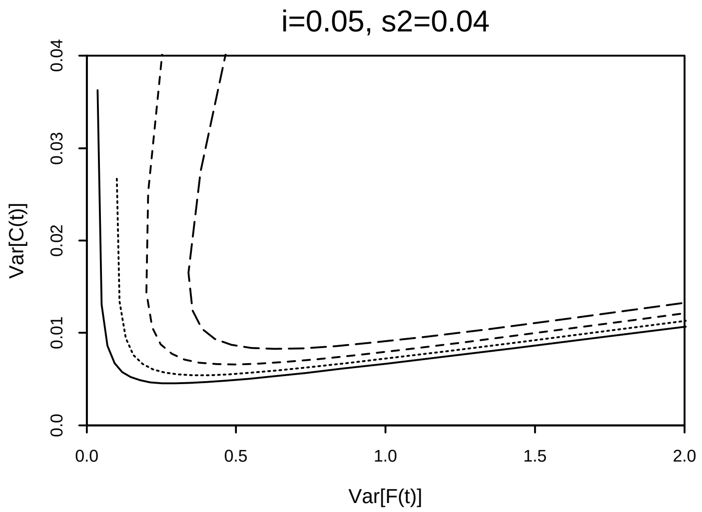

Figure 6: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Increasing the delay period increases the variance of both $F(t)$ and $C(t)$.

2.6 The funding method

Recall the equilibrium equation relating AL to NC

$$AL = (1 + i_v)(AL + NC - B)$$

If we increase AL then NC balances this by falling (this is because benefits are paid from contributions plus surplus interest on the fund, which has increased). Furthermore, AL is determined by the funding method. The normal ordering which we find is

$$AL_{CUC} < AL_{PUC} < AL_{EAN}$$

where the subscripts represent the Current Unit Credit (CUC), Projected Unit Credit (PUC) and Entry Age Normal (EAN) methods, these being the three main funding methods appropriate for a stable membership.

The Attained Age Method has the same actuarial liability as the Projected Unit Credit Method but normally has a higher normal contribution rate which is appropriate for a closed fund, but which will give systematic rise to surplus when the fund has a stable membership. In such a case the equilibrium equation is, therefore, not satisfied. Instead the system has a higher equilibrium fund size which depends on the method and period of amortization.

The variances of $F(t)$ and $C(t)$ are both proportional to AL^2. This means that a more secure funding method (higher AL) gives rise to greater variability, suggesting that a method with a low actuarial liability is to be preferred. Clearly this is not a prudent strategy. It jeopardizes member's security and is more likely to violate statutory solvency requirements.

This problem can be overcome by a number of methods, including:

- the use of the normalized variances $Var[F(t)]/E[F(t)]^2$ and $Var[C(t)]/E[F(t)]^2$;
- the use of further fund objectives (for example, by conditioning on the mean fund size being at a specified level).

2.7 The strength of the valuation basis

So far we have concentrated on the case where the valuation rate of interest, i_v, is equal to the mean long term rate of interest, i. It is common, however, for valuations to be carried out on a strong (occasionally weak) basis: that is, to set $i_v < i$ (or $i_v > i$). This gives rise to a wider variety of results.

Recall that

$$\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2
\end{aligned}$$

We concentrate on the variance of the contribution rate and look for the existence of a minimum with respect to the period of amortization, m. There are a number of cases to consider.

1. **Strong basis:** $i_v < i$ $(v_v > v_1)$

   (these are currently observations, and not proved)

   (a) $E(C_t)$ is an increasing function of k for $k > 1-\sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1-\sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

   From this we can see that for $k > k^*$ both the expected value and the variance of the contribution rate are increasing so that increasing k above k^* is not worthwhile. If k is decreased then we trade off a lower contribution rate for a higher variance. The optimal value therefore depends on the pension fund's utility function or objectives. This goes slightly against the conclusions of Dufresne who indicates that k^* would be the *minimum* acceptable value of k.

   For some values of k the mean contribution rate will be negative, indicating that the fund is large enough to pay for itself and at times requiring refunds to the employer. Although this seems an ideal situation, the reality is that the company must first have built up the fund to this level. It would also be likely to violate statutory surplus regulations.

   It is possible to have smaller expected fund levels and higher contribution rates, but these do not arise if the projected unit method is used in the calculation of the funding rate and using a conservative valuation rate of interest.

2. **Best estimate:** $i_v = i$ $(v_v = v_1)$

   The results of Dufresne (1989) hold.

   (a) $E(C_t)$ is a constant function of k for $k > 1-\sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1-\sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

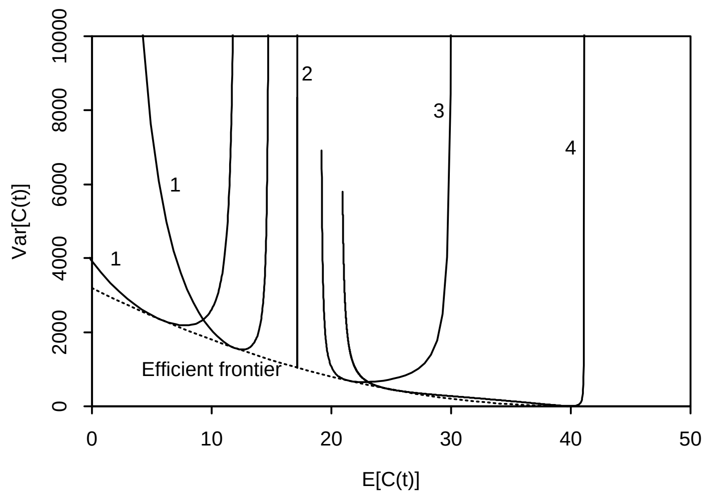

Figure 7: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against $E[C(t)]$ for different valuation rates of interest. Moving from left to right the curves represent: $i_v = 0.03, 0.04$ (type 1, strong basis); $i_v = 0.05$ (type 2, best estimate basis); $i_v = 0.06$ (type 3, weak basis); $i_v = 0.07$ (type 4, very weak basis). The dotted line is the efficient frontier.

3. **Weak basis:** $i < i_v < \sqrt{(1+i)^2+\sigma^2} - 1 \;\; (v_1 > v_v > \sqrt{v_2})$

   (a) $E(C_t)$ is a decreasing function of k for $k > 1 - \sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1 - \sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

   This time we find that it may be acceptable to increase k above k^*, trading off lower contributions for higher variability.

4. **Very weak basis:** $\sqrt{(1+i)^2+\sigma^2} - 1 < i_v \;\; (\sqrt{v_2} > v_v)$

   (a) $E(C_t)$ is a decreasing function of k for $k > 1 - v_v$ at which point it equals B and the scheme is funded on a pay as you go basis. For $1 - v_v > k > 1 - \sqrt{v_2}$ $E(C_t)$ is still a decreasing function.

   (b) $Var(C_t)$ has a minimum equal to zero at $k = 1 - v_v$. This is because the scheme is now funded on a pay as you go basis and contributions equal the constant B.

   (c) $Var(F_t)$ has a local minimum at $k = 1$, a maximum at some $1 - v_v < k^* < 1$ and a global minimum equal to zero at $k = 1 - v_v$ when the fund stays constant at zero.

The efficient frontier

Pooling these results together we can determine a curve $m(\mu_c)$ where

$$m(\mu_C) = \min\{Var(C_t) : E(C_t) = \mu_C, 1 > k > \max(1 - v_v, 1 - \sqrt{v_2}), v_v < 1\}$$

That is, $m(\mu_C)$ gives us the minimum variance attainable for a given mean contribution rate. In fact, it can be shown that $m(\mu_C)$ is convex (quadratic).

These different types of outcome are illustrated in Figure 7, with $i = 0.05$ and $\sigma^2 = 0.2^2$.

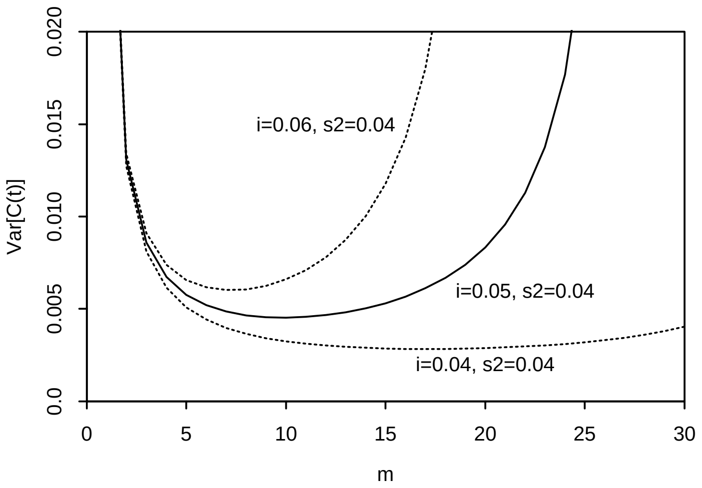

Figure 8: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for different long term rates of return. The valuation rate of interest is fixed.

2.8 Sensitivity testing

In carrying out such analyses it is important to realize that the model for the rate of return including its parameter values are uncertain. First, the model we use here is only one of a range of possible models of varying complexity which all fit past data reasonably well. All of these models are, however, only an approximation to a much more complex reality. Second, the parameter values which we have used (here $i = 0.05$ and $\sigma^2 = 0.04$) are not known with certainty: for example i could equally well be 0.04 or 0.06.

In fact this can have a very significant effect on level the variability. Figures 8 and 9 illustrate this point. i is allowed to take the values 0.04, 0.05 and 0.06. In Figure 8 the effect on $Var[C(t)]$ is very significant, particularly for larger values of m. However, these results are distorted by the fact that when $i \neq i_v$ the mean fund size ($E[F(t)]$) depends on m. The normalized variance of $C(t)$ is plotted in Figure 9 and the effect can be seen to be reduced but still significant.

A change in the value of i of 1% makes a difference in m^* of about 2 years (for example, moving from $i = 0.05$ to $i = 0.06$ changes m^* from 10 to 8).

The result of these changes is not as significant as might first appear. For example, suppose we settled upon $m^* = 10$ on the basis that $i = 0.05$. If in fact the long term mean turned out to be $i = 0.06$ then amortizing over 10 years would only turn out to have been only marginally worse than if the true optimum $m^* = 8$ had been used. The fact that the actual variance of the contribution rate was perhaps 20% higher than that expected is irrelevant since the lower value would never, in fact, have been attainable.

Figure 10 shows the effects of uncertainty in σ^2 (with σ^2 taking the values 0.03, 0.04 and 0.05). The effect is again substantial, but much more uniform over the whole range of values for m. This is because σ^2 has a much more direct effect on the variance of the fund size and the contribution rate. However, as with uncertainty in i, the normalized variance is relatively stable over a range of values about the minimum, so choosing the wrong value of m will only marginally increase the long term variance.

The point to take in from this section is that we need to take care in ensuring that we look at the right quantities. We therefore need to compare the *actual* outcome based on the decision which

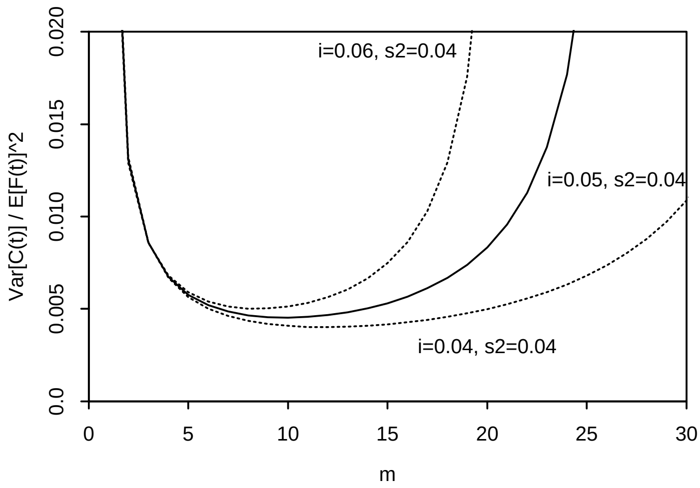

Figure 9: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]/E[F(t)]^2$ plotted against m for different long term rates of return. The valuation rate of interest is fixed.

was based on incorrect assumptions with the outcome which would have *actually* happened had the decision been based on the correct assumptions. Here the differences have been shown to be minimal but if we were to find that they were significant then we may need to look carefully at our estimates to see if they can be refined and improved upon.

2.9 Objectives

We have already discussed that within the efficient region for m ($1 \leq m \leq m^*$) there is a trade off between higher variance of $F(t)$ and higher variance of $C(t)$. To settle on an optimal spread period therefore requires a specific objective or utility function. For example, we may be concerned about containing the fund size within a specified band (bounded below, say, by the minimum solvency level and above by a statutory surplus limit). We could accomodate this by specifying that $E[F(t)]$ lie in the middle of this band and that the standard deviation of $F(t)$ be no more than 10% of this mean fund size. In this case the optimum would be m^{**} which pushes the variance of $F(t)$ up to the maximum level allowable or m^* if this is lower.

If a proper optimum is to be found then the fund must have a well defined objective which will allow optimization to take place. Examples of some objectives are:

- Minimize $Var[C(t)]$ subject to $Var[F(t)] \leq V_{max}$;
- Minimize $Var[C(t)]$ subject to $E[F(t)] = \mu_F$;
- Minimize the variance of the present value of all future contributions (that is, $\sum_{t=0}^{\infty} v^t C(t)$) subject to ......;
- Maximize $E[u(F(t))]$ where $u(f)$ is utility function which depends on the fund size. For example, if $u(f) = -(f - f_0)^2$ then $E[u(F(t))] = -Var[F(t)] - (E[F(t)] - f_0)^2$, the second term being a penalty for deviation of the mean from the target of f_0.

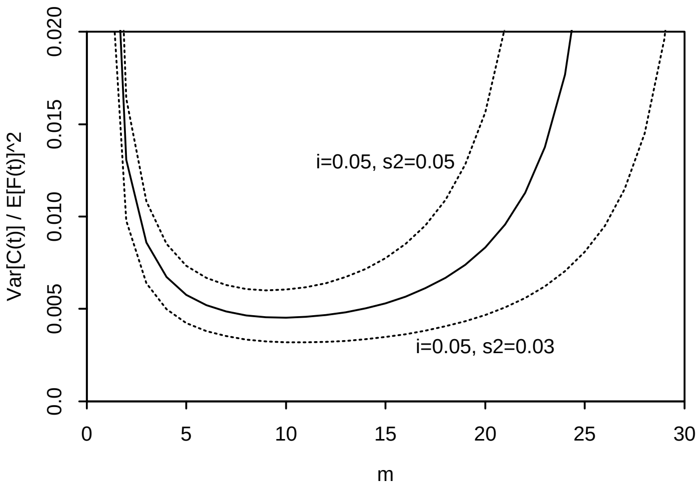

Figure 10: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]/E[F(t)]^2$ plotted against m for varying levels of volatility in the rate of return. The valuation rate of interest is fixed.

Care should be taken when formulating an objective. For example, the last of these makes less sense if $E[F(t)]$ is constant for all values of m (that is if $i_v = i$); and constraints should have reasonable rather than extreme values.

2.10 Other stochastic investment models

We have used the simplest stochastic interest model here (independent and identically distributed returns) which allows us to obtain some intuitively appealing analytical results. A wide variety of more complex models are used in practice for which analytical results are not possible. However, it is expected that similar qualitative results should be available.

Autoregressive time series models: Haberman (1993a) has investigated the use of the AR(1) time series model:

$$
\begin{aligned}
\delta(t) &= \delta + \alpha(\delta(t-1) - \delta) + \nu Z(t) \\
\text{where } \delta(t) &= \log(1 + i(t)) \\
Z(t) &\sim N(0,1) \\
|\alpha| &< 1 \text{ is the autoregressive parameter} \\
\delta &= \text{long term mean rate of return} \\
\nu^2 &= \text{variance parameter} \\
\text{Hence } E[\delta(t)] &= \delta \\
Var[\delta(t)] &= \sigma^2 = \frac{\nu^2}{1-\alpha^2} \\
E[1+i(t)] &= e^{\delta + \frac{1}{2}\sigma^2} \\
Var[1+i(t)] &= e^{2\delta+\sigma^2}\left(e^{\sigma^2} - 1\right)
\end{aligned}
$$

It has been found that $\alpha > 0$ (positively correlated returns) decreases the value of m^* (for example, with $E[i(t)] = 0.05$ and $Var[i(t)] = 0.2^2$ m^* falls from 10 to 5 when α is changed from 0 (independent and identically distributed returns) to only 0.1). More likely is the case $\alpha < 0$ (a high return one year is followed by a low return the next year) which increases the value of m^*.

Note that such models seem more appropriate to fixed interest investments: past equity data do not show any significant signs of autocorrelation from one year to the next.

In summary the most widely used stochastic interest models are

- Independent and identically distributed returns: for example, Waters (1978), Dufresne (1990), Papachristou and Waters (1991), Parker (1993 a,b, 1994 a,b) and Aebi *et al.* (1994) give but a few examples.
- Simple autoregressive models, such as the $AR(1)$ time series model, and the Ornstein-Uhlenbeck process: for example, Dhaene (1989), Parker (1993 a,b, 1994 a,b) and Norberg and Møller (1994).
- Models for the term structure of interest rates: for example, Boyle (1978, 1980), Brennan and Schwarz (1979), Albrecht (1985), Cox, Ingersoll and Ross (1985), Beekman and Shiu (1988), Heath, Jarrow and Morton (1990, 1992), Reitano (1991), Sercu (1991) and Longstaff and Schwarz (1992).
- Models with several asset classes: for example, Wilkie (1987, 1992, 1994), and Chan (1994).

The last two of these classes are the most appropriate for the purposes of making an asset allocation decision. In an objective based setting, however, the asset allocation strategy must be considered simulataneously with other factors which are within our control (see the example in the next section).

Increasing complexity means that we need to resort to stochastic simulation in most of these cases.

2.11 Example: A two asset model

Suppose that the fund has two assets in which it can invest. The return in year t on asset j ($j = 1,2$) is $i_j(t)$ with

$$\begin{aligned} E[i_j(t)] &= i_j \text{ for } j = 1,2 \\ Cov[i_j(t), i_k(t)] &= c_{jk} = c_{kj} \quad j,k = 1,2 \end{aligned}$$

Suppose asset 1 carries a lower risk and a lower return: that is, $i_1 < i_2$ and $c_{11} < c_{22}$.

Let $i(t)$ be the overall return during year t, and suppose that a proportion p of the fund is invested in asset 1. Then

$$
\begin{aligned}
E[i(t)] &= pi_1 + (1-p)i_2 = \mu(p) \ \text{ say} \\
Var[i(t)] &= Var[pi_1(t) + (1-p)i_2(t)] \\
&= Var[pi_1(t)] + Var[(1-p)i_2(t)] + 2Cov[pi_1(t), (1-p)i_2(t)] \\
&= p^2 c_{11} + (1-p)^2 c_{22} + 2p(1-p)c_{12} \\
&= \sigma^2(p) \ \text{ say}
\end{aligned}
$$

(This is following the approach of Modern Portfolio Theory.)

We now put this new mean and variance into the original equations:

$$
\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
\text{where } \ v_1 &= \frac{1}{E[1+i(t)]} = \frac{1}{1+\mu(p)} \\
v_2 &= \frac{1}{E[(1+i(t))^2]} = \frac{1}{(1+\mu(p))^2+\sigma^2(p)}
\end{aligned}
$$

We now have at our disposal:

- the period of amortization;
- valuation basis;
- asset mix.

We have seen from looking at the strength of the valuation basis that a wide range of fund sizes can be attained. Optimal choices must therefore be made with reference to some specific objectives. For example,

$$
\begin{aligned}
\text{minimize} \quad & Var[C(t)] \\
\text{subject to} \quad & E[F(t)] = AL' \\
& Var[F(t)] \leq (0.1AL')^2
\end{aligned}
$$

where AL' is, for example, a statutory minimum plus 20%.

To find an appropriate solution one must now use numerical methods to optimize over the factors within our control. The process of optimization may proceed as follows:

1. Fix the asset proportion and the valuation rate of interest (p and i_v). Then k (therefore m) is determined by the constraint on $E[F(t)]$:

$$E[F(t)] = \frac{(1-k-v_v)}{(1-k-v_1)} AL(i_v) = AL'$$

2. Find the range of values of i_v for which $Var[F(t)] \leq (0.1AL')^2$, and within that range which i_v minimizes $Var[C(t)]$. Let this minimum be $M(p)$.

3. Minimize $M(p)$ over $0 \leq p \leq 1$.

4. Check that the optimal values are reasonable: for exampe, is i_v reasonable when compared with $E[i(t)] = \mu(p^*)$; is m^* reasonable; is p^* acceptable? If the answer to any of these questions is no then we should ask ourselves why and reformulate the objectives accordingly.

2.12 Constraints on strategies

We have already mentioned in Sections 2.6 and 2.9 that our optimal strategy may be influenced by statutory funding levels. These may be

- a minimum solvency requirement;
- a maximum surplus regulation.

Different countries have different regulations for what happens when one of these limits is breached. Typically, however, there may be a requirement to amortize the difference between the limit and the current fund size over a shorter period than normal (in the UK and Canada this is 5 years).

Another constraint may be a limit on the ability of the employer to take a refund from the fund. If no refund at all is possible then ultimately the fund will reach a stage where the fund becomes large enough to be self funding (that is, interest exceeds benefit outgo) beyond which point the fund will grow exponentially out of control. This is a certain event in a stochastic environment. More common is a (statutory) constraint that contribution refunds may only be made while the asset/liability ration remains above a specified level.

When such constraints are in place exact analyses are no longer possible. Instead numerical investigations are necessary.

2.13 Salary growth and price inflation

We have already illustrated how salary growth can be incorporated into these models. This is done by indexing the actuarial libility, the normal contribution rate and the benefit outgo in line with the total salary roll $S(t)$, and treating $i(t)$ as a real rate of return.

Salary inflation can be adequately modelled by an autoregressive process of order 1 or alternatively it can be linked to price inflation (for example, see Section 3 and Wilkie, 1994).

Problems arise when benefit outgo is not proportional to the total salary roll. For example, if pensions are paid from the fund but linked to a prices index then benefit outgo is equal to a mixture of past salary rolls increased in line with the appropriate price index.

This can be approached in two ways: by carrying out a simulation study (described in the next section); or by assuming that pensions are matched at the date of retirement by index-linked securities. In the latter case

$$
\begin{aligned}
B(t) &= B \times S(t) \times A(t) \\
\text{where } B &= \text{base pension benefit} \\
S(t) &= \text{salary index} \\
A(t) &= \text{real annuity rate at time } t
\end{aligned}
$$

The annuity rate $A(t)$ is itself governed by a random process: for example, $A(1), A(2), \ldots$ may be independent and identically distributed positive random variables.

2.14 Simulation methods

Two simulation methods are available.

Method 1: (Ergodic method)

All of the interest rate processes described are examples of *ergodic* processes (for example, see Karlin and Taylor, 1975). A consequence of this (amongst other properties) is that the fund process will satisfy

$$
\begin{aligned}
\bar{f}_n &= \frac{1}{n}\sum_{t=1}^{n} F(t) \to E[F(t)] \text{ almost surely as } n \to \infty \\
s_n^2 &= \frac{1}{n}\sum_{t=1}^{n} \left(F(t) - \bar{f}_n\right)^2 \to Var[F(t)] \text{ almost surely as } n \to \infty
\end{aligned}
$$

(If salary growth is allowed for, then $F(t)$ above should be replaced by the asset/liability ratio $F(t)/AL(t)$.)

This means that a single, long simulation run of the pension plan will give us good estimates of the means and variances of the quantities of interest. Rough calculations suggest that this simulation should be of at least 2000 years.

The simulation should be repeated for each combination of decisions being examined. For consistency and efficiency the same realization of the interest rate process should be used for each combination of decisions.

Method 2: Repeated simulation

The objective of the fund may, amongst other things, aim to minimize variance over a short period, say 10 years, rather than over the longer term. Repeated simulation is more appropriate here: that is, simulate the fund for 10 years, given appropriate initial conditions; and then repeat this, say, 200 or more times. For consistency and efficiency the same 200 scenarios of the interest rate process should be used for each combination of decisions.

3 Defined Contribution Pension Plans

Defined contribution pension plans are becoming of ever increasing importance and as such they require some long overdue investigation in order that their reliability as a pensions vehicle can be improved upon. The principal distinctions with defined benefit pension plans are that benefits are no longer based upon final salary but depend on past contribution levels and past investment returns thereby passing investment risk from the employer to the individual members.

Whereas an employer as sponsor of a defined benefit plan is able to smooth out good and bad years' investment returns, defined contribution pension plan members are rather more at the mercy of variations in returns from one year to the next. For example, Knox (1993) carried out a simulation study using a simple model which illustrated the high degree of uncertainty in the final amount of a defined contribution pension relative to final salary. This risk is well known and is a major criticism of the defined contribution set-up. Further work is therefore required to see if this risk can be reduced.

Defined contribution pension plans can be divided into two categories:

- those sponsored by an employer;
- those taken out by individuals with an insurer and with no (or only indirect) involvement on the part of an employer (Retirement Savings Plan).

From a statistical standpoint, this is an artificial distinction. Any decision which can be applied to one type should be applicable to the other: for example, the use of investment strategies which depend on the age of the individual.

3.1 Objectives

Clearly defined objectives are perhaps even more important in the decision making process associated with a defined contribution pension plan than a defined benefit pension plan. Different, member oriented objectives are required and the situation may be complicated further by the possibility that different members may have different objectives and utility functions.

An objective is most likely to be defined in terms of the the amount of pension at retirement *as a proportion of final salary* rather than as an absolute amount. Thus we define

$$\begin{aligned} P(t) &= \text{pension on retirement at time } t \\ S(t) &= \text{salary at time } t \\ \pi(t) &= P(t)/S(t) \\ &= \text{pension as a proportion of final salary} \end{aligned}$$

Now $P(t)$ depends on past contributions, past investment returns and annuity rates at retirement. If contributions are paid at the start of each year then

$$\begin{aligned}
P(t) &= \frac{1}{A(t)} \sum_{s=0}^{t} \rho(s) S(s) \frac{F(t)}{F(s)} \\
\text{where } \rho(s) &= \text{contribution rate at time } s \\
\frac{F(t)}{F(s)} &= \text{accumulation at time } t \text{ of an investment of 1 at time } s \\
A(t) &= \text{annuity factor applied on retirement at time } t
\end{aligned}$$

Normally it will be assumed that the contribution rate $\rho(t)$ is constant through time, although this could be used as a method of reducing uncertainty.

Each of the processes $F(t)$, $S(t)$ and $A(t)$ are random. This will exaggerate the level of uncertainty at retirement unless a suitable strategy can be found which can use one process to offset the effects of another. For example, by investing in fixed interest bonds, a fall in bond prices close to retirement will be offset by a fall in the value of $A(t)$, the cost of purchasing an annuity.

Objectives may be divided into two categories

(A) ones in which the member is told of his or her pension only at the date of retirement;

(B) ones in which the member is given advance notice of the (likely) future amount of pension and then expects the final pension to be as close to this as possible (or not too much less than).

Possible objectives of type A are:

- maximize $E[\pi(t)]$;
- maximize $E[\pi(t)]$ subject to $Var[\pi(t)] = \sigma_\pi^2$;
- maximize $Var[\pi(t)]$;
- maximize $Var[\pi(t)]$ subject to $E[\pi(t)] = \mu_\pi$;
- minimize $Pr(\pi(t) < \pi_{\min})$;
- maximize $E[u(\pi(t))]$ where $u(\cdot)$ is some utility function.

Objectives of type B include

- minimize $E[(\pi(t) - \hat{\pi}(t))^2 \mid H_s]$ where H_s gives us the history of the fund up until time t and $\hat{\pi}(t)$ is the estimated future pension based on H_s;
- maximize $E[u(\pi(t)) \mid H_s, \hat{\pi}(t)]$.

It is questionable whether some such objectives may be reasonable. For example, suppose an objective results in a strategy which locks into a given level of pension some time in advance of retirement. The problem with this is that the level which we lock into may be just as variable as the pension which could be obtained had the fund been left alone until the date of retirement. So is it really in the member's best interests to lock into a pension at too early a stage?

3.2 Investment strategies

It may be difficult to examine all possible investment strategies. However, an appropriate starting point may be to examine a small number of possibilities. For example,

- strategies which are fixed through time:
  - equities only
  - equities and matching options
  - fixed interest bonds
  - equities, fixed interest bonds and cash
  - index linked bonds
  - equities, matching options, fixed interest bonds and cash
  - etc.
- strategies which vary through time:
  - equities switching into fixed interest bonds over the last 5 years
  - fixed interest bonds
  - equities and matching options
  - equities, matching options, fixed interest bonds and cash
  - etc.
- strategies which vary through time and depend on the past history of the fund.

3.3 A simple example

Here we look at a simple example which illustrates the fallacy of an early switch into fixed interest bonds.

We simplify the situation by considering a fund which is now of size $F(0)$ and which will receive no further contributions. We are interested in the lump sum which this fund will produce at retirement as a proportion of final salary.

Three options are available:

- a zero-coupon fixed interest investment which provides a guaranteed lump sum L at retirement;
- investment in long-term index linked bonds;
- investment in equities.

The model we will use is described in the Appendix. The model and its parameters were found to fit UK experience reasonably well.

The measure of risk for each option (the variance of the logarithm of the lump sum as a proportion of final salary) is plotted in Figure 11. We can see that although the fixed pension fares

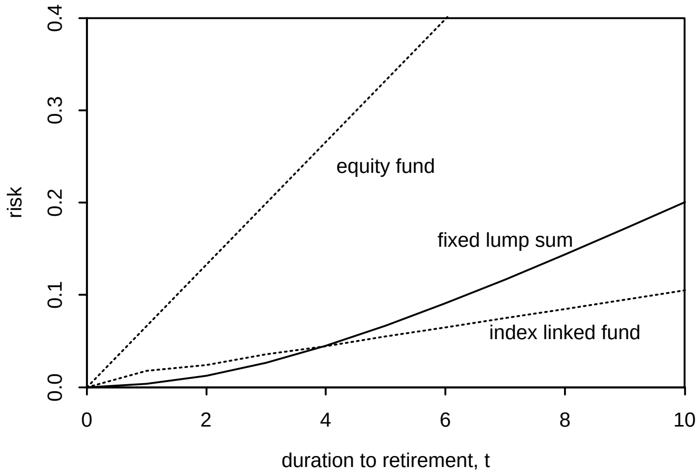

Figure 11: Risk relative to policyholder's salary for three different investment strategies. Risk is measured as $Var[L(t)/S(t)]$ where $L(t)$ is the lump sum at retirement and $S(t)$ is the final salary.

better early on the index linked option clearly becomes lower risk later on. (Note that this does not take account of uncertainty in the initial lump sum which would arise had we been considering the situation part of the way through a policy's lifetime.) The equity fund is, perhaps not surprisingly, well above the other two in terms of risk, but will also attract a reasonable risk premium. It is also likely that a fixed interest investment attracts a small risk premium over an index-linked investment so at later durations the ordering of the risks is in the order we might expect.

4 References

Aebi, M., Embrechts, P., and Mikosch, T. (1994) Stochastic discounting, aggregate claims and Mallows metric. *Advances in Applied Probability* **26** , 183-206.

Albrecht, P. (1985) A note on immunization under a general stochastic equilibrium model of the term structure. *Insurance: Mathematics and Economics* **4**, 239-245.

Beekman, J.A., and Shiu, E.S.W. (1988) Stochastic models for bond prices, function space integrals and immunization theory. *Insurance: Mathematics and Economics* **7**, 163-173.

Black, F. and Jones, R. (1987) Simplifying portfolio insurance for corporate pension plans. *Journal of Portfolio Management* **Summer**, 33-37.

Black, F. and Perold, A. (1992) Theory of constant portfolio insurance. *Journal of Economics and Control* **16**, 403-426.

Boyle, P.P. (1978) Immunization under stochastic models of the term structure. *Journal of the Institute of Actuaries* **105**, 177-188.

Boyle, P.P. (1980) Recent models of the term structure of interest rates with actuarial applications. *21st International Congress of Actuaries* , 95-104.

Brennan, M.J., and Schwarz, E.S. (1979) A continuous time approach to the pricing of bonds. *Journal of Banking and Finance* **3**, 135-155.

Cairns, A.J.G. (1995) Stochastic pension fund modelling in continuous time. *In preparation* , .

Cairns, A.J.G. and Parker, G. (1995) Stochastic pension fund modelling. *In preparation* , .

Chan, T. (1994) Some applications of Lévy processes to stochastic investment models for actuarial use. *submitted to ASTIN Bulletin* , .

Cox, J.C., Ingersoll, J.E., and Ross, S.A. (1985) A theory of the term structure of interest rates. *Econometrica* **53**, 385-407.

Dhaene, J. (1989) Stochastic interest rates and autoregressive integrated moving average processes. *ASTIN Bulletin* **19**, 131-138.

Dufresne, D. (1988) Moments of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries* **115**, 535-544.

Dufresne, D. (1989a) Stability of pension systems when rates of return are random. *Insurance: Mathematics and Economics* **8**, 71-76.

Dufresne, D. (1989b) Weak convergence of random growth processes with applications to insurance. *Insurance: Mathematics and Economics* **8**, 187-201.

Dufresne, D. (1990) The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal* , 39-79.

Dufresne, D. (1992) On discounting when rates of return are random. *24th International*

Congress of Actuaries, Montreal **1**, 27-44.

Heath, D., Jarrow, R., and Morton, A. (1990) Bond pricing and the term structure of interest rates: a discrete time approximation. *Journal of Financial and Quantitative Analysis* **25**, 419-440.

Haberman, S. (1992) Pension funding with time delays: a stochastic approach. *Insurance: Mathematics and Economics* **11**, 179-189.

Haberman, S. (1993a) Pension funding with time delays and autoregressive rates of investment return. *Insurance: Mathematics and Economics* **13**, 45-56.

Haberman, S. (1993b) Pension funding: the effect of changing the frequency of valuations. *Insurance: Mathematics and Economics* **13**, 263-270.

Haberman, S. (1994) Autoregressive rates of return and the variability of pension fund contributions and fund levels for a defined benefit pension scheme. *Insurance: Mathematics and Economics* **14**, 219-240.

Heath, D., Jarrow, R., and Morton, A. (1992) Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**, 77-105.

Karlin, S. and Taylor, H.M. (1975) *A first course in stochastic processes: 2nd Edition.* Academic Press, New York.

Knox, D. (1993) A critique of defined contribution using a simulation approach. Research paper number 7, Centre for Actuarial Studies, The University of Melbourne.

Longstaff, F.A., and Schwartz, E.S. (1992) Interest rate volatility and the term structure: a two factor general equilibrium model. *The Journal of Finance* **47**, 1259-1282.

Norberg, R., and Møller, C.M. (1994) Thiele's differential equation by stochastic interest of diffusion type. *to appear in Scandinavian Actuarial Journal* , .

Papachristou, D., and Waters, H.R. (1991) Some remarks concerning interest rates in relation to long term insurance policies. *Scandinavian Actuarial Journal* **2**, 103-117.

Parker, G. (1993a) Two stochastic approaches for discounting actuarial functions. *Proceedings of the XXIV ASTIN Colloquium* , 367-389.

Parker, G. (1993b) Distribution of the present value of future cash flows. *3rd AFIR, Rome* **2**, 831-843.

Parker, G. (1994a) Stochastic analysis of an insurance portfolio. *4th AFIR, Orlando* **1**, 49-66.

Parker, G. (1994b) Moments of the present value of a portfolio of policies. *to appear in Scandinavian Actuarial Journal* , .

Reitano, R.R. (1991a) Multivariate duration analysis. *Transactions of the Society of Actuaries* **43**, 335-391.

Sercu, P. (1991) Bond options and bond portfolio insurance. *Insurance: Mathematics and Economics* **10**, 203-230.

Waters, H.R. (1978) The moments and distributions of actuarial functions. *Journal of the Institute of Actuaries* **105**, 61-75.

Wilkie, A.D. (1987) Stochastic investment models – theory and practice. *Insurance: Mathematics and Economics* **6**, 65-83.

Wilkie, A.D. (1992) Stochastic investment models for XXIst century actuaries. *24th International Congress of Actuaries, Montreal* **5**, 119-137.

Wilkie, A.D. (1994) Stochastic modelling of long-term investment risks. *submitted to the IMA Journal of Mathematics Applied in Business and Finance* , .

Zimbidis, A. and Haberman, S. (1993) Delay, feedback and variability of pension contributions and fund levels. *Insurance: Mathematics and Economics* **13**, 271-285.

5 Appendix

$$\begin{aligned}
S(t) &= \text{salary at time } t \\
F_e(t) &= \text{equities fund at time } t \\
F_{il}(t) &= \text{index-linked fund at time } t \\
\delta_s(t) &= \log[S(t)/S(t-1)] \\
\delta_e(t) &= \log[F_e(t)/F_e(t-1)] \\
\delta_{il}(t) &= \log[F_{il}(t)/F_{il}(t-1)]
\end{aligned}$$

$$\begin{aligned}
\text{with } \delta_s(t) &= \delta_p(t) + \delta_{rs}(t) \\
\delta_e(t) &= \delta_p(t) + \delta_{rs}(t) + \delta_{re}(t) \\
\delta_{il}(t) &= \delta_p(t) + \delta_{ril}(t)
\end{aligned}$$

$$\begin{aligned}
\delta_p(t) &= \text{force of price inflation between } t-1 \text{ and } t \\
&= \delta_p + \alpha_p(\delta_p(t-1) - \delta_p) + \sigma_p Z_p(t) \\
\delta_{rs}(t) &= \text{real salary growth rate} \\
&= \delta_{rs} + \alpha_{rs}(\delta_{rs}(t-1) - \delta_{rs}) + \sigma_{rs} Z_{rs}(t) \\
\delta_{re}(t) &= \text{real equities rate of return over salaries} \\
&= \delta_{re} + \sigma_{re} Z_{re}(t) \\
\delta_{ril}(t) &= \text{real index linked return} \\
&= \delta_{ril} + \alpha_{ril}(\delta_{ril}(t-1) - \delta_{ril}) + \sigma_{ril} Z_{ril}(t)
\end{aligned}$$

where $Z_p(t)$, $Z_{rs}(t)$, $Z_{re}(t)$ and $Z_{ril}(t)$ (for $t = 0, 1, 2, \ldots$) are independent and identically distributed sequences of standard Normal random variables.

Now let

$$
\begin{aligned}
y_p(t) &= \sum_{s=1}^{t} \delta_p(s) \\
y_{rs}(t) &= \sum_{s=1}^{t} \delta_{rs}(s) \\
y_{re}(t) &= \sum_{s=1}^{t} \delta_{re}(s) \\
y_{ril}(t) &= \sum_{s=1}^{t} \delta_{ril}(s) \\
\text{Then } E[y_p] &= \delta_p.t \\
Var[y_p(t)] &= \frac{\sigma_p^2}{(1-\alpha_p)^2}\left[t - \frac{2\alpha_p(1-\alpha_p^t)}{(1-\alpha_p)} + \frac{\alpha_p^2(1-\alpha_p^{2t})}{(1-\alpha_p^2)}\right] \\
E[y_{rs}] &= \delta_{rs}.t \\
Var[y_{rs}(t)] &= \frac{\sigma_{rs}^2}{(1-\alpha_{rs})^2}\left[t - \frac{2\alpha_{rs}(1-\alpha_{rs}^t)}{(1-\alpha_{rs})} + \frac{\alpha_{rs}^2(1-\alpha_{rs}^{2t})}{(1-\alpha_{rs}^2)}\right] \\
E[y_{re}(t)] &= \delta_{re}.t \\
Var[y_{re}(t)] &= \sigma_{re}^2.t \\
E[y_{ril}] &= \delta_{ril}.t \\
Var[y_{ril}(t)] &= \frac{\sigma_{ril}^2}{(1-\alpha_{ril})^2}\left[t - \frac{2\alpha_{ril}(1-\alpha_{ril}^t)}{(1-\alpha_{ril})} + \frac{\alpha_{ril}^2(1-\alpha_{ril}^{2t})}{(1-\alpha_{ril}^2)}\right]
\end{aligned}
$$

We also define

$$
\begin{aligned}
F_e(t) &= \exp[y_p(t) + y_{re}(t)] \\
F_{il}(t) &= \exp[y_p(t) + y_{ril}(t)] \\
S(t) &= \exp[y_p(t) + y_{rs}(t)]
\end{aligned}
$$

We are interested in the three quantities

$$
\begin{aligned}
L_1 &= L/S(t) \\
L_2 &= F_{il}(t)/S(t) \\
L_3 &= F_e(t)/S(t)
\end{aligned}
$$

Of particular interest is the level of risk associated with each option which we measure by taking the variance of the logarithm of each quantity.

$$
\begin{aligned}
Var[\log L_1] &= Var[y_p(t)] + Var[y_{rs}(t)] \\
Var[\log L_2] &= Var[y_{rs}(t)] + Var[y_{ril}(t)] \\
Var[\log L_3] &= Var[y_{re}(t)]
\end{aligned}
$$

These variances are described in the main text.

Parameter values

type, θ	δ_θ	α_θ	σ_θ^2
prices, *p*	0.05	0.7	0.05^2
real salary, *rs*	0.02	0.4	0.03^2
real index-linked, *ril*	0.036	-0.5	0.13^2
real equity, *re*	0.036		0.26^2

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A SURVEY OF BEHAVIORAL FINANCE

Nicholas Barberis
Richard Thaler

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A Survey of Behavioral Finance
Nicholas Barberis and Richard Thaler
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ABSTRACT

Behavioral finance argues that some financial phenomena can plausibly be understood using models in which some agents are not fully rational. The field has two building blocks: limits to arbitrage, which argues that it can be difficult for rational traders to undo the dislocations caused by less rational traders; and psychology, which catalogues the kinds of deviations from full rationality we might expect to see. We discuss these two topics, and then present a number of behavioral finance applications: to the aggregate stock market, to the cross-section of average returns, to individual trading behavior, and to corporate finance. We close by assessing progress in the field and speculating about its future course.

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1 Introduction

The traditional finance paradigm, which underlies many of the other articles in this handbook, seeks to understand financial markets using models in which agents are "rational". Rationality means two things. First, when they receive new information, agents update their beliefs correctly, in the manner described by Bayes' law. Second, given their beliefs, agents make choices that are normatively acceptable, in the sense that they are consistent with Savage's notion of Subjective Expected Utility (SEU).

This traditional framework is appealingly simple, and it would be very satisfying if its predictions were confirmed in the data. Unfortunately, after years of effort, it has become clear that basic facts about the aggregate stock market, the cross-section of average returns and individual trading behavior are not easily understood in this framework.

Behavioral finance is a new approach to financial markets that has emerged, at least in part, in response to the difficulties faced by the traditional paradigm. In broad terms, it argues that some financial phenomena can be better understood using models in which some agents are *not* fully rational. More specifically, it analyzes what happens when we relax one, or both, of the two tenets that underlie individual rationality. In some behavioral finance models, agents fail to update their beliefs correctly. In other models, agents apply Bayes' law properly but make choices that are normatively questionable, in that they are incompatible with SEU.[1]

This review essay evaluates recent work in this rapidly growing field. In Section 2, we consider the classic objection to behavioral finance, namely that even if some agents in the economy are less than fully rational, rational agents will prevent them from influencing security prices for very long, through a process known as arbitrage. One of the biggest successes of behavioral finance is a series of theoretical papers showing that in an economy where rational and irrational traders interact, irrationality *can* have a substantial and long-lived impact on prices. These papers, known as the literature on "limits to arbitrage," form

[1]It is important to note that most models of asset pricing use the Rational Expectations Equilibrium framework (REE), which assumes not only individual rationality but also *consistent beliefs* (Sargent, 1993). Consistent beliefs means that agents' beliefs are correct: the subjective distribution they use to forecast future realizations of unknown variables is indeed the distribution that those realizations are drawn from. This requires not only that agents process new information correctly, but that they have *enough* information about the structure of the economy to be able to figure out the correct distribution for the variables of interest.

Behavioral finance departs from REE by relaxing the assumption of individual rationality. An alternative departure is to retain individual rationality but to relax the consistent beliefs assumption: while investors apply Bayes' law correctly, they lack the information required to know the actual distribution variables are drawn from. This line of research is sometimes referred to as the literature on bounded rationality, or on structural uncertainty. For example, a model in which investors do not know the growth rate of an asset's cash flows but learn it as best as they can from available data, would fall into this class. Although the literature we discuss also uses the term bounded rationality, the approach is quite different.

one of the two buildings blocks of behavioral finance.

To make sharp predictions, behavioral models often need to specify the form of agents' irrationality. How exactly do people misapply Bayes law or deviate from SEU? For guidance on this, behavioral economists typically turn to the extensive experimental evidence compiled by cognitive psychologists on the biases that arise when people form *beliefs*, and on people's *preferences*, or on how they make decisions, given their beliefs. Psychology is therefore the second building block of behavioral finance, and we review the psychology most relevant for financial economists in Section 3.[2]

In Sections 4-8, we consider specific applications of behavioral finance: to understanding the aggregate stock market, the cross-section of average returns, and the pricing of closed-end funds in Sections 4, 5 and 6 respectively; to understanding how particular groups of investors choose their portfolios and trade over time in Section 7; and to understanding the financing and investment decisions of firms in Section 8. Section 9 takes stock and suggests directions for future research.[3]

2 Limits to Arbitrage

2.1 Market Efficiency

In the traditional framework where agents are rational and there are no frictions, a security's price equals its "fundamental value." This is the discounted sum of expected future cash flows, where in forming expectations, investors correctly process all available information, and where the discount rate is consistent with a normatively acceptable preference specification. The hypothesis that actual prices reflect fundamental values is the Efficient Markets Hypothesis (EMH). Put simply, under this hypothesis, "prices are right," in that they are set by agents who understand Bayes' law and have sensible preferences. In an efficient market, there is "no free lunch": no investment strategy can earn excess risk-adjusted average returns, or average returns greater than are warranted for its risk.

Behavioral finance argues that some features of asset prices are most plausibly interpreted as deviations from fundamental value, and that these deviations are brought about by the presence of traders who are not fully rational. A long-standing objection to this view that goes back to Friedman (1953) is that rational traders will quickly undo any dislocations

[2]The idea, now widely adopted, that behavioral finance rests on the two pillars of limits to arbitrage and investor psychology is originally due to Shleifer and Summers (1990).

[3]We draw readers' attention to two other recent surveys of behavioral finance. Shleifer (2000) provides a particularly detailed discussion of the theoretical and empirical work on limits to arbitrage, which we summarize in Section 2. Hirshleifer's (2001) survey is closer to ours in terms of material covered, although we devote less space to asset pricing, and more to corporate finance and individual investor behavior. We also organize the material somewhat differently.

caused by irrational traders. To illustrate the argument, suppose that the fundamental value of a share of Ford is $20. Imagine that a group of irrational traders becomes excessively pessimistic about Ford's future prospects and through its selling, pushes the price to $15. Defenders of the EMH argue that rational traders, sensing an attractive opportunity, will buy the security at its bargain price and at the same time, hedge their bet by shorting a "substitute" security, such as General Motors, that has similar cash flows to Ford in future states of the world. The buying pressure on Ford shares will then bring their price back to fundamental value.

Friedman's line of argument is initially compelling, but it has not survived careful theoretical scrutiny. In essence, it is based on two assertions. First, as soon as there is a deviation from fundamental value – in short, a mispricing – an attractive investment opportunity is created. Second, rational traders will immediately snap up the opportunity, thereby correcting the mispricing. Behavioral finance does not take issue with the second step in this argument: when attractive investment opportunities come to light, it is hard to believe that they are not quickly exploited. Rather, it disputes the first step. The argument, which we elaborate on in Sections 2.2 and 2.3., is that even when an asset is wildly mispriced, strategies designed to correct the mispricing can be both risky and costly, rendering them unattractive. As a result, the mispricing can remain unchallenged.

It is interesting to think about common finance terminology in this light. While irrational traders are often known as "noise traders," rational traders are typically referred to as "arbitrageurs." Strictly speaking, an arbitrage is an investment strategy that offers riskless profits at no cost. Presumably, the rational traders in Friedman's fable became known as arbitrageurs because of the belief that a mispriced asset immediately creates an opportunity for riskless profits. Behavioral finance argues that this is *not* true: the strategies that Friedman would have his rational traders adopt are not necessarily arbitrages; quite often, they are very risky.

An immediate corollary of this line of thinking is that "prices are right" and "there is no free lunch" are *not* equivalent statements. While both are true in an efficient market, "no free lunch" can also be true in an inefficient market: just because prices are away from fundamental value does not necessarily mean that there are any excess risk-adjusted average returns for the taking. In other words,

$$\boxed{\text{“prices are right”} \Rightarrow \text{“no free lunch”}}$$

but

$$\boxed{\text{“no free lunch”} \not\Rightarrow \text{“prices are right”}.}$$

This distinction is important for evaluating the ongoing debate on market efficiency. First, many researchers still point to the inability of professional money managers to beat the market as strong evidence of market efficiency (Rubinstein, 2000, Ross, 2001). Underlying this argument, though, is the assumption that "no free lunch" implies "prices are right." If, as we argue in Sections 2.2 and 2.3., this link is broken, the performance of money managers tells us nothing about whether prices reflect fundamental value.

Second, while some researchers accept that there is a distinction between "prices are right" and "there is no free lunch," they believe that the debate should be more about the latter statement than about the former. We disagree with this emphasis. As economists, our ultimate concern is that capital be allocated to the most promising investment opportunities. Whether this is true or not depends much more on whether prices are right than on whether there are any free lunches for the taking.

2.2 Theory

In the previous section, we emphasize the idea that when a mispricing occurs, strategies designed to correct it can be both risky and costly, thereby allowing the mispricing to survive. Here we discuss some of the risks and costs that have been identified. In our discussion, we return to the example of Ford, whose fundamental value is $20, but which has been pushed down to $15 by pessimistic noise traders.

Fundamental Risk

The most obvious risk an arbitrageur faces if he buys Ford's stock at $15 is that a piece of bad news about Ford's fundamental value causes the stock to fall further, leading to losses. Of course, arbitrageurs are well aware of this risk, which is why they short a substitute security such as General Motors at the same time that they buy Ford. The problem is that substitute securities are rarely perfect, and often highly imperfect, making it impossible to remove all the fundamental risk. Shorting General Motors protects the arbitrageur somewhat from adverse news about the car industry as a whole, but still leaves him vulnerable to news that is specific to Ford – news about defective tires, say.[4]

Noise Trader Risk

Noise trader risk, an idea introduced by De Long et al. (1990a) and studied further by Shleifer and Vishny (1997), is the risk that the mispricing being exploited by the arbitrageur worsens in the short run. Even if General Motors is a *perfect* substitute security for Ford, the arbitrageur still faces the risk that the pessimistic investors causing Ford to be undervalued in the first place become even more pessimistic, lowering its price even further. Once one has granted the possibility that a price can be different from its fundamental value, then one must also grant the possibility that future price movements will increase the divergence.

[4] Another problem is that even if a substitute security exists, it may itself be mispriced. This can happen in situations involving industry-wide mispricing: in that case, the only stocks with similar future cash flows to the mispriced one are themselves mispriced.

Noise trader risk matters because it can force arbitrageurs to liquidate their positions early, bringing them potentially steep losses. To see this, note that most real-world arbitrageurs – in other words, professional portfolio managers – are not managing their own money, but rather managing money for other people. In the words of Shleifer and Vishny (1997), there is "a separation of brains and capital."

This agency feature has important consequences. Investors, lacking the specialized knowledge to evaluate the arbitrageur's strategy, may simply evaluate him based on his returns. If a mispricing that the arbitrageur is trying to exploit worsens in the short run, generating negative returns, investors may decide that he is incompetent, and withdraw their funds. If this happens, the arbitrageur will be forced to liquidate his position prematurely. Fear of such premature liquidation makes him less aggressive in combating the mispricing in the first place.

These problems can be severely exacerbated by creditors. After poor short-term returns, creditors, seeing the value of their collateral erode, will call their loans, again triggering premature liquidation.

In these scenarios, the forced liquidation is brought about by the worsening of the mispricing itself. This need not always be the case. For example, in their efforts to remove fundamental risk, many arbitrageurs sell securities short. Should the original owner of the borrowed security want it back, the arbitrageur may again be forced to close out his position if he cannot find other shares to borrow. The risk that this occurs during a temporary worsening of the mispricing makes the arbitrageur more cautious from the start.

Implementation Costs

Well-understood transaction costs such as commissions, bid-ask spreads and price impact can make it less attractive to exploit a mispricing. Since shorting is often essential to the arbitrage process, we also include short-sales constraints in the implementation costs category. These refer to anything that makes it less attractive to establish a short position than a long one. The simplest such constraint is the fee charged for borrowing a stock. In general these fees are small – D'Avolio (2002) finds that for most stocks, they range between 10 and 15 basis points – but they can be much larger; in some cases, arbitrageurs may not be able to find shares to borrow at *any* price. Other than the fees themselves, there can be legal constraints: for a large fraction of money managers – many pension fund and mutual fund managers in particular – short-selling is simply not allowed.[5]

[5]The presence of per-period transaction costs like lending fees can expose arbitrageurs to another kind of risk, *horizon risk*, which is the risk that the mispricing takes so long to close that any profits are swamped by the accumulated transaction costs. This applies even when the arbitrageur is certain that no outside party will force him to liquidate early. Abreu and Brunnermeier (2002) study a particular type of horizon risk, which they label *synchronization risk*. Suppose that the elimination of a mispricing requires the participation of a sufficiently large number of separate arbitrageurs. Then in the presence of per-period transaction costs,

We also include in this category the cost of finding and learning about a mispricing, as well as the cost of the resources needed to exploit it (Merton, 1987). Finding mispricing, in particular, can be a tricky matter. It was once thought that if noise traders influenced stock prices to any substantial degree, their actions would quickly show up in the form of predictability in returns. Shiller (1984) and Summers (1986) demonstrate that this argument is completely erroneous, with Shiller (1984) calling it "one of the most remarkable errors in the history of economic thought." They show that even if noise trader demand is so strong as to cause a large and persistent mispricing, it may generate so little predictability in returns as to be virtually undetectable.

In contrast, then, to straightforward-sounding textbook arbitrage, real world arbitrage entails both costs and risks, which under some conditions will limit arbitrage and allow deviations from fundamental value to persist. To see what these conditions are, consider two cases.

Suppose first that the mispriced security does *not* have a close substitute. By definition then, the arbitrageur is exposed to fundamental risk. In this case, sufficient conditions for arbitrage to be limited are (i) that arbitrageurs are risk averse and (ii) that the fundamental risk is systematic, in that it cannot be diversified by taking many such positions. Condition (i) ensures that the mispricing will not be wiped out by a single arbitrageur taking a large position in the mispriced security. Condition (ii) ensures that the mispricing will not be wiped out by a large number of investors each adding a *small* position in the mispriced security to their current holdings. The presence of noise trader risk or implementation costs will only limit arbitrage further.

Even if a perfect substitute does exist, arbitrage can still be limited. The existence of the substitute security immunizes the arbitrageur from fundamental risk. We can go further and assume that there are no implementation costs, so that only noise trader risk remains. De Long et al. (1990a) show that noise trader risk is powerful enough, that even with this single form of risk, arbitrage can sometimes be limited. The sufficient conditions are similar to those above, with one important difference. Here arbitrage will be limited if: (i) arbitrageurs are risk averse *and have short horizons* and (ii) the noise trader risk is systematic. As before, condition (i) ensures that the mispricing cannot be wiped out by a single, large arbitrageur, while condition (ii) prevents a large number of small investors from exploiting the mispricing. The central contribution of Shleifer and Vishny (1997) is to point out the real world relevance of condition (i): the possibility of an early, forced liquidation means that many arbitrageurs effectively have short horizons.

In the presence of certain implementation costs, conditions (ii) may not even be necessary.

arbitrageurs may hesitate to exploit the mispricing because they don't know how many *other* arbitrageurs have heard about the opportunity, and therefore how long they will have to wait before prices revert to correct values.

If it is costly to learn about a mispricing, or the resources required it to exploit it are expensive, that may be enough to explain why a large number of different individuals do not intervene in an attempt to correct the mispricing.

It is also important to note that for particular types of noise trading, arbitrageurs may prefer to trade in the *same* direction as the noise traders, thereby exacerbating the mispricing, rather than against them. For example, De Long et al. (1990b) consider an economy with positive feedback traders, who buy more of an asset this period if it performed well last period. If these noise traders push an asset's price above fundamental value, arbitrageurs do not sell or short the asset. Rather, they *buy* it, knowing that the earlier price rise will attract more feedback traders next period, leading to still higher prices, at which point the arbitrageurs can exit at a profit.

So far, we have argued that it is not easy for arbitrageurs like hedge funds to exploit market inefficiencies. However, hedge funds are not the only market participants trying to take advantage of noise traders: firm managers also play this game. If a manager believes that investors are overvaluing his firm's shares, he can benefit the firm's existing shareholders by issuing extra shares at attractive prices. The extra supply this generates could potentially push prices back to fundamental value.

Unfortunately, this game entails risks and costs for managers, just as it does for hedge funds. Issuing shares is an expensive process, both in terms of underwriting fees and time spent by company management. Moreover, the manager can rarely be *sure* that investors are overvaluing his firm's shares. If he issues shares, thinking that they are overvalued when in fact they are not, he incurs the costs of deviating from his target capital structure, without getting any benefits in return.

2.3 Evidence

From the theoretical point of view, there is reason to believe that arbitrage is a risky process and therefore that it is only of limited effectiveness. But is there any *evidence* that arbitrage is limited? In principle, any example of persistent mispricing is immediate evidence of limited arbitrage: if arbitrage were not limited, the mispricing would quickly disappear. The problem is that while many pricing phenomena can be interpreted as deviations from fundamental value, it is only in a few cases that the presence of a mispricing can be established beyond any reasonable doubt. The reason for this is what Fama (1970) dubbed the "joint hypothesis problem." In order to claim that the price of a security differs from its properly discounted future cash flows, one needs a model of "proper" discounting. Any test of mispricing is therefore inevitably a *joint* test of mispricing and of a model of discount rates, making it difficult to provide definitive evidence of inefficiency.

In spite of this difficulty, researchers have uncovered a number of financial market phenomena that are almost certainly mispricings, and persistent ones at that. These examples show that arbitrage is indeed limited, and also serve as interesting illustrations of the risks and costs described earlier.

Twin Shares

In 1907, Royal Dutch and Shell Transport, at the time completely independent companies, agreed to merge their interests on a 60:40 basis while remaining separate entities. Shares of Royal Dutch, which are primarily traded in the U.S. and in the Netherlands, are a claim to 60 percent of the total cash flow of the two companies, while Shell, which trades primarily in the U.K., is a claim to the remaining 40 percent. If prices equal fundamental value, the market value of Royal Dutch equity should always be 1.5 times the market value of Shell equity. Remarkably, it isn't.

Figure 1, taken from Froot and Dabora's (1999) analysis of this case, shows the ratio of Royal Dutch equity value to Shell equity value relative to the efficient markets benchmark of 1.5. The picture provides strong evidence of a persistent inefficiency. Moreover, the deviations are not small. Royal Dutch is sometimes 35 percent underpriced relative to parity, and sometimes 15 percent overpriced.

This evidence of mispricing is simultaneously evidence of limited arbitrage, and it is not hard to see why arbitrage might be limited in this case. If an arbitrageur wanted to exploit this phenomenon – and several hedge funds, Long Term Capital Management included, did try to – he would buy the relatively undervalued share and short the other. Table 1 summarizes the risks facing the arbitrageur. Since one share is a good substitute for the other, fundamental risk is nicely hedged: news about fundamentals should affect the two shares equally, leaving the arbitrageur immune. Nor are there any major implementation costs to speak of: shorting shares of either company is an easy matter.

The one risk that remains is noise trader risk. Whatever investor sentiment is causing one share to be undervalued relative to the other could also cause that share to become *even more* undervalued in the short term. The graph shows that this danger is very real: an arbitrageur buying a 10 percent undervalued Royal Dutch share in March 1983 would have seen it drop still further in value over the next six months. As discussed earlier, when a mispriced security has a perfect substitute, arbitrage can still be limited if (i) arbitrageurs are risk averse and have short horizons and (ii) the noise trader risk is systematic, or the arbitrage requires specialized skills, or there are costs to learning about such opportunities. It is very plausible that both (i) and (ii) are true, thereby explaining why the mispricing persisted for so long. It took until 2001 for the shares to finally sell at par.

This example also provides a nice illustration of the distinction between "prices are right" and "no free lunch" discussed in Section 2.1. While prices in this case are clearly *not* right, there are no easy profits for the taking.

Index Inclusions

Every so often, one of the companies in the S&P 500 is taken out of the index because of a merger or bankruptcy, and is replaced by another firm. Two early studies of such index inclusions, Harris and Gurel (1986) and Shleifer (1986), document a remarkable fact: when a stock is added to the index, it jumps in price by an average of 3.5 percent, and much of this jump is permanent. In one dramatic illustration of this phenomenon, when Yahoo was added to the index, its shares jumped by 24 percent in a single day.

The fact that a stock jumps in value upon inclusion is once again clear evidence of mispricing: the price of the share changes even though its fundamental value does not. Standard and Poor's emphasizes that in selecting stocks for inclusion, they are simply trying to make their index representative of the U.S. economy, not to convey any information about the level or riskiness of a firm's future cash flows.[6]

This example of a deviation from fundamental value is also evidence of limited arbitrage. When one thinks about the risks involved in trying to exploit the anomaly, its persistence becomes less surprising. An arbitrageur needs to short the included security and to buy as good a substitute security as he can. This entails considerable fundamental risk because individual stocks rarely have good substitutes. It also carries substantial noise trader risk: whatever caused the initial jump in price – in all likelihood, buying by S&P 500 index funds – may continue, and cause the price to rise still further in the short run; indeed, Yahoo went from \$115 prior to its S&P inclusion announcement to \$210 a month later.

Wurgler and Zhuravskaya (2002) provide additional support for the limited arbitrage view of S&P 500 inclusions. They hypothesize that the jump upon inclusion should be particularly large for those stocks with the worst substitute securities, in other words, for those stocks for which the arbitrage is riskiest. By constructing the best possible substitute portfolio for each included stock, they are able to test this, and find strong support. Their analysis also shows just how hard it is to find good substitute securities for individual stocks. For most regressions of included stock returns on the returns of the best substitute securities, the R^2 is below 25 percent.

Internet Carve-Outs

In March 2000, 3Com sold 5 percent of its wholly owned subsidiary Palm Inc. in an initial public offering, retaining ownership of the remaining 95 percent. After the IPO, a

[6] After the initial studies on index inclusions appeared, some researchers argued that the price increase might be rationally explained through information or liquidity effects. While such explanations cannot be completely ruled out, the case for mispricing was considerably strengthened by Kaul, Mehrotra and Morck (2000). They consider the case of the TS300 index of Canadian equities, which in 1996 changed the weights of some of its component stocks to meet an innocuous regulatory requirement. The reweighting was accompanied by significant price effects. Since the affected stocks were *already* in the index at the time of the event, information and liquidity explanations for the price jumps are extremely implausible.

shareholder of 3Com indirectly owned 1.5 shares of Palm. 3Com also announced its intention to spin off the remainder of Palm within 9 months, at which time they would give each 3Com shareholder 1.5 shares of Palm.

At the close of trading on the first day after the IPO, Palm shares stood at \$95, putting a lower bound on the value of 3Com at \$142. In fact, 3Com's price was \$81, implying a market valuation of 3Com's substantial businesses outside of Palm of about -\$60 per share!

This situation surely represents a severe mispricing, and it persisted for several weeks. To exploit it, an arbitrageur could buy one share of 3Com, short 1.5 shares of Palm, and wait for the spin-off, thus earning certain profits at no cost. This strategy entails no fundamental risk and no noise trader risk. Why, then, is arbitrage limited? Lamont and Thaler (2002), who analyze this case in detail, argue that implementation costs played a major role. Many investors who tried to borrow Palm shares to short were either told by their broker that no shares were available, or else were quoted a very high borrowing price. This barrier to shorting was not a legal one, but one that arose endogenously in the marketplace: such was the demand for shorting Palm, that the supply of Palm shorts was unable to meet it. Arbitrage was therefore limited, and the mispricing persisted.[7]

Some financial economists react to these examples by arguing that they are simply isolated instances with little broad relevance.[8] We think this is an overly complacent view. The "twin shares" example illustrates that in situations where arbitrageurs face only one type of risk – noise trader risk – securities can become mispriced by almost 35 percent. This suggests that if a typical stock trading on the NYSE or NASDAQ becomes subject to investor sentiment, the mispricing could be an order of magnitude larger. Not only would arbitrageurs face noise trader risk in trying to correct the mispricing, but fundamental risk as well, not to mention implementation costs.

3 Psychology

The theory of limited arbitrage shows that if irrational traders cause deviations from fundamental value, rational traders will often be powerless to do anything about it. In order to say more about the structure of these deviations, behavioral models often assume a specific form of irrationality. For guidance on this, economists turn to the extensive experimental evidence compiled by cognitive psychologists on the systematic biases that arise when people

[7]See also Mitchell, Pulvino and Stafford (2002) and Ofek and Richardson (2001) for further discussion of such "negative stub" situations, in which the market value of a company is less than the sum of its publicly traded parts.

[8]During a discussion of these issues at a University of Chicago seminar, one economist argued that these examples are "the tip of the iceberg," to which another retorted that "they *are* the iceberg."

form *beliefs*, and on people's *preferences*.[9]

In this section, we summarize the psychology that may be of particular interest to financial economists. Our discussion of each finding is necessarily brief. For a deeper understanding of the phenomena we touch on, we refer the reader to the surveys of Camerer (1995) and Rabin (1998) and to the edited volumes of Kahneman, Slovic and Tversky (1982), Kahneman and Tversky (2000) and Gilovich, Griffin and Kahneman (2002).

3.1 Beliefs

A crucial component of any model of financial markets is a specification of how agents form expectations. We now summarize what psychologists have learned about how people appear to form beliefs in practice.

Overconfidence. Extensive evidence shows that people are overconfident in their judgments. This appears in two guises. First, the confidence intervals people assign to their estimates of quantities – the level of the Dow in a year, say – are far too narrow. Their 98 percent confidence intervals, for example, include the true quantity only about 60 percent of the time (Alpert and Raiffa, 1982). Second, people are poorly calibrated when estimating probabilities: events they think are certain to occur actually occur only around 80 percent of the time, and events they deem impossible occur approximately 20 percent of the time (Fischhoff, Slovic and Lichtenstein, 1977).[10]

Optimism and Wishful Thinking. Most people display unrealistically rosy views of their abilities and prospects (Weinstein, 1980). Typically, over 90 percent of those surveyed think they are above average in such domains as driving skill, ability to get along with people and sense of humor. They also display a systematic planning fallacy: they predict that tasks (such as writing survey papers) will be completed much sooner than they actually are (Buehler, Griffin and Ross, 1994).

Representativeness. Kahneman and Tversky (1974) show that when people try to deter-

[9]We emphasize, however, that behavioral models do not *need* to make extensive psychological assumptions in order to generate testable predictions. In Section 6, we discuss Lee, Shleifer and Thaler's (1991) theory of closed-end fund pricing. That theory makes numerous crisp predictions using only the assumptions that there are noise traders with correlated sentiment in the economy, and that arbitrage is limited.

[10]Overconfidence may in part stem from two other biases, self-attribution bias and hindsight bias. Self-attribution bias refers to people's tendency to ascribe any success they have in some activity to their own talents, while blaming failure on bad luck, rather than on their ineptitude. Doing this repeatedly will lead people to the pleasing but erroneous conclusion that they are very talented. For example, investors might become overconfident after several quarters of investing success (Gervais and Odean, 2001). Hindsight bias is the tendency of people to believe, after an event has occurred, that they predicted it before it happened. If people think they predicted the past better than they actually did, they may also believe that they can predict the future better than they actually can.

mine the probability that a data set A was generated by a model B, or that an object A belongs to a class B, they often use the representativeness heuristic. This means that they evaluate the probability by the degree to which A reflects the essential characteristics of B.

Much of the time, representativeness is a helpful heuristic, but it can generate some severe biases. The first is *base rate neglect.* To illustrate, Kahneman and Tversky present this description of a person named Linda:

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

When asked which of "Linda is a bank teller" (statement A) and "Linda is a bank teller and is active in the feminist movement" (statement B) is more likely, subjects typically assign greater probability to B. This is, of course, impossible. Representativeness provides a simple explanation. The description of Linda *sounds* like the description of a feminist – it is representative of a feminist – leading subjects to pick B. Put differently, while Bayes law says that

$$p(\text{statement B}|\text{description}) = \frac{p(\text{description}|\text{statement B})p(\text{statement B})}{p(\text{description})},$$

people apply the law incorrectly, putting too much weight on $p(\text{description}|\text{statement B})$, which captures representativeness, and too little weight on the base rate, $p(\text{statement B})$.

Representativeness also leads to another bias, *sample size neglect.* When judging the likelihood that a data set was generated by a particular model, people often fail to take the size of the sample into account: after all, a small sample can be just as representative as a large one. Six tosses of a coin resulting in three heads and three tails are as representative of a fair coin as 500 heads and 500 tails are in a total of 1000 tosses. Representativeness implies that people will find the two sets of tosses equally informative about the fairness of the coin, even though the second set is much more so.

Sample size neglect means that in cases where people do not initially know the data-generating process, they will tend to infer it too quickly on the basis of too few data points. For instance, they will come to believe that a financial analyst with four good stock picks is talented because four successes are not representative of a bad or mediocre analyst. It also generates a "hot hand" phenomenon, whereby sports fans become convinced that a basketball player who has made three shots in a row is on a hot streak and will score again, even though there is no evidence of a hot hand in the data (Gilovich, Vallone and Tversky, 1985). This belief that even small samples will reflect the properties of the parent population is sometimes known as the "law of small numbers" (Rabin, 2002).

In situations where people *do* know the data-generating process in advance, the law of small numbers generates a gambler's fallacy effect. If a fair coin generates five heads in a row, people will say that "tails are due". Since they believe that even a short sample should be representative of the fair coin, there have to be more tails to balance out the large number of heads.

Conservatism. While representativeness leads to an underweighting of base rates, there are situations where base rates are *over*-emphasized relative to sample evidence. In an experiment run by Edwards (1968), there are two urns, one containing 3 blue balls and 7 red ones, and the other containing 7 blue balls and 3 red ones. A random draw of 12 balls, with replacement, from one of the urns yields 8 reds and 4 blues. What is the probability the draw was made from the first urn? While the correct answer is 0.97, most people estimate a number around 0.7, apparently overweighting the base rate of 0.5.

At first sight, the evidence of conservatism appears at odds with representativeness. However, there may be a natural way in which they fit together. It appears that if a data sample is representative of an underlying model, then people overweight the data. However, if the data is not representative of any salient model, people react too little to the data and rely too much on their priors. In Edwards' experiment, the draw of 8 red and 4 blue balls is not particularly representative of either urn, possibly leading to an overreliance on prior information.

Belief Perseverance. There is much evidence that once people have formed an opinion, they cling to it too tightly and for too long (Lord, Ross and Lepper, 1979). At least two effects appear to be at work. First, people are reluctant to search for evidence that contradicts their beliefs. Second, even if they find such evidence, they treat it with excessive skepticism. Some studies have found an even stronger effect, known as confirmation bias, whereby people misinterpret evidence that goes against their hypothesis as actually being in their favor. In the context of academic finance, belief perseverance predicts that if people start out believing in the Efficient Markets Hypothesis, they may continue to believe in it long after compelling evidence to the contrary has emerged.

Anchoring. Kahneman and Tversky (1974) argue that when forming estimates, people often start with some initial, possibly arbitrary value, and then adjust away from it. Experimental evidence shows that the adjustment is often insufficient. Put differently, people "anchor" too much on the initial value.

In one experiment, subjects were asked to estimate the percentage of United Nations' countries that are African. More specifically, before giving a percentage, they were asked whether their guess was higher or lower than a randomly generated number between 0 and 100. Their subsequent estimates were significantly affected by the initial random number. Those who were asked to compare their estimate to 10, subsequently estimated 25 percent, while those who compared to 60, estimated 45 percent.

Availability Biases. When judging the probability of an event – the likelihood of getting mugged in Chicago, say – people often search their memories for relevant information. While this is a perfectly sensible procedure, it can produce biased estimates because not all memories are equally retrievable or "available", in the language of Kahneman and Tversky (1974). More recent events and more salient events – the mugging of a close friend, say – will weigh more heavily and distort the estimate.

Economists are sometimes wary of this body of experimental evidence because they believe (i) that people, through repetition, will learn their way out of biases; (ii) that experts in a field, such as traders in an investment bank, will make fewer errors; and (iii) that with more powerful incentives, the effects will disappear.

While all these factors can attenuate biases to some extent, there is little evidence that they wipe them out altogether. The effect of learning is often muted by errors of application: when the bias is explained, people often understand it, but then immediately proceed to violate it again in specific applications. Expertise, too, is often a hindrance rather than a help: experts, armed with their sophisticated models, have been found to exhibit *more* overconfidence than laymen, particularly when they receive only limited feedback about their predictions. Finally, in a review of dozens of studies on the topic, Camerer and Hogarth (1999) conclude that while incentives can sometimes reduce the biases people display, "no replicated study has made rationality violations disappear purely by raising incentives" (p.7).

3.2 Preferences

Prospect Theory

An essential ingredient of any model trying to understand asset prices or trading behavior is an assumption about investor preferences, or about how investors evaluate risky gambles. The vast majority of models assume that investors evaluate gambles according to the expected utility framework, EU henceforth. The theoretical motivation for this goes back to Von Neumann and Morgenstern (1947), VNM henceforth, who show that if preferences satisfy a number of plausible axioms – completeness, transitivity, continuity, and independence – then they can be represented by the expectation of a utility function.

Unfortunately, experimental work in the decades after VNM has shown that people systematically violate EU theory when choosing among risky gambles. In response to this, there has been an explosion of work on so-called non-EU theories, all of them trying to do a better job of matching the experimental evidence. Some of the better known models include weighted-utility theory (Chew and MacCrimmon 1979, Chew 1983), implicit EU (Chew 1989, Dekel 1986), disappointment aversion (Gul 1991), regret theory (Bell, 1982, Loomes and Sugden, 1982), rank-dependent utility theories (Quiggin 1982, Segal 1987, 1989, Yaari 1987), and prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman, 1992).

Should financial economists be interested in any of these alternatives to expected utility? It may be that EU theory is a good approximation to how people evaluate a risky gamble like the stock market, even if it does not explain attitudes to the kinds of gambles studied in experimental settings. On the other hand, the difficulty the EU approach has encountered in trying to explain basic facts about the stock market suggests that it may be worth taking a closer look at the experimental evidence. Indeed, recent work in behavioral finance has argued that some of the lessons we learn from violations of EU are central to understanding a number of financial phenomena.

Of all the non-EU theories, prospect theory may be the most promising for financial applications, and we discuss it in detail. The reason we focus on this theory is, quite simply, that it is the most successful at capturing the experimental results. In a way, this is not surprising. Most of the other non-EU models are what might be called quasi-normative, in that they try to capture some of the anomalous experimental evidence by slightly weakening the VNM axioms. The difficulty with such models is that in trying to achieve two goals – normative and descriptive – they end up doing an unsatisfactory job at both. In contrast, prospect theory has no aspirations as a normative theory: it simply tries to capture people's attitudes to risky gambles as parsimoniously as possible. Indeed, Tversky and Kahneman (1986) argue convincingly that normative approaches are doomed to failure, because people routinely make choices that are simply impossible to justify on normative grounds, in that they violate dominance or invariance.

Kahneman and Tversky (1979), KT henceforth, lay out the original version of prospect theory, designed for gambles with at most two non-zero outcomes. They propose that when offered a gamble

$$(x, p; y, q),$$

to be read as "get outcome x with probability p, outcome y with probability q", where $x \leq 0 \leq y$ or $y \leq 0 \leq x$, people assign it a value of

$$\pi(p)v(x) + \pi(q)v(y), \tag{1}$$

where v and π are shown in Figure 2. When choosing between different gambles, they pick the one with the highest value.

This formulation has a number of important features. First, utility is defined over gains and losses rather than over final wealth positions, an idea first proposed by Markowitz (1952). This fits naturally with the way gambles are often presented and discussed in everyday life. More generally, it is consistent with the way people perceive attributes such as brightness, loudness, or temperature relative to earlier levels, rather than in absolute terms. Kahneman and Tversky (1979) also offer the following violation of EU as evidence that people focus on gains and losses. Subjects are asked:[11]

[11] All the experiments in Kahneman and Tversky (1979) are conducted in terms of Israeli currency. The

In addition to whatever you own, you have been given 1000. Now choose between

$$A = (1000, 0.5)$$
$$B = (500, 1).$$

B was the more popular choice. The same subjects were then asked:

In addition to whatever you own, you have been given 2000. Now choose between

$$C = (-1000, 0.5)$$
$$D = (-500, 1).$$

This time, C was more popular.

Note that the two problems are identical in terms of their final wealth positions and yet people choose differently. The subjects are apparently focusing only on gains and losses. Indeed, when they are not given any information about prior winnings, they choose B over A and C over D.

The second important feature is the shape of the value function v, namely its concavity in the domain of gains and convexity in the domain of losses. Put simply, people are risk averse over gains, and risk-seeking over losses. Simple evidence for this comes from the fact just mentioned, namely that in the absence of any information about prior winnings[12]

$$B \succ A,\ C \succ D.$$

The v function also has a kink at the origin, indicating a greater sensitivity to losses than to gains, a feature known as *loss aversion*. Loss aversion is introduced to capture aversion to bets of the form:

$$E = (110, \frac{1}{2}; -100, \frac{1}{2}).$$

It may seem surprising that we need to depart from the expected utility framework in order to understand attitudes to gambles as simple as E, but it is nonetheless true. In a remarkable paper, Rabin (2000) shows that if an expected utility maximizer rejects gamble E at all wealth levels, then he will also reject

$$(20000000, \frac{1}{2}; -1000, \frac{1}{2}),$$

an utterly implausible prediction. The intuition is simple: if a smooth, increasing, and concave utility function defined over final wealth has sufficient local curvature to reject E

authors note that at the time of their research, the median monthly family income was about 3000 Israeli lira.

[12]In this section $G_1 \succ G_2$ should be read as "a statistically significant fraction of Kahneman and Tversky's subjects preferred G_1 to G_2."

over a wide range of wealth levels, it must be an extraordinarily concave function, making the investor extremely risk averse over large stakes gambles.

The final piece of prospect theory is the nonlinear probability transformation. Small probabilities are overweighted, so that $\pi(p) > p$. This is deduced from KT's finding that

$$(5000, 0.001) \succ (5, 1)$$

and

$$(-5, 1) \succ (-5000, 0.001),$$

together with the earlier assumption that v is concave (convex) in the domain of gains (losses). Moreover, people are more sensitive to differences in probabilities at higher probability levels. For example, the following pair of choices,

$$(3000, 1) \succ (4000, 0.8; 0, 0.2)$$

and

$$(4000, 0.2; 0, 0.8) \succ (3000, 0.25),$$

which violate EU theory, imply

$$\frac{\pi(0.25)}{\pi(0.2)} < \frac{\pi(1)}{\pi(0.8)}.$$

The intuition is that the 20 percent jump in probability from 0.8 to 1 is more striking to people than the 20 percent jump from 0.2 to 0.25. In particular, people place much more weight on outcomes that are certain relative to outcomes that are merely probable, a feature sometimes known as the "certainty effect".

Along with capturing experimental evidence, prospect theory also simultaneously explains preferences for insurance and for buying lottery tickets. Although the concavity of v in the region of gains generally produces risk aversion, for lotteries which offer a small chance of a large gain, the overweighting of small probabilities in Figure 2 dominates, leading to risk-seeking. Along the same lines, while the convexity of v in the region of losses typically leads to risk-seeking, the same overweighting of small probabilities introduces risk aversion over gambles which have a small chance of a large loss.

Based on additional evidence, Tversky and Kahneman (1992) propose a generalization of prospect theory which can be applied to gambles with more than two outcomes. Specifically, if a gamble promises outcome x_i with probability p_i, Tversky and Kahneman (1992) propose that people assign the gamble the value

$$\sum_i \pi_i v(x_i) \tag{2}$$

where

$$v = \begin{array}{ll} x^\alpha & \text{if } x \geq 0 \\ -\lambda(-x)^\alpha & \text{if } x < 0 \end{array}$$

and

$$\begin{aligned} \pi_i &= w(P_i) - w(P_i^*) \\ w(P) &= \frac{P^\gamma}{(P^\gamma + (1-P)^\gamma)^{1/\gamma}}. \end{aligned}$$

Here, P_i (P_i^*) is the probability that the gamble will yield an outcome at least as good as (strictly better than) x_i. Tversky and Kahneman (1992) use experimental evidence to estimate $\alpha = 0.88$, $\lambda = 2.25$, and $\gamma = 0.65$. Note that λ is the coefficient of loss aversion, a measure of the relative sensitivity to gains and losses. Over a wide range of experimental contexts λ has been estimated in the neighborhood of 2.

Earlier in this section, we saw how prospect theory could explain why people made different choices in situations with identical final wealth levels. This illustrates an important feature of the theory, namely that it can accommodate the effects of problem description, or of *framing*. Such effects are powerful. There are numerous demonstrations of a 30 to 40 percent shift in preferences depending on the wording of a problem. No normative theory of choice can accommodate such behavior since a first principle of rational choice is that choices should be independent of the problem description or representation.

Framing refers to the way a problem is posed for the decision maker. In many actual choice contexts the decision maker also has flexibility in how to think about the problem. For example, suppose that a gambler goes to the race track and wins \$200 in her first bet, but then loses \$50 on her second bet. Does she code the outcome of the second bet as a loss of \$50 or as a reduction in her recently won gain of \$200? In other words, is the utility of the second loss $v(-50)$ or $v(150) - v(200)$? The process by which people formulate such problems for themselves is called *mental accounting* (Thaler, 1999). Mental accounting matters because in prospect theory, v is nonlinear.

One important feature of mental accounting is *narrow framing*, which is the tendency to treat individual gambles separately from other portions of wealth. In other words, when offered a gamble, people often evaluate it as if it is the only gamble they face in the world, rather than merging it with pre-existing bets to see if the new bet is a worthwhile addition.

Redelmeier and Tversky (1992) provide a simple illustration, based on the gamble

$$F = (2000, \frac{1}{2}; -500, \frac{1}{2}).$$

Subjects in their experiment were asked whether they were willing to take this bet; 57 percent said they would not. They were then asked whether they would prefer to play F five times or six times; 70 percent preferred the six-fold gamble. Finally they were asked:

Suppose that you have played F five times but you don't yet know your wins and losses. Would you play the gamble a sixth time?

60 percent rejected the opportunity to play a sixth time, reversing their preference from the earlier question. This suggests that some subjects are framing the sixth gamble narrowly, segregating it from the other gambles. Indeed, the 60 percent rejection level is very similar to the 57 percent rejection level for the one-off play of F.

Ambiguity Aversion

Our discussion so far has centered on understanding how people act when the outcomes of gambles have known, objective probabilities. In reality, probabilities are rarely objectively known. To handle these situations, Savage (1964) develops a counterpart to expected utility known as subjective expected utility, SEU henceforth. Under certain axioms, preferences can be represented by the expectation of a utility function, this time weighted by the individual's subjective probability assessment.

Experimental work in the last few decades has been as unkind to SEU as it was to EU. The violations this time are of a different nature, but they may be just as relevant for financial economists.

The classic experiment was described by Ellsberg (1961). Suppose that there are two urns, 1 and 2. Urn 2 contains a total of 100 balls, 50 red and 50 blue. Urn 1 also contains 100 balls, again a mix of red and blue, but the subject does not know the proportion of each.

Subjects are asked to choose one of the following two gambles, each of which involves a possible payment of \$100, depending on the color of a ball drawn at random from the relevant urn

$$
\begin{aligned}
a_1 &: \text{ a ball is drawn from Urn 1, \$100 if red, \$0 if blue} \\
a_2 &: \text{ a ball is drawn from Urn 2, \$100 if red, \$0 if blue.}
\end{aligned}
$$

Subjects are then also asked to choose between the following two gambles:

$$
\begin{aligned}
b_1 &: \text{ a ball is drawn from Urn 1, \$100 if blue, \$0 if red} \\
b_2 &: \text{ a ball is drawn from Urn 2, \$100 if blue, \$0 if red.}
\end{aligned}
$$

a_2 is typically preferred to a_1, while b_2 is chosen over b_1. These choices are inconsistent with SEU: the choice of a_2 implies a subjective probability that *fewer* than 50 percent of the balls in Urn 1 are red, while the choice of b_2 implies the opposite.

The experiment suggests that people do not like situations where they are uncertain about the probability distribution of a gamble. Such situations are known as situations of ambiguity, and the general dislike for them, as ambiguity aversion.[13] SEU does not allow

[13]An early discussion of this aversion can be found in Knight (1921), who defines risk as a gamble with

agents to express their degree of confidence about a probability distribution and therefore cannot capture such aversion.

Ambiguity aversion appears in a wide variety of contexts. For example, a researcher might ask a subject for his estimate of the probability that a certain team will win its upcoming football match, to which the subject might respond 0.4. The researcher then asks the subject to imagine a chance machine, which will display 1 with probability 0.4 and 0 otherwise, and asks whether the subject would prefer to bet on the football game – an ambiguous bet – or on the machine, which offers no ambiguity. In general, people prefer to bet on the machine, illustrating aversion to ambiguity.

Heath and Tversky (1991) argue that in the real world, ambiguity aversion has much to do with how competent an individual feels he is at assessing the relevant distribution. Ambiguity aversion over a bet can be strengthened by highlighting subjects' feelings of incompetence, either by showing them other bets in which they have more expertise, or by mentioning other people who are more qualified to evaluate the bet (Fox and Tversky, 1995).

Further evidence that supports the competence hypothesis is that in situations where people feel especially competent in evaluating a gamble, the opposite of ambiguity aversion, namely a "preference for the familiar," has been observed. In the example above, people chosen to be especially knowledgeable about football often prefer to bet on the outcome of the game than on the chance machine. Just as with ambiguity aversion, such behavior cannot be captured by SEU.

4 Application: The Aggregate Stock Market

Researchers studying the aggregate U.S. stock market have identified a number of interesting facts about its behavior. Three of the most striking are:

(i) The *Equity Premium*. The stock market has historically earned a high excess rate of return. For example, using annual data from 1871-1993, Campbell and Cochrane (1999) report that the average log return on the S&P 500 index is 3.9 percent higher than the average log return on short term commercial paper.

(ii) *Volatility*. Stock returns and price-dividend ratios are both highly variable. In the same data set, the annual standard deviation of excess log returns on the S&P 500 is 18 percent, while the annual standard deviation of the log price-dividend ratio is 0.27.

(iii) *Predictability*. Stock returns are forecastable. Using monthly, real, equal-weighted NYSE returns from 1941-1986, Fama and French (1988) show that the dividend-price ratio is able

known distribution and uncertainty as a gamble with unknown distribution, and suggests that people dislike uncertainty more than risk.

to explain 27 percent of the variation of cumulative stock returns over the subsequent four years.[14]

All three of these facts can be labelled puzzles. Fact (i) has been known as the equity premium puzzle since the work of Mehra and Prescott (1985) (see also Hansen and Singleton, 1983). Campbell (1999) calls (ii) the volatility puzzle and we refer to (iii) as the predictability puzzle. The reason they are called puzzles is that they are hard to rationalize in a simple consumption-based model.

To see this, consider the following endowment economy, which we come back to a number of times in this section. There are an infinite number of identical investors, and two assets: a risk-free asset in zero net supply, with gross return $R_{f,t}$ between time t and $t+1$, and a risky asset – the stock market – in fixed positive supply, with gross return R_{t+1} between time t and $t+1$. The stock market is a claim to a perishable stream of dividends $\{D_t\}$, where

$$\frac{D_{t+1}}{D_t} = e^{g_D + \sigma_D \varepsilon_{t+1}}, \tag{3}$$

and where each period's dividend can be thought of as one component of a consumption endowment C_t, where

$$\frac{C_{t+1}}{C_t} = e^{g_C + \sigma_C \eta_{t+1}} \tag{4}$$

and

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix}\right), \text{ i.i.d. over time.} \tag{5}$$

Investors choose consumption C_t and an allocation S_t to the risky asset to maximize

$$E_0 \sum_{t=0}^{\infty} \rho^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{6}$$

subject to the standard budget constraint.[15] Using the Euler equation of optimality,

$$1 = \rho E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right], \tag{7}$$

it is straightforward to derive expressions for stock returns and prices. The details are in the Appendix.

We can now examine the model's quantitative predictions for the parameter values in Table 2. The endowment process parameters are taken from U.S. data spanning the 20th century, and are standard in the literature. It is also standard to start out by considering *low*

[14] These three facts are widely agreed on, but they are not completely uncontroversial. A large literature has debated the statistical significance of the time series predictability, while others have argued that the equity premium is overstated due to survivorship bias (Brown, Goetzmann and Ross, 1995).

[15] For $\gamma = 1$, we replace $C_t^{1-\gamma}/1-\gamma$ with $\log(C_t)$.

values of γ. The reason is that when one computes, for various values of γ, how much wealth an individual would be prepared to give up to avoid a large-scale timeless wealth gamble, low values of γ match best with introspection as to what the answers should be (Mankiw and Zeldes, 1991). We take $\gamma = 1$, which corresponds to log utility.

In an economy with these parameter values, the average log return on the stock market would be just 0.1 percent higher than the risk-free rate, not the 3.9 percent observed historically. The standard deviation of log stock returns would be only 12 percent, not 18 percent, and the price-dividend ratio would be constant (implying, of course, that the dividend-price ratio has no forecast power for future returns).

It is useful to recall the intuition for these results. In an economy with power utility preferences, the equity premium is determined by risk aversion γ and by risk, measured as the covariance of stock returns and consumption growth. Since consumption growth is very smooth in the data, this covariance is very low, thus predicting a very low equity premium. Stocks simply do not appear risky to investors with the preferences in (6) and with low γ, and therefore do not warrant a large premium. Of course, the equity premium predicted by the model can be increased by using higher values of γ. However, other than making counterintuitive predictions about individuals' attitudes to large-scale gambles, this would also predict a counterfactually high risk-free rate, a problem known as the risk-free rate puzzle (Weil, 1989).

To understand the volatility puzzle, note that in the simple economy described above, both discount rates and expected dividend growth are constant over time. A direct application of the present value formula implies that the price-dividend ratio, P/D henceforth, is constant. Since

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = \frac{1 + P_{t+1}/D_{t+1}}{P_t/D_t} \frac{D_{t+1}}{D_t}, \tag{8}$$

it follows that

$$r_{t+1} = \Delta d_{t+1} + \text{const.} \equiv d_{t+1} - d_t + \text{const.}, \tag{9}$$

where lower case letters indicate log variables. The standard deviation of log returns will therefore only be as high as the standard deviation of log dividend growth, namely 12 percent.

The particular volatility puzzle seen here illustrates a more general point, first made by Shiller (1981) and LeRoy and Porter (1981), namely that it is difficult to explain the historical volatility of stock returns with *any* model in which investors are rational and discount rates are constant.

To see the intuition, consider the identity in equation (8) again. Since the volatility of log dividend growth is only 12 percent, the only way for a model to generate an 18 percent volatility of log returns is to introduce variation in the P/D ratio. But if discount rates are constant, a quick glance at a present-value formula shows that the only way to do that is to introduce variation in investors' forecasts of the dividend growth rate: a higher forecast raises the P/D ratio, a lower forecast brings it down. There is a catch here, though: if investors are rational, their expectations for dividend growth must, on average, be confirmed. In other words, times of higher (lower) P/D ratios should, on average, be followed by higher (lower) cash-flow growth. Unfortunately, price-dividend ratios are *not* reliable forecasters of dividend growth, neither in the U.S. nor in most international markets (see Campbell, 1999, for recent evidence).

Shiller and LeRoy and Porter's results shocked the profession when they first appeared. At the time, most economists felt that discount rates *were* close to constant over time, apparently implying that stock market volatility could only be fully explained by appealing to investor irrationality. Today, it is well understood that rational variation in discount rates can help explain the volatility puzzle, although we argue later that models with irrational beliefs also offer a plausible way of thinking about the data.

Both the rational and behavioral approaches to finance have made progress in understanding the three puzzles singled out at the start of this section. The advances on the rational side are well described in other articles in this handbook. Here, we discuss the behavioral approaches, starting with the equity premium puzzle and then turning to the volatility puzzle.

We do not consider the predictability puzzle separately, because in any model with a stationary P/D ratio, a resolution of the volatility puzzle is simultaneously a resolution of the predictability puzzle. To see this, recall from equation (8) that any model which captures the empirical volatility of returns must involve variation in the P/D ratio. Moreover, for a model to be a *satisfactory* resolution of the volatility puzzle, it should not make the counterfactual prediction that P/D ratios forecast subsequent dividend growth. Now suppose that the P/D ratio is higher than average. The only way it can return to its mean is if cash flows D subsequently go up, or if prices P fall. Since the P/D ratio is not allowed to forecast cash flows, it must forecast lower returns, thereby explaining the predictability puzzle.

4.1 The Equity Premium Puzzle

The core of the equity premium puzzle is that even though stocks appear to be an attractive asset – they have high average returns and a low covariance with consumption growth – investors appear very unwilling to hold them. In particular, they appear to demand a substantial risk premium in order to hold the market supply.

To date, behavioral finance has pursued two approaches to this puzzle. Both are based on preferences: one relies on prospect theory, the other on ambiguity aversion. In essence, both approaches try to understand what it is that is missing from the popular preference specification in (6) that makes investors fear stocks so much, leading them to charge a high premium in equilibrium.

Prospect Theory

One of the earliest papers to link prospect theory to the equity premium is Benartzi and Thaler (1995), BT henceforth. They study how an investor with prospect theory-type preferences allocates his financial wealth between T-Bills and the stock market. Prospect theory argues that when choosing between gambles, people compute the gains and losses for each one and select the one with the highest prospective utility. In a financial context, this suggests that people may choose a portfolio allocation by computing, for each allocation, the potential gains and losses in the value of their holdings, and then taking the allocation with the highest prospective utility. In other words, they choose ω, the fraction of financial wealth in stocks, to maximize

$$E_{\pi}\ v[(1-\omega)R_{f,t+1}+\omega R_{t+1}-1], \tag{10}$$

where π and v are defined in (2). In particular, v captures loss aversion, the experimental finding that people are more sensitive to losses than to gains. $R_{f,t+1}$ and R_{t+1} are the gross returns on T-Bills and the stock market between t and $t+1$, respectively, making the argument of v the return on financial wealth. The distributions of $R_{f,t+1}$ and R_{t+1} are obtained by bootstrapping historical U.S. data.[16]

In order to implement this model, BT need to stipulate how often investors evaluate their portfolios. In other words, how long is the time interval between t and $t+1$? To see why this matters, compare two investors: energetic Nick who calculates the gains and losses in his portfolio every day, and laid-back Dick who looks at his portfolio only once per decade. Since, on a daily basis, stocks go down in value almost as often as they go up, the loss aversion built into v makes stocks appear unattractive to Nick. In contrast, loss aversion does not have much effect on Dick's perception of stocks since, at ten year horizons, stocks offer only a small risk of losing money. Rather than simply pick an evaluation interval, BT calculate how often investors would have to evaluate their portfolios to make them roughly indifferent between stocks and bonds. In other words, they compute how often investors would need to evaluate their gains and losses so that even in the face of the large historical equity premium, they would still be happy to hold the market supply of bonds and stocks.

When they solve (10) using the parametric forms for π and v estimated in experimental settings, BT find the answer to be a year, and argue that this is indeed a natural evaluation period for investors to use. The way people frame gains and losses is plausibly influenced by the way information is presented to them. Since we receive our most comprehensive mutual fund reports once a year, and do our taxes once a year, it is not unreasonable that gains and losses might be expressed as annual changes in value.

[16] In (2), π and v are defined over discrete, not continuous distributions. Benartzi and Thaler (1995) therefore summarize the historical distributions of T-Bills and stocks as discrete histograms before applying (2).

This, in turn, suggests a simple way of understanding the high historical equity premium. If investors get utility from annual changes in financial wealth and are loss averse over these changes, their fear of a major drop in financial wealth will lead them to demand a high premium as compensation. BT call the combination of loss aversion and frequent evaluations *myopic loss aversion*.

BT's result is only *suggestive* of a solution to Mehra and Prescott's equity premium puzzle. As emphasized at the start of this section, that puzzle is in large part a consumption puzzle: given the low volatility of consumption growth, why are investors so reluctant to buy a high return asset, stocks, especially when that asset's covariance with consumption growth is so low? Since BT do not consider an intertemporal model with consumption choice, they cannot address this issue directly.

To see if prospect theory can in fact help with the equity premium puzzle, Barberis, Huang and Santos (2001), BHS henceforth, make a first attempt at building it into a dynamic equilibrium model of stock returns. A simple version of their model, an extension of which we consider later, examines an economy with the same structure as the one described at the start of Section 4, but in which investors have the preferences

$$E_0 \sum_{t=0}^{\infty} \left[\rho^t \frac{C_t^{1-\gamma}}{1-\gamma} + b_0 \overline{C}_t^{-\gamma} \widehat{v}(X_{t+1}) \right]. \tag{11}$$

The investor gets utility from consumption, but over and above that, he gets utility from changes in the value of his holdings of the risky asset between t and $t+1$, denoted here by X_{t+1}. Motivated by BT's findings, BHS define the unit of time to be a year, so that gains and losses are measured annually.

The utility from these gains and losses is determined by $\widehat{v}$ where

$$\widehat{v}(X) = \begin{cases} X & \text{for} \quad X \geq 0 \\ 2.25X & \phantom{\text{for}} \quad X < 0 \end{cases}. \tag{12}$$

The 2.25 factor comes from Tversky and Kahneman's (1992) experimental study of attitudes to timeless gambles. This functional form is simpler than the one used by BT, v. It captures loss aversion, but ignores other elements of prospect theory, such as the concavity (convexity) over gains (losses) and the probability transformation. In part this is because it is difficult to incorporate all these features into a fully dynamic framework; but also, it is based on BT's observation that it is mainly loss aversion that drives their results.[17]

[17]The $b_0 \overline{C}_t^{-\gamma}$ coefficient on the loss aversion term is a scaling factor which ensures that risk premia in the economy remain stationary even as aggregate wealth increases over time. It involves per capita consumption $\overline{C}_t$ which is exogeneous to the investor, and so does not affect the intuition of the model. The constant b_0 controls the importance of the loss aversion term in the investor's preferences; setting $b_0 = 0$ reduces the model to the much studied case of power utility over consumption. As $b_0 \to \infty$, the investor's decisions are driven primarily by concern about gains and losses in financial wealth, as assumed by BT.

BHS show that loss aversion can indeed provide a partial explanation of the high Sharpe ratio on the aggregate stock market. However, how much of the Sharpe ratio it can explain depends heavily on the importance of the second source of utility in (11), or in short, on b_0. As a way of thinking about this parameter, BHS note that when $b_0 = 0.7$, the psychological pain of losing \$100 in the stock market, captured by the second term, is roughly equal to the consumption-related pain of having to consume \$100 less, captured by the first term. For this b_0, the Sharpe ratio of the risky asset is 0.11, about a third of its historical value.

BT and BHS are both effectively assuming that investors engage in narrow framing, both cross-sectionally and temporally. Even if they have many forms of wealth, both financial and non-financial, they still get utility from changes in the value of one specific component of their total wealth: financial wealth in the case of BT, and stock holdings in the case of BHS. And even if investors have long investment horizons, they still evaluate their portfolio returns on an annual basis.

The assumption about cross-sectional narrow framing can be motivated in a number of ways. The simplest possibility is that it captures non-consumption utility, such as regret. Regret is the pain we feel when we realize that we would be better off if we had not taken a certain action in the past. If the investor's stock holdings fall in value, he may regret the specific decision he made to invest in stocks. Such feelings are naturally captured by defining utility directly over changes in the investors' financial wealth or in the value of his stock holdings.

Another possibility is that while people actually care only about consumption-related utility, they are boundedly rational. For example, suppose that they are concerned that their consumption might fall below some habit level. They know that the right thing to do when considering a stock market investment is to merge the stock market risk with other pre-existing risks that they face – labor income risk, say – and then to compute the likelihood of consumption falling below habit. However, this calculation may be too complex. As a result, people may simply focus on gains and losses in stock market wealth alone, rather than on gains and losses in total wealth.

What about temporal narrow framing? We suggested above that the way information is presented may lead investors to care about annual changes in financial wealth even if they have longer investment horizons. To provide further evidence for this, Thaler, Tversky, Kahneman and Schwartz (1997) provide an *experimental* test of the idea that the manner in which information is presented affects the frame people adopt in their decision-making.[18]

In their experiment, subjects are asked to imagine that they are portfolio managers for a small college endowment. One group of subjects – Group I, say – is shown monthly observations on two funds, Fund A and Fund B. Returns on Fund A (B) are drawn from a normal distribution calibrated to mimic bond (stock) returns as closely as possible, although

[18] See also Gneezy and Potters (1997) for a similar experiment.

subjects are not given this information. After each monthly observation, subjects are asked to allocate their portfolio between the two funds over the next month. They are then shown the realized returns over that month, and asked to allocate once again.

A second group of investors – Group II – is shown exactly the same series of returns, except that it is aggregated at the annual level; in other words, these subjects do not see the monthly fund fluctuations, but only cumulative annual returns. After each annual observation, they are asked to allocate their portfolio between the two funds over the next year.

A final group of investors – Group III – is shown exactly the same data, this time aggregated at the five-year level, and they too are asked to allocate their portfolio after each observation.

After going through a total of 200 months worth of observations, each group is asked to make one final portfolio allocation, which is to apply over the next 400 months. Thaler et al. (1997) find that the average final allocation chosen by subjects in Group I is much lower than that chosen by people in Groups II and III. This result is consistent with the idea that people code gains and losses based on how information is presented to them. Subjects in Group I see monthly observations and hence more frequent losses. If they adopt the monthly distribution as a frame, they will be more wary of stocks and will allocate less to them.

Ambiguity Aversion

In Section 3, we presented the Ellsberg paradox as evidence that people dislike ambiguity, or situations where they are not sure what the probability distribution of a gamble is. This is potentially very relevant for finance, as investors are often uncertain about the distribution of a stock's return.

Following the work of Ellsberg, many models of how people react to ambiguity have been proposed; Camerer and Weber (1992) provide a comprehensive review. One of the more popular approaches is to suppose that when faced with ambiguity, people entertain a range of possible probability distributions and act to maximize the minimum expected utility under any candidate distribution. In effect, people behave as if playing a game against a malevolent opponent who picks the actual distribution of the gamble so as to leave them as worse off as possible. Such a decision rule was first axiomatized by Gilboa and Schmeidler (1989). Epstein and Wang (1994) showed how such an approach could be incorporated into a dynamic asset pricing model, although they did not try to assess the quantitative implications of ambiguity aversion for asset prices.

Quantitative implications *have* been derived using a closely related framework known as robust control. In this approach, the agent has a reference probability distribution in mind, but wants to ensure that his decisions are good ones even if the reference model is misspecified to some extent. Here too, the agent essentially tries to guard against a "worst-case" misspecification. Anderson, Hansen and Sargent (1998) show how such a framework can be used for portfolio choice and pricing problems, even when state equations and objective functions are nonlinear.

Maenhout (1999) applies the Anderson et al. framework to the specific issue of the equity premium. He shows that if investors are concerned that their model of stock returns is misspecified, they will charge a substantially higher equity premium as compensation for the perceived ambiguity in the probability distribution. He notes, however, that to explain the full 3.9 percent equity premium requires an unreasonably high concern about misspecification. At best then, ambiguity aversion is only a partial resolution of the equity premium puzzle.

4.2 The Volatility Puzzle

Before turning to behavioral work on the volatility puzzle, it is worth thinking about how rational approaches to this puzzle might proceed. Since, in the data, the volatility of returns is higher than the volatility of dividend growth, equation (8) makes it clear that we have to make up the gap by introducing variation in the price-dividend ratio. What are the different ways we might do this? A useful framework for thinking about this is a version of the present value formula originally derived by Campbell and Shiller (1988). Starting from

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}, \tag{13}$$

where P_t is the value of the stock market at time t, they use a log-linear approximation to show that the log price-dividend ratio can be written

$$p_t - d_t = E_t \sum_{j=0}^{\infty} \rho^t \Delta d_{t+1+j} - E_t \sum_{j=0}^{\infty} \rho^t r_{t+1+j} + E_t \lim_{j \to \infty} \rho^j (p_{t+j} - d_{t+j}) + \text{const.}, \tag{14}$$

where lower case letters represent log variables – $p_t = \log P_t$, for example – and where $\Delta d_{t+1} = d_{t+1} - d_t$.

If the price-dividend ratio is stationary, so that the third term on the right is zero, this equation shows clearly that there are just two reasons price-dividend ratios can move around: changing expectations of future dividend growth or changing discount rates. Discount rates, in turn, can change because of changing expectations of future risk-free rates, changing forecasts of risk or changing risk aversion.

While there appear to be many ways of introducing variation in the P/D ratio, it has become clear that most of them cannot form the basis of a rational explanation of the volatility puzzle. We cannot use changing forecasts of dividend growth to drive the P/D ratio: restating the argument of Shiller (1981) and Le Roy and Porter (1981), if these forecasts are indeed rational, it must be that P/D ratios predict cash-flow growth in the time series, which they do not.[19] Nor can we use changing forecasts of future risk-free rates: again, if the forecasts are rational, P/D ratios must predict interest rates in the time series, which they do not. Even changing forecasts of risk cannot work, as there is little evidence that P/D ratios predict changes in risk in the time series. The only story that remains is therefore one about changing risk aversion, and this is the idea behind the Campbell and Cochrane (1999) model of aggregate stock market behavior. They propose a habit formation framework in which changes in consumption relative to habit lead to changes in risk aversion and hence variation in P/D ratios. This variation helps to plug the gap between the volatility of dividend growth and the volatility of returns.

Some rational approaches try to introduce variation in the P/D ratio through the third term on the right in equation (14). Since this requires investors to expect explosive growth in P/D ratios forever, they are known as models of rational bubbles. The idea is that prices are high today because they are expected to be higher next period; and they are higher next period because they are expected to be higher the period after that, and so on, forever. While such a model might initially seem appealing, a number of papers, most recently Santos and Woodford (1997), show that the conditions under which rational bubbles can survive are extremely restrictive.[20]

We now discuss some of the behavioral approaches to the volatility puzzle, grouping them by whether they focus on beliefs or on preferences.

Beliefs

One possible story is that investors believe that the mean dividend growth rate is more variable than it actually is. When they see a surge in dividends, they are too quick to believe that the mean dividend growth rate has increased. Their exuberance pushes prices up relative to dividends, adding to the volatility of returns.

A story of this kind can be derived as a direct application of representativeness and in particular, of the version of representativeness known as the law of small numbers, whereby people expect even short samples to reflect the properties of the parent population. If the investor sees many periods of good earnings, the law of small numbers leads him to believe that earnings growth has gone up, and hence that earnings will continue to be high in the

[19] There is an imporant caveat to the statement that changing cash-flow forecasts cannot be the basis of a satisfactory solution to the volatility puzzle. A large literature on structural uncertainty and learning, in which investors do not know the parameters of the cash-flow process but learn them over time, has had some success in matching the empirical volatility of returns (Brennan and Xia, 2001, Veronesi, 1999). In these models, variation in price-dividend ratios comes precisely from changing forecasts of cash-flow growth. While these forecasts are not subsequently confirmed in the data, investors are not considered irrational – they simply don't have enough data to infer the correct model. In related work, Barksy and De Long (1993) generate return volatility in an economy where investors forecast cash flows using a model that is wrong, but not easily rejected with available data.

[20] Brunnermeier (2001) provides a comprehensive review of this literature.

future. After all, the earnings growth rate cannot be "average". If it were, then according to the law of small numbers, earnings should *appear* average, even in short samples: some good earnings news, some bad earnings news, but not several good pieces of news in a row.

Another belief-based story relies more on private, rather than public information, and in particular, on overconfidence about private information. Suppose that an investor has seen public information about the economy, and has formed a prior opinion about future cash-flow growth. He then does some research on his own and becomes overconfident about the information he gathers: he overestimates its accuracy and puts too much weight on it relative to his prior. If the private information is positive, he will push prices up too high relative to current dividends, again adding to return volatility.[21]

Price-dividend ratios and returns might also be excessively volatile because investors extrapolate *past returns* too far into the future when forming expectations of future returns. Such a story might again be based on representativeness and the law of small numbers. The same argument for why investors might extrapolate past cash flows too far into the future can be applied here to explain why they might do the same thing with past returns.

The reader will have noticed that we do not cite any specific papers in connection with these behavioral stories. This is because these ideas were originally put forward in papers whose primary focus is explaining *cross-sectional* anomalies such as the value premium, even though they also apply here in a natural way. In brief, many of those papers – which we discuss in detail in Section 5 – generate certain cross-sectional anomalies by building excessive time series variation into the price-earnings ratios of individual stocks. It is therefore not surprising that the mechanisms proposed there might also explain the substantial time series variation in *aggregate*-level price-earnings ratios. In fact, it is perhaps satisfying that these behavioral theories simultaneously address both aggregate and firm-level evidence.

We close this section with a brief mention of "money illusion", the confusion between real and nominal values first discussed by Fisher (1928), and more recently investigated by Shafir et al. (1997). In financial markets, Modigliani and Cohn (1979) and more recently, Ritter and Warr (2002), have argued that part of the variation in P/D ratios and returns may be due to investors mixing real and nominal quantities when forecasting future cash flows. The value of the stock market can be determined by discounted real cash flows at real rates, or nominal cash flows at nominal rates. At times of especially high or especially low inflation though, it is possible that some investors mistakenly discount *real* cash flows

[21] Campbell (2000), among others, notes that behavioral models based on cash-flow forecasts often ignore potentially important interest rate effects. If investors are forecasting excessively high cash-flow growth, pushing up prices, interest rates should also rise, thereby dampening the price rise. One response is that interest rates are governed by expectations about *consumption* growth, and in the short run, consumption and dividends can be somewhat delinked: even if dividend growth is expected to be high, this need not necessarily trigger an immediate interest rate response. Alternatively, one can try to specify investors' expectations in such a way that interest rate effects become less important. Cecchetti, Lam and Mark (2000) take a step in this direction.

at *nominal* rates. If inflation increases, so will the nominal discount rate. If investors then discount the *same* set of cash flows at this higher rate, they will push the value of the stock market down. Of course, this calculation is incorrect: the same inflation which pushes up the discount rate should also push up future cash flows. On net, inflation should have little effect on market value. Such real vs. nominal confusion may therefore cause excessive variation in P/D ratios and returns and seems particularly relevant to understanding the low market valuations during the high inflation years of the 1970s, as well as the high market valuations during the low inflation 1990s.

Preferences

Barberis, Huang and Santos (2001) show that a straightforward extension of the version of their model discussed in Section 4.1 can explain both the equity premium and volatility puzzles. To do this, they appeal to experimental evidence about dynamic aspects of loss aversion. This evidence suggests that the degree of loss aversion is not the same in all circumstances but depends on prior gains and losses. In particular, Thaler and Johnson (1990) find that after prior gains, subjects take on gambles they normally do not, and that after prior losses, they refuse gambles that they normally accept. The first finding is sometimes known as the "house money effect", reflecting gamblers' increasing willingness to bet when ahead. One interpretation of this evidence is that losses are less painful after prior gains because they are cushioned by those gains. However, after being burned by a painful loss, people may become more wary of additional setbacks.[22]

To capture these ideas, Barberis, Huang and Santos (2001) modify the utility function in (11) to

$$E_0 \sum_{t=0}^{\infty} \left[\rho^t \frac{C_t^{1-\gamma}}{1-\gamma} + b_0 \overline{C}_t^{-\gamma} \widetilde{v}(X_{t+1}, z_t) \right]. \tag{15}$$

Here, z_t is a state variable that tracks past gains and losses on the stock market. For any fixed z_t, the function $\widetilde{v}$ is a piecewise linear function similar in form to $\widehat{v}$, defined in (12). However, the investors' sensitivity to losses is no longer constant at 2.25, but is determined by z_t, in a way that reflects the experimental evidence described above.

A model of this kind can help explain the volatility puzzle. Suppose that there is some good cash-flow news. This pushes the stock market up, generating prior gains for investors, who are now less scared of stocks: any losses will be cushioned by the accumulated gains. They therefore discount future cash flows at a lower rate, pushing prices up still further relative to current dividends and adding to return volatility.

[22] It is important to distinguish Thaler and Johnson's (1990) evidence from other evidence presented by Kahneman and Tversky (1979) and discussed in Section 3, showing that people are risk averse over gains and risk seeking over losses. One set of evidence pertains to one-shot gambles, the other to sequences of gambles. Kahneman and Tversky's (1979) evidence suggests that people are willing to take risks in order to avoid a loss; Thaler and Johnson's (1990) evidence suggests that if these efforts are unsuccessful and the investor suffers an unpleasant loss, he will *subsequently* act in a more risk averse manner.

5 Application: The Cross-section of Average Returns

While the behavior of the aggregate stock market is not easy to understand from the rational point of view, promising rational models have nonetheless been developed and can be tested against behavioral alternatives. Empirical studies of the behavior of *individual* stocks have unearthed a set of facts which is altogether more frustrating for the rational paradigm. Many of these facts are about the *cross-section* of average returns: they document that one group of stocks earns higher average returns than another. These facts have come to be known as "anomalies" because they cannot be explained by the simplest and most intuitive model of risk and return in the financial economist's toolkit, the Capital Asset Pricing Model, or CAPM.

We now outline some of the more salient findings in this literature and then consider some of the rational and behavioral approaches in more detail.

The Size Premium

This anomaly was first documented by Banz (1981). We report the more recent findings of Fama and French (1992). Every year from 1963 to 1990, Fama and French group all stocks traded on the NYSE, Amex, and Nasdaq into deciles based on their market capitalization, and then measure the average return of each decile over the next year. They find that for this sample period, the average return of the smallest stock decile is 0.74 percent per month higher than the average return of the largest stock decile. This is certainly an anomaly relative to the CAPM: while stocks in the smallest decile do have higher betas, the difference in risk is not enough to explain the difference in average returns.[23]

Long-term Reversals

Every three years from 1926 to 1982, De Bondt and Thaler (1985) rank all stocks traded on the NYSE by their prior three year cumulative return and form two portfolios: a "winner" portfolio of the 35 stocks with the best prior record and a "loser" portfolio of the 35 worst performers. They then measure the average return of these two portfolios over the three years subsequent to their formation. They find that over the whole sample period, the average annual return of the loser portfolio is higher than the average return of the winner portfolio by about 8 percent per year.

The Predictive Power of Scaled-price Ratios

These anomalies, which are about the cross-sectional predictive power of variables like the book-to-market (B/M) and earnings-to-price (E/P) ratios, where some measure of fun-

[23] The last decade of data has served to reduce the size premium considerably. Gompers and Metrick (2001) argue that this is due to demand pressure for large stocks resulting from the growth of institutional investors, who prefer such stocks.

damentals is scaled by price, have a long history in finance going back at least to Graham (1949), and more recently Dreman (1977), Basu (1983) and Rosenberg, Reid, and Lanstein (1985). We concentrate on Fama and French's (1992) more recent evidence.

Every year, from 1963 to 1990, Fama and French group all stocks traded on the NYSE, AMEX, and Nasdaq into deciles based on their book-to-market ratio, and measure the average return of each decile over the next year. They find that the average return of the highest-B/M-ratio decile, containing so called "value" stocks, is 1.53 percent per month higher than the average return on the lowest-B/M-ratio decile, "growth" or "glamour" stocks, a difference much higher than can be explained through differences in beta between the two portfolios. Repeating the calculations with the earnings-price ratio as the ranking measure produces a difference of 0.68 percent per month between the two extreme decile portfolios, again an anomalous result.[24]

Momentum

Every month from January 1963 to December 1989, Jegadeesh and Titman (1993) group all stocks traded on the NYSE into deciles based on their prior six month return and compute average returns of each decile over the six months after portfolio formation. They find that the decile of biggest prior winners outperforms the decile of biggest prior losers by an average of 10 percent on an annual basis.

Comparing this result to De Bondt and Thaler's (1985) study of prior winners and losers illustrates the crucial role played by the length of the prior ranking period. In one case, prior winners continue to win; in the other, they perform poorly.[25] A challenge to both behavioral and rational approaches is to explain why extending the formation period switches the result in this way.

There is some evidence that tax-loss selling creates seasonal variation in the momentum effect. Stocks with poor performance during the year may later be subject to selling by investors keen to realize losses that can offset capital gains elsewhere. This selling pressure means that prior losers continue to lose, enhancing the momentum effect. At the turn of the year, though, the selling pressure eases off, allowing prior losers to rebound and weakening the momentum effect. A careful analysis by Grinblatt and Moskowitz (1999) finds that on net, tax-loss selling may explain part of the momentum effect, but by no means all of it. In any case, while selling a stock for tax purposes is rational, a model of predictable price movements based on such behavior is not. Roll (1983) calls such explanations "stupid" since

[24]Ball (1978) and Berk (1995) point out that the size premium and the scaled-price ratio effects emerge naturally in any model where investors apply different discount rates to different stocks: if investors discount a stock's cash flows at a higher rate, that stock will typically have a lower market capitalization and a lower price-earnings ratio, but also higher returns. Note, however, that this view does not shed any light on whether the variation in discount rates is rationally justifiable or not.

[25]In fact, De Bondt and Thaler (1985) also report that one-year big winners outperform one-year big losers over the following year, but do not make much of this finding.

investors would have to be stupid not to buy in December if prices were going to increase in January.

A number of studies have examined stock returns following important corporate announcements, a type of analysis known as an event study. Jay Ritter's chapter in this volume discusses many of these studies in detail; here, we summarize them briefly.

Event Studies of Earnings Announcements

Every quarter from 1974 to 1986, Bernard and Thomas (1989) group all stocks traded on the NYSE and AMEX into deciles based on the size of the surprise in their most recent earnings announcement. "Surprise" is measured relative to a simple random walk model of earnings. They find that on average, over the 60 days after the earnings announcement, the decile of stocks with surprisingly good news outperforms the decile with surprisingly bad news by an average of about 4 percent, a phenomenon known as post-earnings announcement drift. Once again, this difference in returns is not explained by differences in beta between the two portfolios. A later study by Chan, Jegadeesh and Lakonishok (1996) measures surprise in other ways – relative to analyst expectations, and by the stock price reaction to the news – and obtains similar results.[26]

Event Studies of Dividend Initiations and Omissions

Michaely, Thaler and Womack (1995) study firms which announced initiation or omission of a dividend payment between 1964 and 1988. They find that on average, the shares of firms initiating (omitting) dividends significantly outperform (underperform) the market portfolio over the year after the announcement.

Event Studies of Stock Repurchases

Ikenberry, Lakonishok and Vermaelen (1995) look at firms which announced a share repurchase between 1980 and 1990, while Mitchell and Stafford (2001) study firms which did either self-tenders or share repurchases between 1960 and 1993. The latter study finds that on average, the shares of these firms outperform a control group matched on size and book-to-market by a substantial margin over the four year period following the event.

Event Studies of Primary and Secondary Offerings

Loughran and Ritter (1995) study firms which undertook primary or secondary equity offerings between 1970 and 1990. They find that the average return of shares of these firms

[26]Vuolteenaho (2002) combines a clean-surplus accounting version of the present value formula with Campbell's (1991) log-linear decomposion of returns to estimate a measure of cash-flow news that is potentially more accurate than earnings announcements. Analogous to the post-earnings announcement studies, he finds that stocks with good cash-flows news subsequently have higher average returns than stocks with disappointing cash-flow news.

over the five year period after the issuance is markedly below the average return of shares of non-issuing firms matched to the issuing firms on size. Brav and Gompers (1997) and Brav, Geczy and Gompers (2001) argue that this anomaly may not be distinct from the scaled-price anomaly listed above: when the returns of event firms are compared to the returns of firms matched on both size and book-to-market, there is very little difference.

Long-term event studies like the last three analyses summarized above raise some thorny statistical problems. In particular, conducting statistical inference with long-term buy-and-hold post-event returns is a treacherous business. Barber and Lyon (1997), Lyon, Barber and Tsai (1999), Brav (2000), Fama (1998), Loughran and Ritter (2000) and Mitchell and Stafford (2001) are just a few of the papers that discuss this topic. Cross-sectional correlation is one important issue: if a certain firm announces a share repurchase shortly after another firm does, their four-year post event returns will overlap and cannot be considered independent. Although the problem is an obvious one, it is not easy to deal with effectively. Some recent attempts to do so, such as Brav (2000), suggest that the anomalous evidence in the event studies on dividend announcements, repurchase announcements, and equity offerings is statistically weaker than initially thought, although how much weaker remains controversial.

A more general concern with *all* the above empirical evidence is data-mining. After all, if we sort and rank stocks in enough different ways, we are bound to discover striking – but completely spurious – cross-sectional differences in average returns.

A first response to the data-mining critique is to note that the above studies do not use the kind of obscure firm characteristics or marginal corporate announcements that would suggest data-mining. Indeed, it is hard to think of an important class of corporate announcements that has *not* been associated with a claim about anomalous post-event returns. A more direct check is to perform out-of-sample tests. Interestingly, a good deal of the above evidence *has* been replicated in other data sets. Fama, French and Davis (2000) show that there is a value premium in the subsample of U.S. data that precedes the data set used in Fama and French (1992), while Fama and French (1998) document a value premium in international stock markets. Rouwenhourst (1998) shows that the momentum effect is alive and well in international stock market data.

If the empirical results are taken at face value, then the challenge to the rational paradigm is to show that the above cross-sectional evidence emerges naturally from a model with fully rational investors. In special cases, models of this form reduce to the CAPM, and we know that this does not explain the evidence. More generally, rational models predict a multifactor pricing structure,

$$\overline{r}_i - r_f = \beta_{i,1}(\overline{F}_1 - r_f) + \ldots + \beta_{i,K}(\overline{F}_K - r_f), \tag{16}$$

where the factors proxy for marginal utility growth and where the loadings $\beta_{i,k}$ come from a time series regression of excess stock returns on excess factor returns,

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,1}(F_{1,t} - r_{f,t}) + \ldots + \beta_{i,K}(F_{K,t} - r_{f,t}) + \varepsilon_{i,t}. \tag{17}$$

To date, it has proved difficult to derive a multi-factor model which explains the cross-sectional evidence, although this remains a major research direction.

Alternatively, one can skip the step of *deriving* a factor model, and simply try a specific model to see how it does. This is the approach of Fama and French (1993, 1996). They show that a certain three factor model does a good job explaining the average returns of portfolios formed on size and book-to-market rankings. Put differently, the α_i intercepts in regression (17) are typically close to zero for their choice of factors. The specific factors they use are the return on the market portfolio, the return on a portfolio of small stocks minus the return on a portfolio of large stocks – the "size" factor – and the return on a portfolio of value stocks minus the return on a portfolio of growth stocks – the "book-to-market" factor. By constructing these last two factors, Fama and French are isolating common factors in the returns of small stocks and value stocks, and their three factor model can be loosely motivated by the idea that this comovement is a systematic risk that is priced in equilibrium.

The low α_i intercepts obtained by Fama and French (1993, 1996) are not necessarily cause for celebration. After all, these authors began their investigation only after it was already known that small stocks and value stocks earn high average returns. Moreover, as Roll (1977) emphasizes, in any specific sample, it is always possible to mechanically construct a one factor model that prices average returns *exactly*.[27] This sounds a cautionary note: just because a factor model happens to work well does not necessarily mean that we are learning anything about the economic drivers of average returns. To be fair, Fama and French (1996) themselves admit that their results can only have their full impact once it is explained what it is about investor preferences and the structure of the economy that leads people to price assets according to their model.

One general feature of the rational approach is that it is loadings or betas, and not firm characteristics that determine average returns. For example, a risk-based approach would argue that value stocks earn high returns not because they have high book-to-market ratios, but because such stocks happen to have a high loading on the book-to-market factor. Daniel and Titman (1997) cast doubt on this specific prediction by performing double sorts of stocks on both book-to-market ratios and loadings on book-to-market factors, and showing that stocks with different loadings but the same book-to-market ratio do *not* differ in their average returns. These results appear quite damaging to rational approach. However, using a longer data set and a different methodology, Fama, French and Davis (2000) claim to reverse Daniel and Titman's findings. We expect further developments on this controversial front.

[27] For any sample of observations on individual returns, choose any one of the ex-post mean-variance efficient portfolios. Roll (1977) shows that there is an exact linear relationship between the sample mean returns of the individual assets and their betas, computed with respect to the mean-variance efficient portfolio.

More generally, rational approaches to the cross-sectional evidence face a number of other obstacles. First, rational models typically measure risk as the covariance of returns with marginal utility of consumption. Stocks are risky if they fail to pay out at times of high marginal utility – in "bad" times – and instead pay out when marginal utility is low – in "good" times. The problem is that for many of the above findings, there is little evidence that the portfolios with anomalously *high* average returns do poorly in bad times, whatever plausible measure of bad times is used. For example, Lakonishok, Shleifer and Vishny (1994) show that in their 1968 to 1989 sample period, value stocks do well relative to growth stocks even when the economy is in recession. Similarly, De Bondt and Thaler (1987) find that their loser stocks have higher betas than winners in up markets and lower betas in down markets – an attractive combination that no one would label "risky".

Second, some of the portfolios in the above studies – the decile of stocks with the lowest book-to-market ratios for example – earn average returns below the risk-free rate. It is not easy to explain why a rational investor would willingly accept a lower return than the T-Bill rate on a volatile portfolio.

Finally, in some of the examples given above, it is not just that one portfolio outperforms another on average. In some cases, the outperformance is present in almost every period of the sample. For example, in Bernard and Thomas' (1989) study, firms with surprisingly good earnings outperform those with surprisingly poor earnings in 46 out of the 50 quarters studied. It is not easy to see any risk here than might justify the outperformance.

There are a number of behavioral models which try to explain some of the above phenomena. We classify them based on whether their mechanism centers on beliefs or on preferences.

5.1 Belief-based Models

Barberis, Shleifer and Vishny (1998), BSV henceforth, argue that much of the above evidence is the result of systematic errors that investors make when they use public information to form expectations of future cash flows. They build a model that incorporates two of the updating biases from Section 3: conservatism, the tendency to underweight new information relative to priors; and representativeness, and in particular the version of representativeness known as the law of small numbers, whereby people expect even short samples to reflect the properties of the parent population.

When a company announces surprisingly good earnings, conservatism means that investors react insufficiently, pushing the price up too little. Since the price is too low, subsequent returns will be higher on average, thereby generating both post-earnings announcement drift and momentum. After a *series* of good earnings announcements, though, representativeness causes people to overreact and push the price up too high. The reason is that after many periods of good earnings, the law of small numbers leads investors to believe that this is a firm with particularly high earnings growth, and hence to forecast high earnings in the future. After all, the firm cannot be "average". If it were, then according the to law of small numbers, its earnings should *appear* average, even in short samples. Since the price is now too high, subsequent returns are too low on average, thereby generating long-term reversals and a scaled-price ratio effect.

To capture these ideas mathematically, BSV consider a model with a representative risk neutral investor in which the true earnings process for all assets is a random walk. Investors, however, do not use the random walk model to forecast future earnings. They think that at any time, earnings are being generated by one of two regimes: a "mean-reverting" regime, in which earnings are more mean-reverting than in reality, and a "trending" regime in which earnings trend more than in reality. The investor believes that the regime generating earnings changes exogenously over time and sees his task as trying to figure out which of the two regimes is currently generating earnings.

This framework offers one way of modelling the updating biases described above. Including a "trending" regime in the model captures the effect of representativeness by allowing investors to put more weight on trends than they should. Conservatism suggests that people may put too little weight on the latest piece of earnings news relative to their prior beliefs. In other words, when they get a good piece of earnings news, they effectively act as if part of the shock will be reversed in the next period, in other words, as if they believe in a "mean-reverting" regime. BSV confirm that for a wide range of parameter values, this model does indeed generate post-earnings announcement drift, momentum, long-term reversals and cross-sectional forecasting power for scaled-price ratios.[28]

Daniel, Hirshleifer and Subrahmanyam (1998, 2001), DHS henceforth, stress biases in the interpretation of *private*, rather than public information. Imagine that the investor does some research on his own to try to determine a firm's future cash flows. DHS assume that he is overconfident about this information; in particular, they argue that investors are more likely to be overconfident about private information they have worked hard to generate than about public information. If the private information is positive, overconfidence means that investors will push prices up too far relative to fundamentals. Future public information will slowly pull prices back to their correct value, thus generating long-term reversals and a scaled-price effect. To get momentum and a post-earnings announcement effect, DHS assume that the public information alters the investor's confidence in his original private information in an asymmetric fashion, a phenomenon known as self-attribution bias: public news which confirms the investor's research strongly increases the confidence he has in that research. Disconfirming public news, though, is given less attention, and the investor's confidence in the private information remains unchanged. This asymmetric response means that initial

[28]Poteshman (2001) finds evidence of a BSV-type expectations formation process in the options market. He shows that when pricing options, traders appear to underreact to individual daily changes in instantaneous variance, while overreacting to longer sequences of increasing or decreasing changes in instantaneous variance.

overconfidence is on average followed by even greater overconfidence, generating momentum.

Chopra, Lakonishok and Ritter (1992) and La Porta et al. (1997) provide compelling evidence that supports the idea that investors make irrational forecasts of future cash flows. If, as BSV and DHS argue, long-term reversals and the predictive power of scaled-price ratios are driven by excessive optimism or pessimism about future cash flows followed by a correction, then most of the correction should occur at those times when investors find out that their initial beliefs were too extreme, in other words, at earnings announcement dates. The data strongly confirms this prediction. Chopra et al. show that after portfolio formation, De Bondt and Thaler's (1985) "winner" portfolio performs particularly poorly in the few days around earnings' announcements. La Porta et al. show that the same is true for a portfolio of growth stocks. It is very hard to give a rational reason for why these portfolios earn such low average returns over just a few days of the year.

Perhaps the simplest way of capturing much of the cross-sectional evidence is positive feedback trading, where investors buy more of an asset which has recently gone up in value (De Long et al., 1990b, Barberis and Shleifer, 2003). If a company's stock price goes up this period on good earnings, positive feedback traders buy the stock in the following period, causing a further price rise. On the one hand, this generates momentum and post-earnings announcement drift. On the other hand, since the price has now risen above what is justified by fundamentals, subsequent returns will on average be too low, generating long-term reversals and a scaled-price ratio effect.

The simplest way of motivating positive feedback trading is extrapolative expectations, where investors' expectations of future returns are based on past returns. This in turn, may be due to representativeness and to the law of small numbers in particular. The same argument made by BSV as to why investors might extrapolate past cash flows too far into the future can be applied here to explain why they might extrapolate past *returns* too far into the future. De Long et al. (1990b) note that institutional features such as portfolio insurance or margin calls can also generate positive feedback trading.

Positive feedback trading also plays a central role in the model of Hong and Stein (1999), although in this case it emerges endogenously from more primitive assumptions. In this model, two boundedly rational groups of investors interact, where bounded rationality means that investors are only able to process a subset of available information. "Newswatchers" make forecasts based on private information, but do not condition on past prices. "Momentum traders" condition only on the most recent price change.

Hong and Stein also assume that private information diffuses slowly through the population of newswatchers. Since these investors are unable to extract each others' private information from prices, the slow diffusion generates momentum. Momentum traders are then added to the mix. Given what they are allowed to condition on, their optimal strategy is to engage in positive feedback trading: a price increase last period is a sign that good private information is diffusing through the economy. By buying, momentum traders hope to profit from the continued diffusion of information. This behavior preserves momentum, but also generates price reversals: since momentum traders cannot observe the extent of news diffusion, they keep buying even after price has reached fundamental value, generating an overreaction that is only later reversed.

These four models differ most in their explanation of momentum. In two of the models – BSV and Hong and Stein (1999) – momentum is due to an initial underreaction followed by a correction. In De Long et al. (1990b) and DHS, it is due to an initial overreaction followed by even more overreaction. Within each pair, the stories are different again.[29]

Hong, Lim and Stein (2000) present supportive evidence for the view of HS that momentum is due simply to slow diffusion of private information through the economy. They argue that the diffusion of information will be particularly slow among small firms and among firms with low analyst coverage, and that the momentum effect should therefore be more prominent there, a prediction they confirm in the data. They also find that among firms with low analyst coverage, momentum is almost entirely driven by prior losers continuing to lose. They argue that this too, is consistent with a diffusion story. If a firm not covered by analysts is sitting on good news, it will do its best to convey the news to as many people as possible, and as quickly as possible; bad news, however, will be swept under the carpet, making its diffusion much slower.

5.2 Belief-based Models with Institutional Frictions

Some authors have argued that models which combine mild assumptions about investor irrationality with institutional frictions may offer a fruitful way of thinking about some of the anomalous cross-sectional evidence.

The institutional friction that has attracted the most attention is short-sale constraints. As mentioned in Section 2.2., these can be thought of as anything which makes investors less willing to establish a short position than a long one. They include the direct cost of shorting, namely the lending fee; the risk that the loan is recalled by the lender at an inopportune moment; as well as legal restrictions: a large fraction of mutual funds are not allowed to short stocks.

Several papers argue that when investors differ in their beliefs, the existence of short-sale constraints can generate deviations from fundamental value and in particular, explain why stocks with high price-earnings ratios earn lower average returns in the cross-section. The simplest way of motivating the assumption of heterogeneous beliefs is overconfidence, which

[29] In particular, the models make different predictions about how individual investors would trade following certain sequences of past returns. Armed with transaction-level data, Hvidkjaer (2001) exploits this to provide initial evidence that may distinguish the theories.

is why that assumption is often thought of as capturing a mild form of irrationality. In the absence of overconfidence, investors' beliefs converge rapidly as they hear each other's opinions and hence deduce each other's private information.

There are at least two mechanisms through which differences of opinion and short-sale constraints can generate price-earnings ratios that are too high, and thereby explain why price-earnings ratios predict returns in the cross-section.

Miller (1977) notes that when investors hold different views about a stock, those with bullish opinions will, of course, take long positions. Bearish investors, on the other hand, want to short the stock, but being unable to do so, they sit out of the market. Stock prices therefore reflect only the opinions of the most optimistic investors which, in turn, means that they are too high and that they will be followed by lower returns.

Harrison and Kreps (1978) and Scheinkman and Xiong (2001) argue that in a dynamic setting, a second, speculation-based mechanism arises. They show that when there are differences in beliefs, investors will be happy to buy a stock for more than its fundamental value in anticipation of being able to sell it later to other investors even more optimistic than themselves. Note that short-sale constraints are essential to this story: in their absence, an investor can profit from another's greater optimism by simply shorting the stock. With short-sale constraints, the only way to do so is to buy the stock first, and then sell it on later.

Both types of models make the intriguing prediction that stocks which investors disagree about more will have higher price-earnings ratios and lower subsequent returns. Three recent papers test this prediction, each using a different measure of differences of opinion.

Dieter, Malloy and Scherbina (2003) use IBES data on analyst forecasts to obtain a direct measure of heterogeneity of opinion. They group stocks into quintiles based on the level of dispersion in analysts' forecasts of current year earnings and confirms that the highest dispersion portfolio earns lower average returns than the lowest dispersion portfolio.

Chen, Hong and Stein (2002) use "breadth of ownership" – defined roughly as the fraction of mutual funds that hold a particular stock – as a proxy for divergence of opinion about the stock. The more dispersion in opinions there is, the more mutual funds will need to sit out the market due to short sales constraints, leading to lower breadth. Chen et al. predict, and confirm in the data, that stocks experiencing a decrease in breadth subsequently have lower average returns compared to stocks whose breadth increases.

Jones and Lamont (2002) use the cost of short-selling a stock – in other words, the lending fee – to measure differences of opinion about that stock. The idea is that if there is a lot of disagreement about a stock's prospects, many investors will want to short the stock, thereby pushing up the cost of doing so. Jones and Lamont confirm that stocks with higher lending fees have higher price-earnings ratios and earn lower subsequent returns. It is interesting to note that their data set spans the years from 1926 to 1933. At that time, there existed a centralized market for borrowing stocks and lending fees were published daily in the Wall Street Journal. Today, by contrast, stock lending is an over-the-counter market, and data on lending fees is harder to come by.

In other related work, Hong and Stein (2003) analyze the implications of short sales constraints and differences of opinion for higher order moments, and show that they lead to skewness. The intuition is that when a stock's price goes down, more information is revealed: by seeing at what point they enter into the market, we learn the valuations of those investors whose pessimistic views could not initially be reflected in the stock price, because of short sales constraints. When the stock market goes up, the sidelined investors stay out of the market and there is less information revelation. This increase in volatility after a downturn is the source of the skewness.

One prediction of this idea is that stocks which investors disagree about more should exhibit greater skewness. Chen, Hong and Stein (2001) test this idea using increases in turnover as a sign of investor disagreement. They show that stocks whose turnover increases subsequently display greater skewness.

5.3 Preferences

Earlier, we discussed Barberis, Huang and Santos (2001) which tries to explain *aggregate* stock market behavior by combining loss aversion and narrow framing with an assumption about how the degree of loss aversion changes over time. Barberis and Huang (2001) show that applying the same ideas to individual stocks can generate the evidence on long-term reversals and on scaled-price ratios. The key idea is that when investors hold a number of different stocks, narrow framing may induce them to derive utility from gains and losses in the value of *individual* stocks. The specification of this additional source of utility is exactly the same as in BHS, except that it is now applied at the individual stock level instead of at the portfolio level: the investor is loss averse over individual stock fluctuations and the pain of a loss on a specific stock depends on that stock's past performance.

To see how this model generates a value premium, consider a stock which has had poor returns several periods in a row. Precisely because the investor focuses on individual stock gains and losses, he finds this painful and becomes especially sensitive to the possibility of further losses on the stock. In effect, he perceives the stock as riskier, and discounts its future cash flows at a higher rate: this lowers its price-earnings ratio and leads to higher subsequent returns, generating a value premium. In one sense, this model is narrower than those in the "beliefs" section, Section 5.1., as it does not claim to address momentum. In another sense, it is broader, in that it simultaneously explains the equity premium and derives the risk-free rate endogenously.

The models we describe in Sections 5.1., 5.2., and 5.3 focus primarily on momentum, long-term reversals, the predictive power of scaled-price ratios and post-earnings announcement drift. What about the other examples of anomalous evidence with which we began Section 5? In Section 7, we argue that the long-run return patterns following equity issuance and repurchases may be the result of rational managers responding to the kinds of noise traders analyzed in the preceding behavioral models. In short, if investors cause prices to swing away from fundamental value, managers may try to time these cycles, issuing equity when it is overpriced, and repurchasing it when it is cheap. In such a world, equity issues will indeed be followed by low returns, and repurchases by high returns. The models we have discussed so far do not, however, shed light on the size anomaly, nor on the dividend announcement event study.

6 Application: Closed-end Funds and Comovement

6.1 Closed-end Funds

Closed-end funds differ from more familiar open-end funds in that they only issue a fixed number of shares. These shares are then traded on exchanges: an investor who wants to buy a share of a closed-end fund must go to the exchange and buy it from another investor at the prevailing price. By contrast, should he want to buy a share of an open-end fund, the fund would create a new share and sell it to him at its net asset value, or NAV, the per share market value of its asset holdings.

The central puzzle about closed-end funds is that fund share prices differ from NAV. The typical fund trades at a discount to NAV of about 10 percent on average, although the difference between price and NAV varies substantially over time. When closed-end funds are created, the share price is typically above NAV; when they are terminated, either through liquidation or open-ending, the gap between price and NAV closes.

A number of rational explanations for the average closed-end fund discount have been proposed. These include expenses, expectations about future fund manager performance, and tax liabilities. These factors can go some way to explaining certain aspects of the closed-end fund puzzle. However, none of them can satisfactorily explain *all* aspects of the evidence. For example, agency costs such as management fees can explain why funds usually sell at discounts, but not why they typically initially sell at a premium, nor why discounts tend to vary from week to week.

Lee, Shleifer and Thaler (1991), LST henceforth, propose a simple behavioral view of these closed-end fund puzzles. They argue that some of the individual investors who are the primary owners of closed-end funds are noise traders, exhibiting irrational swings in their expectations about future fund returns. Sometimes they are too optimistic, while at other times, they are too pessimistic. Changes in their sentiment affect fund share prices and hence also the difference between prices and net asset values.[30]

This view provides a clean explanation of all aspects of the closed-end fund puzzle. Owners of closed-end funds have to contend with two sources of risk: fluctuations in the value of the funds' assets, and fluctuations in noise trader sentiment. If this second risk is systematic – we return to this issue shortly – rational investors will demand compensation for it. In other words, they will require that the fund's shares trade at a discount to NAV.

This also explains why new closed-end funds are often sold at a premium. Entrepreneurs will choose to create closed-end funds at times of investor exuberance, when they know that they can sell fund shares for more than they are worth. On the other hand, when a closed-end fund is liquidated, rational investors no longer have to worry about changes in noise trader sentiment because they know that at liquidation, the fund price will equal NAV. They therefore no longer demand compensation for this risk, and the fund price rises towards NAV.

An immediate prediction of the LST view is that prices of closed-end funds should comove strongly, even if the cash-flow fundamentals of the assets held by the funds do not: if noise traders become irrationally pessimistic, they will sell closed-end funds across the board, depressing their prices regardless of cash-flow news. LST confirm in the data that closed-end fund discounts are highly correlated.

The LST story depends on noise trader risk being systematic. There is good reason to think that it is. If the noise traders who hold closed-end funds also hold other assets, then negative changes in sentiment, say, will drive down the prices of closed-end funds *and* of their other holdings, making the noise trader risk systematic. To check this, LST compute the correlation of closed-end fund discounts with another group of assets primarily owned by individuals, small stocks. Consistent with the noise trader risk being systematic, they find a significant positive correlation.

6.2 Comovement

The LST model illustrates that behavioral models can make interesting predictions not only about the *average* level of returns, but also about patterns of comovement. In particular, it explains why the prices of closed-end funds comove so strongly, and also why closed-end funds as a class comove with small stocks. This raises the hope that behavioral models might be able to explain other puzzling instances of comovement as well.

[30] For the noise traders to affect the *difference* between price and NAV rather than just price, it must be that they are more active traders of closed-end fund shares than they are of assets owned by the funds. As evidence for this, LST point out that while funds are primarily owned by individual investors, the funds' assets are not.

Before studying this in more detail, it is worth setting out the traditional view of return comovement. The simplest rational explanation of return comovement is that it is due to cash-flow comovement: there will be a common factor in the returns of a group of assets if there is a common factor in news about their future earnings. There is little doubt that many instances of return comovement can be explained by cash flows: stocks in the automotive industry move together primarily because their earnings are correlated.

The closed-end fund evidence shows that cash-flow view of comovement is at best, incomplete: in that case, the prices of closed-end funds comove even though their fundamentals do not.[31] Other evidence is just as puzzling. Froot and Dabora (1999) study "twin stocks", which are claims to the same cash-flow stream, but are traded in different locations. The Royal Dutch/Shell pair, discussed in Section 2, is perhaps the best known example. If return comovement is simply a reflection of cash-flow comovement, these two stocks should be perfectly correlated. In fact, as Froot and Dabora show, Royal Dutch comoves strongly with the S&P 500 index of U.S. stocks, while Shell comoves with the FTSE index of U.K. stocks.

Fama and French (1993) uncover salient common factors in the returns of small stocks, as well as in the returns on value stocks. In order to test the rational view of comovement, Fama and French (1995) investigate whether these strong common factors can be traced to common factors in the earnings of these stocks. While they do uncover a common factor in the earnings of small stocks, as well as in the earnings of value stocks, these cash-flow factors are weaker than the factors in returns and there is little evidence that the return factors are driven by the cash-flow factors. Once again, there appears to be comovement in returns that has little to do with cash-flow comovement.[32]

In response to this evidence, researchers have begun to posit behavioral theories of comovement. LST is one such theory. To state their argument more generally, they start by observing that many investors choose to trade only a subset of all available securities. As these investors' risk aversion or sentiment changes, they alter their exposure to the particular securities they hold, thereby inducing a common factor in the returns of these securities. Put differently, this "habitat" view of comovement predicts that there will be a common factor in the returns of securities that are the primary holdings of a specific subset of investors,

[31] Bodurtha et al. (1993) and Hardouvelis et al. (1994) provide further interesting examples of a delinking between cash-flow comovement and return comovement in the closed-end fund market. They study closed-end *country* funds, whose assets trade in a different location from the funds themselves and find that the funds comove as much with the national stock market in the country where they are traded as with the national stock market in the country where their *assets* are traded. For example, a closed-end fund invested in German equities but traded in the U.S. typically comoves as much with the U.S. stock market as with the German stock market.

[32] In principle, comovement can also be rationally generated through changes in discount rates. However, changes in interest rates or risk aversion induce a common factor in the returns on *all* stocks, and do not explain why a particular group of stocks comoves. A common factor in news about the risk of certain assets may also be a source of comovement for those assets, but there is little direct evidence to support such a mechanism in the case of small stocks or value stocks.

such as individual investors. This story seems particularly appropriate for thinking about closed-end funds, and also for Froot and Dabora's evidence.

A second behavioral view of comovement was recently proposed by Barberis and Shleifer (2003). They argue that to simplify the portfolio allocation process, many investors first group stocks into categories such as small-cap stocks or automotive industry stocks, and then allocate funds across these various categories. If these categories are also adopted by noise traders, then as these traders move funds from one category to another, the price pressure from their coordinated demand will induce common factors in the returns of stocks that happen to be classified into the same category, even if those stocks' cash flows are largely uncorrelated. In particular, this view predicts that when an asset is added to a category, it should begin to comove more with that category than before.

Barberis, Shleifer and Wurgler (2001) test this "category" view of comovement by taking a sample of stocks that has been added to the S&P 500, and computing the betas of these stocks with the S&P 500 both before and after they are included. Based on both univariate and multivariate regressions, they show that upon inclusion, a stock's beta with the S&P 500 rises significantly, as does the fraction of its variance that is explained by the S&P 500, while its beta with stocks outside the index falls.[33] This result does not sit well with the cash-flow view of comovement – addition to the S&P 500 carries no information about the covariance of a stock's cash flows with other stocks' cash flows – but emerges naturally from a model where prices are affected by category-level demand shocks. Little is known, at this point, about how investors form categories in the first place, but an intriguing start on this problem is provided by Mullainathan (2000).

7 Application: Investor Behavior

Behavioral finance has also had some success in explaining how certain groups of investors behave, and in particular, what kinds of portfolios they choose to hold and how they trade over time. The goal here is less controversial than in the previous three sections: it is simply to explain the actions of certain investors, and not necessarily to claim that these actions also affect prices. Two factors make this type of research increasingly important. First, now that the costs of entering the stock market have fallen, more and more individuals are investing in equities. Second, the world-wide trend toward defined contribution retirement savings plans, and the possibility of individual accounts in social security systems mean that individuals are more responsible for their own financial well-being in retirement. It is therefore natural to ask how well they are handling these tasks.

We now describe some of the evidence on the actions of investors and the behavioral ideas that have been used to explain it.

[33] Similar results from univariate regressions can also be found in earlier work by Vijh (1994).

Insufficient Diversification

A large body of evidence suggests that investors diversify their portfolio holdings much less than is recommended by normative models of portfolio choice.

First, investors exhibit a pronounced "home bias". French and Poterba (1991) report that investors in the U.S., Japan and the U.K. allocate 94 percent, 98 percent, and 82 percent of their overall equity investment, respectively, to *domestic* equities. It has not been easy to explain this fact on rational grounds (Lewis, 1999). Indeed, normative portfolio choice models that take human capital into account typically advise investors to *short* their national stock market, because of its high correlation with their human capital (Baxter and Jermann, 1997).

Some studies have found an analog to home bias *within* countries. Using an especially detailed data set from Finland, Grinblatt and Keloharju (2001) find that investors in that country are much more likely to hold and trade stocks of Finnish firms which are located close to them geographically, which use their native tongue in company reports, and whose chief executive shares their cultural background. Huberman (2001) studies the geographic distribution of shareholders of U.S. Regional Bell Operating Companies (RBOCs) and finds that investors are much more likely to hold shares in their local RBOC than in out-of-state RBOCs. Finally, studies of allocation decisions in 401(k) plans find a strong bias towards holding own company stock: over 30 percent of defined contribution plan assets in large U.S. companies are invested in employer stock, much of this representing voluntary contributions by employees (Benartzi, 2001).

In Section 3, we discussed evidence showing that people dislike ambiguous situations, where they feel unable to specify a gamble's probability distribution. Often, these are situations where they feel that they have little competence in evaluating a certain gamble. On the other hand, people show an excessive liking for familiar situations, where they feel they are in a better position than others to evaluate a gamble.

Ambiguity and familiarity offer a simple way of understanding the different examples of insufficient diversification. Investors may find their national stock markets more familiar – or less ambiguous – than foreign stock indices; they may find firms situated close to them geographically more familiar than those located further away; and they may find their employer's stock more familiar than other stocks.[34] Since familiar assets are attractive, people invest heavily in those, and invest little or nothing at all in ambiguous assets. Their portfolios therefore appear undiversified relative to the predictions of standard models that ignore the investor's degree of confidence in the probability distribution of a gamble.

Not all evidence of home bias should be interpreted as a preference for the familiar.

[34] Particularly relevant to this last point is survey data showing that people consider their own company stock less risky than a diversified index (Driscoll et al., 1995).

Coval and Moskowitz (1999) show that U.S. mutual fund managers tend to hold stocks whose company headquarters are located close to their funds' headquarters. However, Coval and Moskowitz's (2001) finding that these local holdings subsequently perform well suggests that an information story is at work here, not a preference for the familiar. It is simply less costly to research local firms and so fund managers do indeed focus on those firms, picking out the stocks with higher expected returns. There is no obvious information-based explanation for the results of French and Poterba (1991), Huberman (2001) or Benartzi (2001), while Grinblatt and Keloharju (2001) argue against such an interpretation of their findings.

Naive Diversification

Benartzi and Thaler (2001) find that when people *do* diversify, they do so in a naive fashion. In particular, they provide evidence that in 401(k) plans, many people seem to use strategies as simple as allocating $1/n$ of their savings to each of the n available investment options, whatever those options are. Some evidence that people think in this way comes from the laboratory. Benartzi and Thaler ask subjects to make an allocation decision in each of the following three conditions: first, between a stock fund and a bond fund; next, between a stock fund and a balanced fund, which invests 50 percent in stocks and 50 percent in bonds; and finally, between a bond fund and a balanced fund. They find that in all three cases, a 50:50 split across the two funds is a popular choice, although of course this leads to very different effective choices between stocks and bonds: the average allocation to stocks in the three conditions was 54 percent, 73 percent and 35 percent respectively.

The $1/n$ diversification heuristic or other naive diversification strategies predicts that in 401(k) plans which offer predominantly stock funds, investors will allocate more to stocks. Benartzi and Thaler test this in a sample of 170 large retirement savings plans. They divide the plans into three groups based on the fraction of funds – low, medium, or high – they offer that are stock funds. The allocation to stocks increases across the three groups, from 49 percent to 60 percent to 64 percent, confirming the initial prediction.

Excessive Trading

One of the clearest predictions of rational models of investing is that there should be very little trading. In a world where rationality is common knowledge, I am reluctant to buy if you are ready to sell. In contrast to this prediction, the volume of trading on the world's stock exchanges is very high. Furthermore, studies of individuals and institutions suggest that both groups trade more than can be justified on rational grounds.

Barber and Odean (2000) examine the trading activity from 1991 to 1996 in a large sample of accounts at a national discount brokerage firm. They find that after taking trading costs into account, the average return of investors in their sample is well below the return of standard benchmarks. Put simply, these investors would do a lot better if they traded less. The underperformance in this sample is largely due to transaction costs. However, there is also some evidence of poor security selection: in a similar data set covering the 1987 to 1993 time period, Odean (1999) finds that the average gross return of stocks that investors buy, over the year after they buy them, is lower than the average gross return of stocks that they sell, over the year after they sell them.

The most prominent behavioral explanation of such excessive trading is overconfidence: people believe that they have information strong enough to justify a trade, whereas in fact the information is too weak to warrant any action. This hypothesis immediately predicts that people who are more overconfident will trade more and, because of transaction costs, earn lower returns. Consistent with this, Barber and Odean (2000) show that the investors in their sample who trade the most earn by far the lowest average returns. Building on evidence that men are more overconfident than women, and using the same data as in their earlier study, Barber and Odean (2001) predict and confirm that men trade more and earn lower returns on average.

Working with the same data again, Barber and Odean (2002a) study the subsample of individual investors who switch from phone-based to online trading. They argue that for a number of reasons, the switch should be accompanied by an increase in overconfidence. First, better access to information and a greater degree of control – both features of an online trading environment – have been shown to increase overconfidence. Moreover, the investors who switch have often earned high returns prior to switching, which may only increase their overconfidence further. If this is indeed the case, they should trade more actively after switching and perform worse. Barber and Odean confirm these predictions.

The Selling Decision

Several studies find that investors are reluctant to sell assets trading at a loss relative to the price at which they were purchased, a phenomenon labelled the "disposition effect" by Shefrin and Statman (1985). Working with the same discount brokerage data used in the Odean (1999) study from above, Odean (1998) finds that the individual investors in his sample are more likely to sell stocks which have gone up in value relative to their purchase price, rather than stocks which have gone down.

It is hard to explain this behavior on rational grounds. Tax considerations point to the selling of losers, not winners.[35] Nor can one argue that investors rationally sell the winners because of information that their future performance will be poor. Odean reports that the average performance of stocks that people sell is better than that of stocks they hold on to.

Two behavioral explanations of these findings have been suggested. First, investors may have an irrational belief in mean-reversion. A second possibility relies on prospect theory and narrow framing. We have used these ingredients before, but this time it is not loss

[35] Odean (1998) does find that in December, investors prefer to sell past losers rather than past winners, but overall, this effect is swamped by a strong preference for selling past winners in the remaining 11 months.

aversion that is central, but rather the concavity (convexity) of the value function in the region of gains (losses).

To see the argument, suppose that a stock that was originally bought at \$50 now sells for \$55. Should the investor sell it at this point? Suppose that the gains and losses of prospect theory refer to the sale price minus the purchase price. In that case, the utility from selling the stock now is $v(5)$. Alternatively, the investor can wait another period, whereupon we suppose that the stock could go to \$50 or \$60 with equal probability; in other words, we abstract from belief-based trading motives by saying that the investor expects the stock price to stay flat. The expected value of waiting and selling next period is then $\frac{1}{2}v(0) + \frac{1}{2}v(10)$. Since the value function v is concave in the region of gains, the investor sells now. In a different scenario, the stock may currently be trading at \$45. This time, the comparison is between $v(-5)$ and $\frac{1}{2}v(-10) + \frac{1}{2}v(0)$, assuming a second period distribution of \$40 and \$50 with equal probability. Convexity of v pushes the investor to wait. Intuitively, by not selling, he is gambling that the stock will eventually break even, saving him from having to experience a painful loss.

The disposition effect is not confined to individual stocks. In an innovative study, Genesove and Mayer (2001) find evidence of a reluctance to sell at a loss in the housing market. They show that sellers whose expected selling price is below their original purchase price, set an asking price that exceeds the asking price of sellers with comparable houses. Moreover, this is not simply wishful thinking on the sellers' part that is later corrected by the market: sellers facing a possible loss do actually transact at considerably higher prices than other sellers.

Coval and Shumway (2000) study the behavior of professional traders in the Treasury Bond futures pit at the CBOT. If the gains and losses of prospect theory are taken to be daily profits and losses, the curvature of the value function implies that traders with profits (losses) by the middle of the trading day will take less (more) risk in their afternoon trading. This prediction is borne out in the data.

Grinblatt and Han (2001) argue that the investor behavior inherent in the disposition effect may be behind a puzzling feature of the cross-section of average returns, namely momentum in stock returns. Due to the concavity of the value function in the region of gains, investors will be keen to sell a stock which has earned them capital gains on paper. The selling pressure that results may initially depress the stock price, generating higher returns later. On the other hand, if the holders of a stock are facing capital losses, convexity in the region of losses means that they will only sell if offered a price premium; the price is therefore initially inflated, generating lower returns later. Grinblatt and Han provide supportive evidence for their story by regressing, in the cross-section, a stock's return on its past 12-month return as well as on a measure of the capital gain or loss faced by its holders. This last variable is computed as the current stock price minus investors' average cost basis, itself inferred from past volume. They find that the capital gain or loss variable steals a substantial amount of explanatory power from the past return.

The Buying Decision

Odean (1999) presents useful information about the stocks the individual investors in his sample choose to buy. Unlike "sells", which are mainly prior winners, "buys" are evenly split between prior winners and losers. Conditioning on the stock being a prior winner (loser) though, the stock is a big prior winner (loser). In other words, a good deal of the action is in the extremes.

Odean argues that the results for stock purchases are in part due to an attention effect. When buying a stock, people do not tend to systematically sift through the thousands of listed shares until they find a good "buy." They typically buy a stock that has caught their attention and perhaps the best attention draw is extreme past performance, whether good or bad.

Among individual investors, attention is less likely to matter for stock sales because of a fundamental way in which the selling decision differs from the buying decision. Due to short sales constraints, when individuals are looking for a stock to sell, they limit their search to those stocks that they currently own. When buying stocks, though, people have a much wider range of possibilities to choose from, and factors more related to attention may enter the decision.

Using the same discount brokerage data as in their earlier papers, Barber and Odean (2002b) test the idea that for individual investors, buying decisions are more driven by attention than are selling decisions. On any particular day, they create portfolios of "attention-getting" stocks using a number of different criteria: stocks with abnormally high trading volume, stocks with abnormally high or low returns, and stocks with news announcements. They find that whichever criterion is used, the individual investors in their sample are more likely to be purchasers of these high-attention stocks than sellers.

8 Application: Corporate Finance

8.1 Security Issuance, Capital Structure and Investment

An important strand of research in behavioral finance asks whether irrational investors such as those discussed in earlier sections affect the financing and investment decisions of firms.

We first address this question theoretically, and ask how a rational manager interested in maximizing true firm value – in other words, the stock price that will prevail once any mispricing has worked its way out of valuations – should act in the face of irrational investors. Stein (1996) provides a useful framework for thinking about this, as well as about other issues that arise in this section. He shows that when a firm's stock price is too high, the rational manager should issue more shares so as to take advantage of investor exuberance. Conversely, when the price is too low, he should repurchase shares. We refer to this model of security issuance as the "market timing" view.

What evidence there is to date on security issuance appears remarkably consistent with this framework. First, at the aggregate level, the share of new equity issues among total new issues – the "equity share" – is higher when the overall stock market is more highly valued. In fact, Baker and Wurgler (2000) show that the equity share is a reliable predictor of future stock returns: a high share predicts low, and sometimes negative stock returns. This is consistent with managers timing the market, issuing more equity at its peaks, just before it sinks back to more realistic valuation levels.

At the individual firm level, a number of papers have shown that the book-to-market ratio of a firm is a good cross-sectional predictor of new equity issuance (see Koracjzyk, Lucas and Macdonald 1991, Jung, Kim and Stulz 1996, Loughran, Ritter and Rydqvist 1994, Pagano, Panneta and Zingales 1998, Baker and Wurgler 2002a). Firms with high valuations issue more equity while those with low valuations repurchase their shares. Moreover, long-term stock returns after an IPO or SEO are low (Loughran and Ritter, 1995), while long term returns after the announcement of a repurchase are high (Ikenberry, Lakonishok and Vermaelen, 1995). Once again, this evidence is consistent with managers timing the market in their own securities.

More support for the market timing view comes from survey evidence. Graham and Harvey (2001) report that 67 percent of surveyed CFOs said that "the amount by which our stock is undervalued or overvalued" was an important consideration when issuing common stock.

The success of the market timing framework in predicting patterns of equity issuance offers the hope that it might also be the basis of a successful theory of capital structure. After all, a firm's capital structure simply represents its cumulative financing decisions over time. Consider, for example, two firms which are similar in terms of characteristics like firm size, profitability, fraction of tangible assets, and current market-to-book ratio, which have traditionally been thought to affect capital structure. Suppose, however, that in the past, the market-to-book ratio of firm A has reached much higher levels than that of firm B. Since, under the market timing theory, managers of firm A may have issued more shares at that time to take advantage of possible overvaluation, firm A may have more equity in its capital structure today.

In an intriguing recent paper, Baker and Wurgler (2002a) confirm this prediction. They show that all else equal, a firm's weighted-average historical market-to-book ratio, where more weight is placed on years in which the firm made an issuance of some kind, whether debt or equity, is a good cross-sectional predictor of the fraction of equity in the firm's capital structure today.

There is some evidence, then, that irrational investor sentiment affects financing decisions. We now turn to the more critical question of whether this sentiment affects actual investment decisions. Once again, we consider the benchmark case in Stein's (1996) model, in which the manager is both rational and interested in maximizing the firm's true value.

Suppose that a firm's stock price is too high. As discussed above, the manager should issue more equity at this point. More subtly, though, Stein shows that he should *not* channel the fresh capital into any actual new investment, but instead keep it in cash or in another fairly priced capital market security. While investors' exuberance means that, in *their* view, the firm has many positive net present value (NPV) projects it could undertake, the rational manager knows that these projects are not, in fact, positive NPV and that in the interest of true firm value, they should be avoided. Conversely, if the manager thinks that his firm's stock price is irrationally low, he should repurchase shares at the advantageously low price but not scale back actual investment. In short, irrational investors may affect the timing of security issuance, but they should not affect the firm's investment plans.

Once we move beyond this simple benchmark case, though, there emerge several channels through which sentiment might affect investment after all. First, the above argument properly applies only to *non-equity dependent* firms; in other words, to firms which because of their ample internal funds and borrowing capacity do not need the equity markets to finance their marginal investments.

For equity-dependent firms, however, investor sentiment and, in particular, excessive investor pessimism, may distort investment: when investors are excessively pessimistic, such firms may have to forgo attractive investment opportunities because it is too costly to finance them with undervalued equity. This thinking leads to a cross-sectional prediction, namely that the investment of equity-dependent firms should be more sensitive to gyrations in stock price than the investment of non-equity dependent firms.

Other than this equity-dependence mechanism, there are other channels through which investor sentiment might distort investment. Consider the case where investors are excessively optimistic about a firm's prospects. Even if a manager is in principle interested in maximizing true value, he faces the danger that if he refuses to undertake projects investors perceive as profitable, they may depress stock prices, exposing him to the risk of a takeover, or more simply, try to have him fired.[36]

Even if the manager is rational, this does not mean he will choose to maximize the firm's

[36]Shleifer and Vishny (2001) argue that in a situation such as this, where the manager feels forced to undertake some kind of investment, the best investment of all may be an acquisition of a less overvalued firm, in other words, one more likely to retain its value in the long run. This observation leads to a parsimonious theory of takeover waves, which predicts, among other things, an increase in stock-financed acquisitions at times of high dispersion in valuations.

true value. The agency literature has argued that some managers may maximize other objectives – the size of their firm, say – as a way of enhancing their prestige. This suggests another channel for investment distortion: managers might use investor exuberance as a cover for doing negative NPV "empire building" projects.

Finally, investor sentiment can also affect investment if managers put some weight on investors' opinions, perhaps because they think investors know something they don't. Managers may then mistake excessive optimism for well-founded optimism and get drawn into making negative NPV investments.

An important goal of empirical research, then, is to try to understand whether sentiment does affect investment, and if so, through which channel. Early studies produced little evidence of investment distortion. In aggregate data, Blanchard, Rhee and Summers (1993) find that movements in price apparently unrelated to movements in fundamentals have only weak forecasting power for future investment: the effects are marginally statistically significant and weak in economic terms. To pick out two particular historical episodes: the rise in stock prices through the 1920s did not lead to a commensurate rise in investment, nor did the crash of 1987 slow investment down appreciably. Morck, Shleifer and Vishny (1993) reach similar conclusions using firm level data, as do Baker and Wurgler (2002a): in their work on capital structure, they show that not only do firms with higher market-to-book ratios in their past have more equity in their capital structure today, but also that the equity funds raised are typically used to increase cash balances and *not* to finance new investment.

More recently though, Polk and Sapienza (2001) report stronger evidence of investment distortion. They identify overvalued firms as firms with high accruals, defined as earnings minus actual cash flow, and as firms with high net issuance of equity. Firms with high accruals may become overvalued if investors fail to understand that earnings are overstating actual cash flows, and Chan et al. (2001) confirm that such firms indeed earn low returns. Overvalued firms may also be identified through their opportunistic issuance of equity, and we have already discussed the evidence that such firms earn low long-run returns. Controlling for actual investment opportunities as accurately as possible, Polk and Sapienza find that the firms they identify as overvalued appear to invest more than other firms, suggesting that sentiment does influence investment.

Further evidence of distortion comes from Baker, Stein and Wurgler's (2001) test of the cross-sectional prediction that equity-dependent firms will be more sensitive to stock price gyrations than will non-equity dependent firms. They identify equity-dependent firms on the basis of their low cash balances, among other measures, and find that these firms have an investment sensitivity to stock prices about three times as high as that of non-equity dependent firms. This study therefore provides initial evidence that for some firms at least, sentiment may distort investment, and that it does so through the equity-dependence channel.

8.2 Dividends

A major open question in corporate finance asks why firms pay dividends. Historically, dividends have been taxed at a higher rate than capital gains. This means that stockholders who pay taxes would always prefer that the firm repurchase shares rather than pay a dividend. Since the tax exempt shareholders would be indifferent between the dividend payment and the share repurchase, the share repurchase is a Pareto improving action. Why then, do investors seem perfectly happy to accept a substantial part of their return in the form of dividends? Or, using behavioral language, why do firms choose to frame part of their return as an explicit payment to stockholders, and in so doing, apparently make some of their shareholders worse off?

Shefrin and Statman (1984) propose a number of behavioral explanations for why investors exhibit a preference for dividends. Their first idea relies on the notion of self-control. Many people exhibit self-control problems. On the one hand, we want to deny ourselves an indulgence, but on the other hand, we quickly give in to temptation: today, we tell ourselves that tomorrow we will not overeat, and yet, when tomorrow arrives, we again eat too much. To deal with self-control problems, people often set rules, such as "bank the wife's salary, and only spend from the husband's paycheck". Another very natural rule people might create to prevent themselves from overconsuming their wealth is "only consume the dividend, but don't touch the portfolio capital". In other words, people may like dividends because dividends help them surmount self-control problems through the creation of simple rules.

A second rationale for dividends is based on mental accounting: by designating an explicit dividend payment, firms make it easier for investors to segregate gains from losses and hence to increase their utility. To see this, consider the following example. Over the course of a year, the value of a firm has increased by \$10 per share. The firm could choose *not* to pay a dividend and return this increase in value to investors as a \$10 capital gain. Alternatively, it could pay a \$2 dividend, leaving an \$8 capital gain. In the language of prospect theory, investors will code the first option as $v(10)$. They may also code the second option as $v(10)$, but the explicit segregation performed by the firm may encourage them to code it as $v(2) + v(8)$. This will, of course, result in a higher perceived utility, due to the concavity of v in the domain of gains.

This manipulation is equally useful in the case of losses. A firm whose value has declined by \$10 per share over the year can offer investors a \$10 capital loss or a \$12 capital loss combined with a \$2 dividend gain. While the first option will be coded as $v(-10)$, the second is more likely to be coded as $v(2) + v(-12)$, again resulting in a higher perceived utility, this time because of the convexity of v in the domain of losses.

The utility enhancing trick in these examples depends on investors segregating the overall gain or loss into different components. The key insight of Shefrin and Statman is that by paying dividends, firms make it easier for investors to perform this segregation.

Finally, Shefrin and Statman argue that by paying dividends, firms help investors avoid regret. Regret is a frustration that people feel when they imagine having taken an action that would have led to a more desirable outcome. It is stronger for errors of commission – cases where people suffer because of an action they took – than for errors of omission – where people suffer because of an action they *failed* to take.

Consider a company which does not pay a dividend. In order to finance consumption, an investor has to sell stock. If the stock subsequently goes up in value, the investor feels substantial regret because the error is one of commission: he can readily imagine how not selling the stock would have left him better off. If the firm had paid a dividend and the investor was able to finance his consumption out of it, a rise in the stock price would not have caused so much regret. This time, the error would have been one of omission: to be better off, the investor would have had to reinvest the dividend.

Shefrin and Statman try to explain why firms pay dividends at all. Another question asks how dividend paying firms decide on the size of their dividend. The classic paper on this subject is Lintner (1956). His treatment is based on extensive interviews with executives of large American companies in which he asked the respondent, often the CFO, how the firm set dividend policy. Based on these interviews Lintner proposed what we would now call a behavioral model. In his model, firms first establish a target dividend payout rate based on notions of fairness, in other words, on what portion of the earnings it is fair to return to shareholders. Then, as earnings increase and the dividend payout ratio falls below the target level, firms increase dividends only when they are confident that they will not have to reduce them in the future.

There are several behavioral aspects to this model. First, the firm is not setting the dividend to maximize firm value or shareholder after-tax wealth. Second, perceptions of fairness are used to set the target payout rate. Third, the asymmetry between an increase in dividends and a decrease is explicitly considered. Although fewer firms now decide to start paying dividends, for those that do Lintner's model appears to be valid to this day (Benartzi, Michaely and Thaler 1997, Fama and French 2001).

Baker and Wurgler (2002b) argue that changes in dividend policy may also reflect changing investor sentiment about dividend-paying firms relative to their sentiment about non-paying firms. They argue that for some investors, dividend-paying firms and non-paying firms represent salient categories and that these investors exhibit changing sentiment about the categories. For instance, when investors become more risk averse, they may prefer dividend-paying stocks because of a confused notion that these firms are less risky (the well-known "bird in the hand" fallacy). If managers are interested in maximizing short-run value, perhaps because it is linked to their compensation, they may be tempted to change their dividend policy in the direction favored by investors.

Baker and Wurgler find some supportive evidence for their theory. They measure relative investor sentiment about dividend-paying firms as the log market-to-book ratio of paying firms minus the log market-to-book ratio of non-paying firms, and find that in the time series, a high value of this measure one year predicts that in the following year, a higher fraction of non-paying firms initiate a dividend and a larger fraction of newly-listed firms choose to pay one. Similar results obtain for other measures of sentiment about dividend-paying firms.

8.3 Models of Managerial Irrationality

The theories we have discussed so far interpret the data as reflecting actions taken by rational managers in response to irrationality on the part of investors. Other papers have argued that some aspects of managerial behavior are the result of irrationality on the part of managers themselves.

Much of Section 2 was devoted to thinking about whether rational agents might be able to correct dislocations caused by irrational traders. Analogously, before we consider models of irrational managers, we should ask to what extent rational agents can undo their effects.

On reflection, it doesn't seem any easier to deal with irrational managers than irrational investors. It is true that many firms have mechanisms in place designed to solve agency problems and to keep the manager's mind focused on maximizing firm value: giving him stock options for example, or saddling him with debt. The problem is that these mechanisms are unlikely to have much of an effect on irrational managers. These managers *think* that they are maximizing firm value, even if in reality, they are not. Since they think that they are already doing the right thing, stock options or debt are unlikely to change their behavior.

In the best known paper on managerial irrationality, Roll (1986) argues that much of the evidence on takeover activity is consistent with an economy in which there are *no* overall gains to takeovers, but in which managers are overconfident, a theory he terms the "hubris hypothesis". When managers think about taking over another firm, they conduct a valuation analysis of that firm, taking synergies into account. If managers are overconfident about the accuracy of their analysis, they will be too quick to launch a bid when their valuation exceeds the market price of the target. Just as overconfidence among individual investors may lead to excessive trading, so overconfidence among managers may lead to excessive takeover activity.

The main predictions of the hubris hypothesis are that there will be a large amount of takeover activity, but that the total combined gain to bidder and target will be zero; that on the announcement of a bid, the price of the target will rise and the value of the bidder will fall by a similar amount. Roll examines the available evidence and concludes that it is impossible to reject any of these predictions.

Heaton (2002) analyses the consequences of managerial optimism whereby managers overestimate the probability that the future performance of their firm will be good. He shows that it can explain pecking order rules for capital structure: since managers are optimistic relative to the capital markets, they believe their equity is undervalued, and are therefore reluctant to issue it unless they have exhausted internally generated funds or the debt market. Managerial optimism can also explain the puzzlingly high correlation of investment and cash flow: when cash flow is low, managers' reluctance to use external markets for financing means that they forgo an unusually large number of projects, lowering investment at the same time.

Malmendier and Tate (2001) test Heaton's model by investigating whether firms with excessively optimistic CEOs display a greater sensitivity of investment to cash flow. They detect excessive optimism among CEOs by examining at what point they exercise their stock options: CEOs who hold on to their options longer than recommended by normative models of optimal exercise are deemed to be have an overly optimistic forecast of their stock's future price. Malmendier and Tate find that the investment of these CEOs' firms is indeed more sensitive to cash flow than the investment of other firms.[37]

9 Conclusion

Behavioral finance is a young field, with its formal beginnings in the 1980s. Much of the research we have discussed was completed in the past five years. Where do we stand? Substantial progress has been made on numerous fronts.

Empirical investigation of apparently anomalous facts. When De Bondt and Thaler's (1985) paper was published, many scholars thought that the best explanation for their findings was a programming error. Since then their results have been replicated numerous times by authors both sympathetic to their view and by those with alternative views. At this stage, we think that most of the empirical facts are agreed upon by most of the profession, although the interpretation of those facts is still in dispute. This is progress. If we all agree that the planets do orbit the sun, we can focus on understanding why.

Limits of Arbitrage. Twenty years ago, many financial economists thought that the Efficient Markets Hypothesis had to be true because of the forces of arbitrage. We now understand that this was a naive view, and that the limits of arbitrage can permit substantial mispricing. It is now also understood by most that the absence of a profitable investment strategy, because of risks and costs, does not imply the absence of mispricing. Prices can be

[37]Another paper which can be included in the managerial irrationality category is Loughran and Ritter's (2002) explanation for why managers issuing shares appear to leave significant amounts of money "on the table," as evidenced by the high average return of IPOs on their first day of trading. The authors note that the IPOs with good first day performance are often those IPOs in which the price has risen far above its filing range, giving the managers a sizeable wealth gain. One explanation is therefore that since managers are already enjoying a major windfall, they do not care too much about the fact that they could have been even wealthier.

very wrong without creating profit opportunities.

Understanding Bounded Rationality. Thanks largely to the work of cognitive psychologists such as Daniel Kahneman and Amos Tversky, we now have a long list of robust empirical findings that catalogue some of the ways in which actual humans form expectations and make choices. There has also been progress in writing down formal models of these processes, with prospect theory being the most notable. Economists once thought that behavior was either rational or impossible to formalize. We now know that models of bounded rationality are both possible and also much more accurate descriptions of behavior than purely rational models.

Behavioral Finance Theory Building. In the past few years there has been a burst of theoretical work modelling financial markets with less than fully rational agents. These papers relax the assumption of complete rationality either through the belief formation process or through the decision-making process. Like the work of psychologists discussed above, these papers are important existence proofs, showing that it is possible to think coherently about asset pricing while incorporating salient aspects of human behavior.

Investor Behavior. We have now begun the important job of trying to document and understand how investors, both amateurs and professionals, make their portfolio choices. Until recently such research was notably absent from the repertoire of financial economists, perhaps because of the mistaken belief that asset pricing can be modeled without knowing anything about the behavior of the agents in the economy.

This is a lot of accomplishment in a short period of time, but we are still much closer to the beginning of the research agenda than we are to the end. We know enough about the perils of forecasting to realize that most of the future progress of the field is unpredictable. Still, we cannot resist venturing a few observations on what may be coming next.

First, much of the work we have summarized is narrow. Models typically capture something about investors' beliefs, or their preferences, or the limits of arbitrage, but not all three. This comment applies to most research in economics, and is a natural implication of the fact that researchers are boundedly rational too. Still, as progress is made we expect theorists to begin to incorporate more than one strand into their models.

An example can, perhaps, illustrate the point. The empirical literature repeatedly finds that the asset pricing anomalies are more pronounced in small and mid-cap stocks than in the large cap sector. It seems likely that this finding reflects limits of arbitrage: the costs of trading smaller stocks are higher, and the low liquidity keeps many potential arbitrageurs uninterested. While this observation may be an obvious one, it has not found its way into formal models. We expect investigation of the interplay between limits of arbitrage and cognitive biases to be an important research area in the coming years.

Second, there are obviously competing behavioral explanations for some of the empirical facts. Some critics view this as a weakness of the field. It is sometimes said that the long list of cognitive biases summarized in Section 3 offer behavioral modelers so many degrees of freedom that anything can be explained. We concede that there are numerous degrees of freedom, but note that rational modelers have just as many options to choose from. As Arrow (1986) has forcefully argued, rationality per se does not yield many predictions. The predictions come from auxiliary assumptions.

There is really only one scientific way to compare alternative theories, behavioral or rational, and that is with empirical tests. One kind of test looks for novel predictions the theory makes. For example, Lee, Shleifer and Thaler (1991) test their model's prediction that small firm returns will be correlated with closed-end fund discounts, while Hong, Lim and Stein (2000) test the implication of the Hong and Stein (1999) model that momentum will be stronger among stocks with thinner analyst coverage.

Another sort of test is to look for evidence that agents actually behave the way a model claims they do. The Odean (1998) and Genesove and Mayer (2001) investigations of the disposition effect using actual market behavior fall into this category. Bloomfield et al. (2002) offers an experimental test of the behavior theorized by Barberis, Shleifer and Vishny (1998). Of course, such tests are never airtight, but we should be skeptical of theories based on behavior that is undocumented empirically. Since behavioral theories claim to be grounded in realistic assumptions about behavior, we hope behavioral finance researchers will continue to give their assumptions empirical scrutiny. We would urge the same upon authors of rational theories.[38]

We have two predictions about the outcome of the exercise of direct tests of the assumptions of economic models. First, we will find out that most of our current theories, both rational and behavioral, are wrong. Second, substantially better theories will emerge.

[38]Directly testing the validity of a model's assumptions is not common practice in economics, perhaps because of Milton Friedman's influential argument that one should evaluate theories based on the validity of their predictions rather than the validity of their assumptions. Whether or not this is sound scientific practice, we note that much of the debate over the past 20 years has occured precisely because the evidence has not been consistent with the theories, so it may be a good time to start worrying about the assumptions. If a theorist wants to claim that fact X can be explained by behavior Y, it seems prudent to check whether people actually do Y.

10 Appendix

We show that for the economy laid out in (3)-(6), there is an equilibrium in which the risk-free rate is constant and given by

$$R_f = \frac{1}{\rho} e^{\gamma g_C - \frac{1}{2}\gamma^2 \sigma_C^2} \tag{18}$$

and in which the price-dividend ratio is a constant f, and satisfies

$$1 = \rho \frac{1+f}{f} e^{g_D - \gamma g_C + \frac{1}{2}(\sigma_D^2 + \gamma^2 \sigma_C^2 - 2\gamma \sigma_C \sigma_D \omega)}. \tag{19}$$

In this equilibrium, returns are therefore given by

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} = \frac{1 + P_{t+1}/D_{t+1}}{P_t/D_t} . \frac{D_{t+1}}{D_t} = \frac{1+f}{f} e^{g_D + \sigma_D \varepsilon_{t+1}}. \tag{20}$$

To see this, start from the Euler equations of optimality, obtained through the usual perturbation arguments,

$$1 = \rho R_f E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \tag{21}$$

$$1 = \rho E_t \left[R_{t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]. \tag{22}$$

Computing the expectation in (21) gives (18). We conjecture that in this economy, there is an equilibrium in which the price-dividend ratio is a constant f, so that returns are given by (20). Substituting this into (22) and computing the expectation gives (19), as required. For given parameter values, the quantitative implications for P/D ratios and returns are now easily computed.

11 References

Abreu, D., and M. Brunnermeier (2002), "Synchronization Risk and Delayed Arbitrage," forthcoming, *Journal of Financial Economics.*

Alpert M., and H. Raiffa (1982), "A Progress Report on the Training of Probability Assessors," in D. Kahneman, P. Slovic and A. Tversky, eds., *Judgment Under Uncertainty: Heuristics and Biases*, Cambridge: Cambridge University Press.

Anderson E., Hansen L., and T. Sargent (1998), "Risk and Robustness in Equilibrium," Working paper, University of Chicago.

Arrow, K. (1986), "Rationality of Self and Others," in R. Hogarth and M. Reder, eds., *Rational Choice*, Chicago: University of Chicago Press.

Baker, M., Stein, J., and J. Wurgler (2001), "When Does the Market Matter? Stock Prices and the Investment of Equity Dependent Firms," Working paper, Harvard University.

Baker, M., and J. Wurgler (2000), "The Equity Share in New Issues and Aggregate Stock Returns," *Journal of Finance* 55, 2219-2257.

Baker, M., and J. Wurgler (2002a), "Market Timing and Capital Structure," *Journal of Finance* 57, 1-32.

Baker, M., and J. Wurgler (2002b), "A Catering Theory of Dividends," Working paper, Harvard University.

Ball, R. (1978), "Anomalies in Relations between Securities' Yields and Yield Surrogates," *Journal of Financial Economics* 6, 103-126.

Banz, R. (1981), "The Relation between Return and Market Value of Common Stocks," *Journal of Financial Economics* 9, 3-18.

Barber, B., and J. Lyon (1997), "Detecting Long-run Abnormal Stock Returns: the Empirical Power and Specification of Test Statistics," *Journal of Financial Economics* 43, 341-372.

Barber, B., and T. Odean (2000), "Trading is Hazardous to Your Wealth: The Common Stock Performance of Individual Investors," *Journal of Finance* 55, 773-806.

Barber, B., and T. Odean (2001), "Boys will be Boys: Gender, Overconfidence, and Common Stock Investment," *Quarterly Journal of Economics* 141, 261-292.

Barber, B., and T. Odean (2002a), "Online Investors: Do the Slow Die First?" *Review of Financial Studies* 15, 455-487.

Barber, B., and T. Odean (2002b), "All that Glitters: The Effect of Attention and News on the Buying Behavior of Individual and Institutional Investors," Working paper, UC Berkeley.

Barberis, N., and M. Huang (2001), "Mental Accounting, Loss Aversion and Individual Stock Returns," *Journal of Finance* 56, 1247-1292.

Barberis, N., Huang M., and T. Santos (2001), "Prospect Theory and Asset Prices," *Quarterly Journal of Economics* 116, 1-53.

Barberis, N., and A. Shleifer (2003), "Style Investing," forthcoming, *Journal of Financial Economics.*

Barberis, N., Shleifer A., and R. Vishny (1998), "A Model of Investor Sentiment," *Journal of Financial Economics* 49, 307-345.

Barberis, N., Shleifer A., and J. Wurgler (2001), "Comovement," Working paper, University of Chicago.

Barsky, R., and B. De Long (1993), "Why Does the Stock Market Fluctuate?" *Quarterly Journal of Economics* 107, 291-311.

Basu, S. (1983), "The Relationship Between Earnings Yield, Market Value and Return for NYSE Common Stocks: Further Evidence," *Journal of Financial Economics* 12, 129-156.

Baxter, M., and U. Jermann (1997), "The International Diversification Puzzle is Worse than You Think," *American Economic Review* 87, 170-180.

Bell, D. (1982), "Regret in Decision Making Under Uncertainty," *Operations Research* 30, 961-981.

Benartzi, S. (2001), "Excessive Extrapolation and the Allocation of 401(k) Accounts to Company Stock," *Journal of Finance* 56, 1747-1764.

Benartzi, S., Michaely R., and R. Thaler (1997), "Do Changes in Dividends Signal the Future or the Past?" *Journal of Finance* 52, 1007-34.

Benartzi, S., and R. Thaler (1995), "Myopic Loss Aversion and the Equity Premium Puzzle," *Quarterly Journal of Economics* 110, 75-92.

Benartzi, S., and R. Thaler (2001), "Naïve Diversification Strategies in Defined Contribution Savings Plans," *American Economic Review* 91, 79-98.

Berk, J. (1995), "A Critique of Size Related Anomalies," *Review of Financial Studies* 8, 275-286.

Bernard, V. and J. Thomas (1989), "Post-Earnings Announcement Drift: Delayed Price Response or Risk Premium?" *Journal of Accounting Research* (Supplement), 1-36.

Blanchard, O., Rhee C., and L. Summers (1993), "The Stock Market, Profit, and Investment," *Quarterly Journal of Economics* 108, 115-136.

Bloomfield, R., and J. Hales (2002), "Predicting the Next Step of a Random Walk: Experimental Evidence of Regime-Shifting Beliefs," *Journal of Financial Economics* 65, 397-414.

Bodurtha J., Kim, D., and C.M. Lee (1993), "Closed-end Country Funds and U.S. Market Sentiment," *Review of Financial Studies* 8, 879-918.

Brav, A. (2000), "Inference in Long-horizon Event Studies," *Journal of Finance* 55, 1979-2016.

Brav A. and P. Gompers (1997), "Myth or Reality? The Long-run Underperformance of Initial Public Offerings: Evidence from Venture and Non-venture-backed Companies," *Journal of Finance* 52, 1791-1821.

Brav, A., Geczy C. and P. Gompers (2000), "Is the Abnormal Return Following Equity Issuances Anomalous?" *Journal of Financial Economics* 56, 209-249.

Brennan, M., and Y. Xia (2001), "Stock Return Volatility and the Equity Premium," *Journal of Monetary Economics* 47, 249-283.

Brown, S., Goetzmann, W. and S. Ross (1995), "Survival," *Journal of Finance* 50, 853-873.

Brunnermeier, M. (2001), *Asset Pricing under Asymmetric Information – Bubbles, Crashes, Technical Analysis, and Herding*, Oxford: Oxford University Press.

Buehler R., Griffin D., and M. Ross (1994), "Exploring the Planning Fallacy: Why People Underestimate their Task Completion Times," *Journality of Personality and Social Psychology* 67, 366-381.

Camerer, C. (1995), "Individual Decision Making," in J. Kagel and A. Roth, eds., *Handbook of Experimental Economics*, Princeton: Princeton University Press.

Camerer, C., and R. Hogarth (1999), "The Effects of Financial Incentives in Experiments: A Review and Capital-Labor Production Framework," *Journal of Risk and Uncertainty* 19, 7-42.

Camerer, C., and M. Weber (1992), "Recent Developments in Modeling Preferences: Uncertainty and Ambiguity," *Journal of Risk and Uncertainty* 5, 325-70.

Campbell J.Y. (1991), "A Variance Decomposition for Stock Returns," *Economic Journal* 101, 157-179.

Campbell, J.Y. (1999), "Asset Prices, Consumption and the Business Cycle," in J. Taylor and M. Woodford, eds., *Handbook of Macroeconomics*, Amsterdam: North-Holland.

Campbell, J.Y. (2000), "Asset Pricing at the Millenium," *Journal of Finance* 55, 1515-1567.

Campbell, J.Y., and J. Cochrane (1999), "By Force of Habit: A Consumption-based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy* 107, 205-51.

Campbell, J.Y. and R. Shiller (1988), "Stock Prices, Earnings and Expected Dividends," *Journal of Finance* 43, 661-676.

Cecchetti, S., Lam P., and N. Mark (2000), "Asset Pricing with Distorted Beliefs: Are Equity Returns Too Good to Be True?" *American Economic Review* 90, 787-805.

Chan, K., Chan L., Jegadeesh N., and J. Lakonishok (2001), "Earnings Quality and Stock Returns," Working paper, University of Illinois.

Chan, L., Jegadeesh N., and J. Lakonishok (1996), "Momentum Strategies," *Journal of Finance* 51, 1681-1713.

Chen, J., Hong H. and J. Stein (2001), "Forecasting Crashes: Trading Volume, Past Returns and Conditional Skewness in Stock Prices," *Journal of Financial Economics* 61, 345-381.

Chen, J., Hong H., and J. Stein (2002), "Breadth of Ownership and Stock Returns", forthcoming, *Journal of Financial Economics*.

Chew, S. (1983), "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica* 51, 1065-1092.

Chew, S. (1989), "Axiomatic Utility Theories with the Betweenness Property," *Annals of Operations Research* 19, 273-98.

Chew, S., and K. MacCrimmon (1979), "Alpha-nu Choice Theory: An Axiomatization of Expected Utility," Working paper, University of British Columbia.

Chopra, N., Lakonishok J., and J. Ritter (1992), "Measuring Abnormal Performance: Do Stocks Overreact?" *Journal of Financial Economics* 31, 235-268.

Coval, J., and T. Moskowitz (1999), "Home Bias at Home: Local Equity Preference in Domestic Portfolios," *Journal of Finance* 54, 2045-73.

Coval, J., and T. Moskowitz (2001), "The Geography of Investment: Informed Trading and Asset Prices," *Journal of Political Economy* 109, 811-841.

Coval, J. and T. Shumway (2000), "Do Behavioral Biases Affect Prices?" Working paper, University of Michigan.

Daniel, K., Hirshleifer D., and A. Subrahmanyam (1998), "Investor Psychology and Security Market Under- and Overreactions," *Journal of Finance* 53, 1839-1885.

Daniel, K., Hirshleifer D., and A. Subrahmanyam (2001), "Overconfidence, Arbitrage and Equilbrium Asset Pricing," *Journal of Finance* 56, 921-965.

Daniel, K. and S. Titman (1997), "Evidence on the Characteristics of Cross-Sectional Variation in Stock Returns," *Journal of Finance* 52, 1-33.

D'Avolio, G. (2002), "The Market for Borrowing Stock," forthcoming, *Journal of Financial Economics*.

De Bondt, W., and R. Thaler (1985), "Does the Stock Market Overreact?" *Journal of Finance* 40, 793-808.

De Bondt, W., and R. Thaler (1987), "Further Evidence on Investor Overreaction and Stock Market Seasonality," *Journal of Finance* 42, 557-581.

De Long, J.B., Shleifer A., Summers L., and R. Waldmann (1990a), "Noise Trader Risk in Financial Markets," *Journal of Political Economy* 98, 703-738.

De Long, J.B., Shleifer A., Summers L., and R. Waldmann (1990b), "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance* 45, 375-395.

Dekel, E. (1986), "An Axiomatic Characterization of Preferences Under Uncertainty: Weakening the Independence Axiom," *Journal of Economic Theory* 40, 304-18.

Diether, K., Malloy C., and A. Scherbina (2003), "Stock Prices and Differences of Opinion: Empirical Evidence that Stock Prices Reflect Optimism," forthcoming, *Journal of Finance*.

Dreman, D. (1977), *Psychology and the Stock Market: Investment Strategy Beyond Random Walk*, New York: Warner Books.

Driscoll, K., Malcolm J., Sirul M., and P. Slotter (1995), "1995 Gallup Survey of Defined Contribution Plan Participants," John Hancock Financial Services.

Edwards, W. (1986), "Conservatism in Human Information Processing," in B. Kleinmutz, ed., *Formal Representation of Human Judgment*, New York: John Wiley and Sons.

Ellsberg, D. (1961), "Risk, Ambiguity, and the Savage Axioms," *Quarterly Journal of Economics* 75, 643-69.

Epstein, L. and T. Wang (1994), "Intertemporal Asset Pricing under Knightian Uncertainty," *Econometrica* 62, 283-322.

Fama, E. (1970), "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance* 25, 383-417.

Fama, E. (1998), "Market Efficiency, Long-Term Returns and Behavioral Finance," *Journal of Financial Economics* 49, 283-307.

Fama, E. and K. French (1988), "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics* 22, 3-25.

Fama, E. and K. French (1992), "The Cross-Section of Expected Stock Returns," *Journal of Finance* 47, 427-465.

Fama, E. and K. French (1993), "Common Risk Factors in the Returns of Bonds and Stocks," *Journal of Financial Economics* 33, 3-56.

Fama, E., and K. French (1995), "Size and Book-to-Market Factors in Earnings and Returns," *Journal of Finance* 50, 131-155.

Fama, E. and K. French (1996), "Multifactor Explanations of Asset Pricing Anomalies," *Journal of Finance* 51, 55-84.

Fama, E. and K. French (1998), "Value vs. Growth: The International Evidence," *Journal of Finance* 53, 1975-1999.

Fama, E., and K. French (2001), "Disappearing Dividends: Changing Firm Characteristics or Lower Propensity to Pay?" *Journal of Financial Economics* 60, 3-43.

Fama, E., French K., and J. Davis (2000), "Characteristics, Covariances and Average Returns 1929-1997," *Journal of Finance* 55, 389-406.

Fischhoff B., Slovic, P., and S. Lichtenstein (1977), "Knowing With Certainty: The Appropriateness of Extreme Confidence," *Journal of Experimental Pyschology: Human Perception and Performance* 3, 552-564.

Fisher, I. (1928), *Money Illusion*, New York: Adelphi.

Fox, C., and A. Tversky (1995), "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics* 110, 585-603.

French, K., and J. Poterba (1991), "Investor Diversification and International Equity Markets," *American Economic Review* 81, 222-226.

Friedman, M. (1953), "The Case for Flexible Exchange Rates," in *Essays in Positive Economics*, Chicago: University of Chicago Press.

Froot, K. and E. Dabora (1999), "How Are Stock Prices Affected by the Location of Trade?" *Journal of Financial Economics* 53, 189-216.

Genesove, D., and C. Mayer (2001), "Loss Aversion and Seller Behavior: Evidence from the Housing Market," *Quarterly Journal of Economics* 116, 1233-1260.

Gervais, S., and T. Odean (2001), "Learning to be Overconfident," *Review of Financial Studies* 14, 1-27.

Gilboa I., and D. Schmeidler (1989), "Maxmin Expected Utility with a Non-unique Prior," *Journal of Mathematical Economics* 18, 141-53.

Gilovich, T., Griffin D., and D. Kahneman eds. (2002), *Heuristics and Biases: The Psychology of Intuitive Judgment*, Cambridge: Cambridge University Press.

Gilovich, T., Vallone, R., and A. Tversky (1985), "The Hot Hand in Basketball: On the Misperception of Random Sequences," *Cognitive Psychology* 17, 295-314.

Gneezy, U., and J. Potters (1997), "An Experiment on Risk Taking and Evaluation Periods," *Quarterly Journal of Economics* 112, 631-645.

Gompers, P. and A. Metrick (2001), "Institutional Investors and Equity Prices," *Quarterly Journal of Economics* 116, 229-259.

Graham, B. (1949), *The Intelligent Investor: A Book of Practical Counsel*, Harper and Row: New York.

Graham, J., and C. Harvey (2001), "The Theory and Practice of Corporate Finance: Evidence from the Field," *Journal of Financial Economics* 60, 187-243.

Grinblatt, M., and B. Han (2001), "The Disposition Effect and Momentum," Working paper, UCLA.

Grinblatt, M., and M. Keloharju (2001), "How Distance, Language, and Culture Influence Stockholdings and Trades," *Journal of Finance* 56, 1053-73.

Grinblatt, M., and T. Moskowitz (1999), "The Cross-section of Expected Returns and its Relation to Past Returns," Working paper, University of Chicago.

Gul, F. (1991), "A Theory of Disappointment in Decision Making under Uncertainty," *Econometrica* 59, 667-86.

Hansen, L., and K. Singleton (1983), "Stochastic Consumption, Risk Aversion and the Temporal Behavior of Asset Returns," *Journal of Political Economy* 91, 249-268.

Hardouvelis, G., La Porta R., and T. Wizman (1994), "What Moves the Discount on Country Equity Funds?" in J. Frankel, ed., *The Internationalization of Equity Markets*, Chicago: The University of Chicago Press.

Harris, L., and E. Gurel (1986), "Price and Volume Effects Associated with Changes in the S&P 500: New Evidence for the Existence of Price Pressure," *Journal of Finance* 41, 851-60.

Harrison, J.M., and D. Kreps (1978), "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics* 92, 323-336.

Heath, C., and A. Tversky (1991), "Preference and Belief: Ambiguity and Competence in Choice under Uncertainty," *Journal of Risk and Uncertainty* 4, 5-28.

Heaton, J.B. (2002), "Managerial Optimism and Corporate Finance," forthcoming, *Financial Managment*.

Hirshleifer, D. (2001), "Investor Psychology and Asset Pricing," *Journal of Finance* 56, 1533-1597.

Hong, H., Lim T., and J. Stein (2000), "Bad News Travels Slowly: Size, Analyst Coverage, and the Profitability of Momentum Strategies," *Journal of Finance* 55, 265-295.

Hong, H., and J. Stein (1999), "A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets," *Journal of Finance* 54, 2143-2184.

Hong, H., and J. Stein (2003), "Differences of Opinion, Short-sale Constraints and Market Crashes," forthcoming, *Review of Financial Studies*.

Huberman, G. (2001), "Familiarity Breeds Investment," *Review of Financial Studies* 14, 659-680.

Hvidkjaer, S. (2001), "A Trade-based Analysis of Momentum," Working paper, University of Maryland.

Ikenberry, D., Lakonishok J., and T. Vermaelen (1995), "Market Underreaction to Open Market Share Repurchases," *Journal of Financial Economics* 39, 181-208.

Jegadeesh, N. and S. Titman (1993), "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance* 48, 65-91.

Jones, C., and O. Lamont (2002), "Short-sale Constraints and Stock Returns," forthcoming, *Journal of Financial Economics*.

Jung, K., Kim Y., and R. Stulz (1996), "Timing, Investment Opportunities, Managerial Discretion, and the Security Issue Decision," *Journal of Financial Economics* 42, 159-185.

Kahneman, D., Slovic P., and A. Tversky eds. (1982), *Judgment Under Uncertainty: Heuristics and Biases*, Cambridge: Cambridge University Press.

Kahneman, D., and A. Tversky (1974), "Judgment Under Uncertainty: Heuristics and Biases," *Science* 185, 1124-31.

Kahneman, D., and A. Tversky (1979), "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica* 47, 263-91.

Kahneman, D., and A. Tversky eds. (2000), *Choices, Values and Frames*, Cambridge: Cambridge University Press.

Kaul, A., Mehrotra V., and R. Morck (2000), "Demand Curves for Stocks Do Slope Down: New Evidence from an Index Weights Adjustment," *Journal of Finance* 55, 893-912.

Knight, F. (1921), *Risk, Uncertainty and Profit*, Boston, New York: Houghton Mifflin.

Korajczyk, R., Lucas D., and R. MacDonald (1991), "The Effects of Information Releases on the Pricing and Timing of Equity Issues," *Review of Financial Studies* 4, 685-708.

La Porta, R., Lakonishok J., Shleifer A., and R. Vishny (1997), "Good News for Value Stocks: Further Evidence on Market Efficiency," *Journal of Finance* 49, 1541-1578.

Lakonishok, J., Shleifer A., and R. Vishny (1994), "Contrarian Investment, Extrapolation and Risk," *Journal of Finance* 49, 1541-1578.

Lamont, O., and R. Thaler (2002), "Can the Market Add and Subtract? Mispricing in Tech Stock Carve-Outs," forthcoming, *Journal of Political Economy.*

Lee, C., Shleifer, A., and R. Thaler (1991), "Investor Sentiment and the Closed-end Fund Puzzle," *Journal of Finance* 46, 75-110.

LeRoy S. and R. Porter (1981), "The Present-value Relation: Tests Based on Implied Variance Bounds," *Econometrica* 49, 97-113.

Lewis, K. (1999), "Trying to Explain Home Bias in Equities and Consumption," *Journal of Economic Literature* 37, 571-608.

Lintner, J. (1956), "Distribution of Incomes of Corporations among Dividends, Retained Earnings and Taxes," *American Economic Review* 46, 97-113.

Loomes G., and R. Sugden (1982), "Regret Theory: An Alternative Theory of Rational Choice Under Uncertainty," *The Economic Journal* 92, 805-824.

Lord, C., Ross L., and M. Lepper (1979), "Biased Assimilation and Attitude Polarization: The Effects of Prior Theories on Subsequently Considered Evidence," *Journal of Personality and Social Psychology* 37, 2098-2109.

Loughran, T., and J. Ritter (1995), "The New Issues Puzzle," *Journal of Finance* 50, 23-50.

Loughran, T., and J. Ritter (2000), "Uniformly Least Powerful Tests of Market Efficiency," *Journal of Financial Economics* 55, 361-389.

Loughran, T., and J. Ritter (2002), "Why Don't Issuers Get Upset About Leaving Money on the Table?" *Review of Financial Studies* 15, 413-443.

Loughran, T., Ritter J., and K. Rydqvist (1994), "Initial Public Offerings: International Insights," *Pacific Basin Finance Journal* 2, 165-199.

Lyon, J., Barber, B. and C. Tsai (1999), "Improved Methods for Tests of Long-run Abnormal Stock Returns," *Journal of Finance* 54, 165-201.

Maenhout, P. (1999), "Robust Portfolio Rules and Asset Pricing," Working paper, Insead.

Malmendier, U., and G. Tate (2001), "CEO Overconfidence and Corporate Investment," Working paper, Harvard University.

Mankiw, N.G., and S. Zeldes (1991), "The Consumption of Stockholders and Non-stockholders," *Journal of Financial Economics* 29, 97-112.

Markowitz, H. (1952), "The Utility of Wealth," *Journal of Political Economy* 60, 151-158.

Mehra, R. and E. Prescott (1985), "The Equity Premium: A Puzzle," *Journal of Monetary Economics* 15, 145-161.

Merton, R. (1987), "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance* 42 483-510.

Michaely, R., Thaler R., and K. Womack (1995), "Price Reactions to Dividend Initiations and Omissions," *Journal of Finance* 50, 573-608.

Miller, E. (1977), "Risk, Uncertainty and Divergence of Opinion," *Journal of Finance* 32, 1151-1168.

Mitchell, M., Pulvino T., and E. Stafford (2000), "Limited Arbitrage in Equity Markets," *Journal of Finance* 57, 551-584.

Mitchell, M., and E. Stafford (2001), "Managerial Decisions and Long-term Stock Price Performance," *Journal of Business* 73, 287-329.

Modigliani, F. and R. Cohn (1979), "Inflation and the Stock Market," *Financial Analysts Journal* 35, 24-44.

Morck, R., Shleifer A., and R. Vishny (1993), "The Stock Market and Investment: Is the Market a Sideshow?" Brookings Papers on Economic Activity.

Mullainathan (2000), "Thinking Through Categories," Working paper, MIT.

Odean, T. (1998), "Are Investors Reluctant to Realize their Losses?" *Journal of Finance* 53, 1775-1798.

Odean, T. (1999), "Do Investors Trade Too Much?" *American Economic Review* 89, 1279-1298.

Ofek, E., and M. Richardson (2001), "Dot-com Mania: Market Inefficiency in the Internet Sector," Working paper, New York University.

Pagano, M., Panetta F., and L. Zingales (1998), "Why do Companies Go Public? An Empirical Analysis," *Journal of Finance* 53, 27-64.

Polk, C., and P. Sapienza (2001), "The Real Effects of Investor Sentiment," Working paper, Northwestern University.

Poteshman, A. (2001), "Underreaction, Overreaction and Increasing Misreaction to Information in the Options Market," *Journal of Finance* 56, 851-876.

Quiggin, J. (1982), "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization* 3, 323-43.

Rabin, M., (1998), "Psychology and Economics," *Journal of Economic Literature* 36, 11-46.

Rabin, M. (2000), "Risk Aversion and Expected Utility Theory: A Calibration Theorem," *Econometrica* 68, 1281-1292.

Rabin, M., (2002), "Inference By Believers in the Law of Small Numbers," *Quarterly Journal of Economics* 117, 775-816.

Redelmeier D. and A. Tversky (1992), "On the Framing of Multiple Prospects," *Psychological Science* 3, 191-193.

Ritter, J., and R. Warr (2002), "The Decline of Inflation and the Bull Market of 1982 to 1997," *Journal of Financial and Quantitative Analysis* 37, 29-61.

Roll, R. (1977), "A Critique of the Asset Pricing Theory's Tests: Part I," *Journal of Financial Economics* 4, 129-174.

Roll, R. (1983), "Vas ist Das?" *Journal of Portfolio Management* 9, 18-28.

Roll, R. (1986), "The Hubris Hypothesis of Corporate Takeovers," *Journal of Business* 59, 197-216.

Rosenberg, B., Reid, K., and R. Lanstein (1985), "Persuasive Evidence of Market Inefficiency," *Journal of Portfolio Management* 11, 9-17.

Ross, S. (2001), *Lectures Notes on Market Efficiency*, Sloan School of Management.

Rouwenhorst, G. (1998), "International Momentum Strategies," *Journal of Finance* 53, 267-284.

Rubinstein, M. (2001), "Rational Markets: Yes or No? The Affirmative Case," *Financial Analysts Journal* (May-June), 15-29.

Santos, M. and M. Woodford (1997), "Rational Asset Pricing Bubbles," *Econometrica* 65, 19-58.

Sargent T. (1993), *Bounded Rationality in Macroeconomics*, Oxford: Oxford University Press.

Savage, L. (1964), *The Foundations of Statistics*, New York: Wiley.

Scheinkman J. and W. Xiong (2001), "Overconfidence and Speculative Bubbles," Working paper, Princeton University.

Segal, U. (1987), "Some Remarks on Quiggin's Anticipated Utility," *Journal of Economic Behavior and Organization* 8, 145-154.

Segal, U. (1989), "Anticipated Utility: A Measure Representation Approach," *Annals of Operations Research* 19, 359-73.

Shafir, E., Diamond, P. and A. Tversky (1997), "Money Illusion," *Quarterly Journal of Economics* 112, 341-374.

Shefrin, H. and M. Statman (1984), "Explaining Investor Preference for Cash Dividends," *Journal of Financial Economics* 13, 253-282.

Shefin, H., and M. Statman (1985), "The Disposition to Sell Winners too Early and Ride Losers too Long," *Journal of Finance* 40, 777-790.

Shiller, R. (1981), "Do Stock Prices Move too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review* 71, 421-436.

Shiller, R. (1984), "Stock Prices and Social Dynamics," *Brookings Papers on Economic Activity* 2, 457-498.

Shleifer, A. (1986), "Do Demand Curves for Stocks Slope Down?" *Journal of Finance* 41, 579-90.

Shleifer, A. (2000), *Inefficient Markets: An Introduction to Behavioral Finance*, Oxford: Oxford University Press.

Shleifer, A., and L. Summers (1990), "The Noise Trader Approach to Finance," *Journal of Economic Perspectives* 4, 19-33.

Shleifer, A., and R. Vishny (1997), "The Limits of Arbitrage," *Journal of Finance* 52, 35-55.

Shleifer, A., and R. Vishny (2001), "Stock Market Driven Acquisitions," Working paper, Harvard University.

Stein, J. (1996), "Rational Capital Budgeting in an Irrational World," *Journal of Business*

69, 429-55.

Summers, L. (1986), "Does the Stock Market Rationally Reflect Fundamental Values?" *Journal of Finance* 41, 591-601.

Thaler, R. (1999), "Mental Accounting Matters," in D. Kahneman and A. Tversky, eds., *Choice, Values and Frames*, Cambridge: Russell Sage Foundation.

Thaler, R., and E. Johnson (1990), "Gambling with the House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice," *Management Science* 36, 643-660.

Thaler, R., Tversky A., Kahneman D., and A. Schwartz (1997), "The Effect of Myopia and Loss Aversion on Risk-Taking: An Experimental Test," *Quarterly Journal of Economics* 112, 647-661.

Tversky, A., and D. Kahneman (1986), "Rational Choice and the Framing of Decisions," *Journal of Business* 59, 251-78.

Tversky, A., and D. Kahneman (1992), "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297-323.

Veronesi, P. (1999), "Stock Market Overreaction to Bad News in Good Times: A Rational Expectations Equilibrium Model," *Review of Financial Studies* 12, 975-1007.

Vijh, A. (1994), "S&P 500 Trading Strategies and Stock Betas," *Review of Financial Studies* 7, 215-251.

Von Neumann, J., and O. Morgenstern (1944), *Theory of Games and Economic Behavior*, Princeton: Princeton University Press.

Vuolteenaho T. (2002), "What Drives Firm-level Stock Returns?" *Journal of Finance* 57, 233-264.

Weil, P. (1989), "The Equity Premium Puzzle and the Risk-free Rate Puzzle," *Journal of Monetary Economics* 24, 401-421.

Weinstein, N. (1980), "Unrealistic Optimism about Future Life Events" *Journal of Personality and Social Psychology* 39, 806-820.

Wurgler, J., and K. Zhuravskaya (2002), "Does Arbitrage Flatten Demand Curves for Stocks?", forthcoming, *Journal of Business.*

Yaari, M. (1987), "The Dual Theory of Choice Under Risk," *Econometrica* 55, 95-115.

Table 1: Arbitrage costs and risks that arise in exploiting mispricing: fundamental risk (FR), noise trader risk (NTR) and implementation costs (IC).

	FR	NTR	IC
Royal Dutch/Shell	$\times$	$\surd$	$\times$
Index Inclusions	$\surd$	$\surd$	$\times$
Palm/3-Com	$\times$	$\times$	$\surd$

Table 2: Parameter values for a simple consumption-based model.

Parameter	
g_C	1.84%
σ_C	3.79%
g_D	1.5%
σ_D	12.0%
ω	0.15
γ	1.0
ρ	0.98

Figure 1. Log deviations from Royal Dutch/Shell parity. Source: Froot and Dabora (1999).

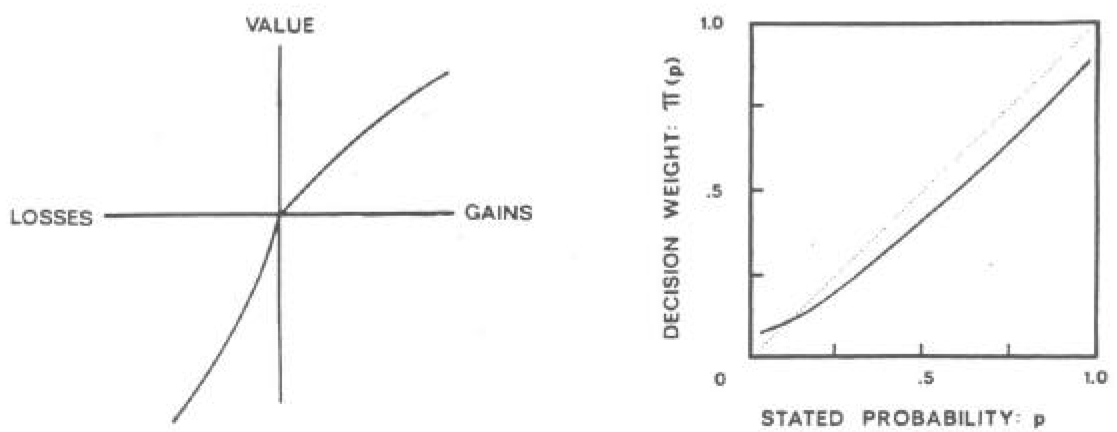

Figure 2. The two panels show Kahneman and Tversky's (1979) proposed value function v and probability weighting function π.

Human Behavior and the Efficiency of the Financial System

by

Robert J. Shiller*

Abstract

Recent literature in empirical finance is surveyed in its relation to underlying behavioral principles, principles which come primarily from psychology, sociology and anthropology. The behavioral principles discussed are: prospect theory, regret and cognitive dissonance, anchoring, mental compartments, overconfidence, over- and underreaction, representativeness heuristic, the disjunction effect, gambling behavior and speculation, perceived irrelevance of history, magical thinking, quasi-magical thinking, attention anomalies, the availability heuristic, culture and social contagion, and global culture.

Theories of human behavior from psychology, sociology, and anthropology have helped motivate much recent empirical research on the behavior of financial markets. In this paper I will survey both some of the most significant theories (for empirical finance) in these other social sciences and the empirical finance literature itself.

Particular attention will be paid to the implications of these theories for the efficient markets hypothesis in finance. This is the hypothesis that financial prices efficiently incorporate all public information and that prices can be regarded as optimal estimates of true investment value at all times. The efficient markets hypothesis in turn is based on more primitive notions that people behave rationally, or accurately maximize expected utility, and are able to process all available information. The idea behind the term "efficient markets hypothesis," a term coined by Harry Roberts (1967),[1] has a long history in financial research, a far longer history than the term itself has. The hypothesis (without the words

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[1]The Roberts (1967) paper has never been published; the fame of his paper apparently owes to the discussion of it in Fama (1970).

efficient markets) was given a clear statement in Gibson (1889), and has apparently been widely known at least since then, if not long before. All this time there has also been tension over the hypothesis, a feeling among many that there is something egregiously wrong with it; for an early example, see MacKay (1841). In the past couple of decades the finance literature, has amassed a substantial number of observations of apparent anomalies (from the standpoint of the efficient markets hypothesis) in financial markets. These anomalies suggest that the underlying principles of rational behavior underlying the efficient markets hypothesis are not entirely correct and that we need to look as well at other models of human behavior, as have been studied in the other social sciences.

The organization of this paper is different from that of other accounts of the literature on behavioral finance (for example, De Bondt and Thaler, 1996 or Fama, 1997): this paper is organized around a list of theories from the other social sciences that are used by researchers in finance, rather than around a list of anomalies. I organized the paper this way because, in reality, most of the fundamental principles that we want to stress here really do seem to be imported from the other social sciences. No surprise here: researchers in these other social sciences have done most of the work over the last century on understanding less-than-perfectly-rational human behavior. Moreover, each anomaly in finance typically has more than one possible explanation in terms of these theories from the other social sciences. The anomalies are observed in complex real world settings, where many possible factors are at work, not in the experimental psychologist's laboratory. Each of their theories contributes a little to our understanding of the anomalies, and there is typically no way to quantify or prove the relevance of any one theory. It is better to set forth the theories from the other social sciences themselves, describing when possible the controlled experiments that demonstrate their validity, and give for each a few illustrations of applications in finance.

Before beginning, it should be noted that theories of human behavior from these other social sciences often have underlying motivation that is different from that of economic theories. Their theories are often intended to be robust to application in a variety of everyday, unstructured experiences, while the economic theories are often intended to be robust in the different sense that, even if the problems the economic agents face become very clearly defined, their behavior will not change after they learn how to solve the problems. Many of the underlying behavioral principles from psychology and other social sciences that are discussed below are unstable and the hypothesized behavioral phenomena may disappear when the situation becomes better structured and people have had a lot of opportunity to learn about it. Indeed, there are papers in the psychology literature claiming that many of the cognitive biases in human judgment under uncertainty uncovered by experimental psychologists will disappear when the experiment is changed so that the probabilities and issues that the experiment raises are explained clearly enough to subjects (see, for example, Gigerenzer, 1991). Experimental subjects can in many cases be convinced, if given proper instruction, that their initial behavior in the experimental situation was irrational, and they will then correct their ways.

To economists, such evidence is taken to be more damning to the theories than it would be by the social scientists in these other disciplines. Apparently economists at large have not fully appreciated the extent to which enduring patterns can be found in this 'unstable' human behavior. The examples below of application of theories from other social sciences to understanding anomalies in financial markets will illustrate.

Each section below, until the conclusion, refers to a theory taken from the literature in psychology, sociology or anthropology. The only order of these sections is that I have placed first theories that seem to have the more concrete applications in finance, leaving some more impressionistic applications to the end. In the conclusion I attempt to put these theories into perspective, and to recall that there are also important strengths in conventional economic theory and in the efficient markets hypothesis itself.

Prospect Theory

Prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) has probably had more impact than any other behavioral theory on economic research. Prospect theory is very influential despite the fact that it is still viewed by much of the economics profession at large as of far less importance than expected utility theory. Among economists, prospect theory has a distinct, though still prominent, second place to expected utility theory for most research.

I should say something first about the expected utility theory that still retains the position of highest honor in the pantheon of economic tools. It has dominated much economic theory so long because the theory offers a parsimonious representation of truly rational behavior under uncertainty. The axioms (Savage, 1954) from which expected utility theory is derived are undeniably sensible representations of basic requirements of rationality. For many purposes, it serves well to base an economic theory on such assumptions of strictly rational behavior, especially if the assumptions of the model are based on simple, robust realities, if the model concerns well-considered decisions of informed people, and if the phenomenon to be explained is one of stable behavior over many repetitions, where learning about subtle issues has a good chance of occurring.

Still, despite the obvious attractiveness of expected utility theory, it has long been known that the theory has systematically mispredicted human behavior, at least in certain circumstances. Allais (1953) reported examples showing that in choosing between certain lotteries, people systematically violate the theory. Kahneman and Tversky (1979) give the following experimental evidence to illustrate one of Allais' examples. When their subjects were asked to choose between a lottery offering a 25% chance of winning 3,000 and a lottery offering a 20% chance of winning 4,000, 65% of their subjects chose the latter, while when subjects were asked to choose between a 100% chance of winning 3,000 and an 80% chance of winning 4,000, 80% chose the former. Expected utility theory predicts that they should not choose differently in these two cases, since the second choice is the same as the first except that all probabilities are multiplied by the same constant. Their preference for the first choice in the lottery when it is certain in this example illustrates what is called the "certainty effect," a preference for certain outcomes.

Prospect theory is a mathematically-formulated alternative to the theory of expected utility maximization, an alternative that is supposed to capture the results of such experimental research. (A prospect is the Kahneman–Tversky name for a lottery as in the

Allais example above.) Prospect theory actually resembles expected utility theory in that individuals are represented as maximizing a weighted sum of "utilities," although the weights are not the same as probabilities and the "utilities" are determined by what they call a "value function" rather than a utility function.

The weights are, according to Kahneman and Tversky (1979) determined by a function of true probabilities which gives zero weight to extremely low probabilities and a weight of one to extremely high probabilities. That is, people behave as if they regard extremely improbable events as impossible and extremely probable events as certain. However, events that are just very improbable (not extremely improbable) are given too much weight; people behave as if they exaggerate the probability. Events that are very probable (not extremely probable) are given too little weight; people behave as if they underestimate the probability. What constitutes an extremely low (rather than very low) probability or an extremely high (rather than very high) probability is determined by individuals' subjective impression and prospect theory is not precise about this. Between the very low and very high probabilities, the weighting function (weights as a function of true probabilities) has a slope of less than one.

This shape for the weighting function allows prospect theory to explain the Allais certainty effect noted just above. Since the 20% and 25% probabilities are in the range of the weighting function where its slope is less than one, the weights people attach to the two outcomes are more nearly equal than are the probabilities, and people tend just to choose the lottery that pays more if it wins. In contrast, in the second lottery choice the 80% probability is reduced by the weighting function while the 100% probability is not; the weights people attach to the two outcomes are more unequal than are the probabilities, and people tend just to choose the outcome that is certain.

If we modify expected utility function only by substituting the Kahneman and Tversky weights for the probabilities in expected utility theory, we might help explain a number of puzzling phenomena in observed human behavior toward risk. For a familiar example, such a modification could explain the apparent public enthusiasm for high-prize lotteries, even though the probability of winning is so low that expected payout of the lottery is not high. It could also explain such phenomenon as the observed tendency for overpaying for airline flight insurance (life insurance policies that one purchases before an airline flight, that has coverage only during that flight), Eisner and Strotz (1961).

The Kahneman–Tversky weighting function may explain observed overpricing of out-of-the-money and in-the-money options. Much empirical work on stock options pricing has uncovered a phenomenon called the "options smile" (see Mayhew, 1995, for a review.). This means that both deep out-of-the-money and deep in-the-money options have relatively high prices, when compared with their theoretical prices using Black–Scholes formulae, while near-the-money options are more nearly correctly priced. Options theorists, accustomed to describing the implied volatility of the stock implicit in options prices, like to state this phenomenon not in terms of option prices but in terms of these implied volatilities. When the implied volatility for options of various strike prices at a point in time derived using the Black–Scholes (1973) formula are plotted, on the vertical axis, against the strike price on the horizontal axis, the curve often resembles a smile. The curve is higher both for low strike price (out-of-the-money) options and for high strike price (in-the-money) options than it is for middle-range strike prices. This options smile might possibly be explained in terms of the distortion in probabilities represented by the Kahneman–Tversky weighting function, since the theory would suggest that people act as if they overestimate the small probability that the price of the underlying crosses the strike price and underestimate the high probability that the price remains on the same side of the strike price. The Kahneman–Tversky weighting function might even explain the down-turned corners of the mouth that some smiles exhibit (see Fortune, 1996) if at these extremes the discontinuities at the extremes of the weighting function become relevant.[2]

We now turn to the other foundation of prospect theory, the Kahneman and Tversky (1979) value function. The value function differs from the utility function in expected utility theory in a very critical respect: the function (of wealth or payout) has a kink in it at a point, the "reference point," the location of which is determined by the subjective impressions of the individual. The reference point is the individual's point of comparison, the "status quo" against which alternative scenarios are contrasted. Taking value as a function of wealth, the Kahneman–Tversky (1979) value function is upward sloping everywhere, but with an abrupt decline in slope at the reference point (today's wealth or whatever measure of wealth that is psychologically important to the subject). For wealth levels above the reference point, the value function is concave downward, just as are conventional utility functions. At the reference point, the value function may be regarded, from the fact that its slope changes abruptly there, as infinitely concave downward. For wealth levels below the reference point, Kahneman and Tversky found evidence that the value function is concave upward, not downward. People are risk lovers for losses, they asserted.

Perhaps the most significant thing to notice about the Kahneman–Tversky value function is just the discontinuity in slope at the reference value, the abrupt downward change in slope as one moves upward past the reference value. Prospect theory does not nail down accurately what determines the location of the reference point, just as it does not nail down accurately, for the weighting function, what is the difference between very high probabilities and extremely high probabilities. The theory does not specify these matters because experimental evidence has not produced any systematic patterns of behavior that can be codified in a general theory. However, the reference point is thought to be determined by some point of comparison that the subject finds convenient, something readily visible or suggested by the wording of a question.

This discontinuity means that, in making choices between risky outcomes, people will behave in a risk averse manner, no matter how small the amounts at stake are. This is a contrast to the prediction of expected utility theory with a utility function of wealth without

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[2]There are other potential explanations of the options smile in terms of nonnormality or jump processes for returns, and these have received the attention in the options literature. Such explanations might even provide a complete rational basis for the smile, though it is hard to know for sure. Since the 1987 stock market crash, the options smile has usually appeared distorted into an options "leer," with the left side of the mouth higher (e.g., the deep out-of-the-money puts are especially overpriced), see Bates (1995), Jackwerth and Rubinstein (1995) and Bates (1991). Public memories of the 1987 crash are apparently at work in producing this "leer."

kinks, for which, since the utility function is approximately linear for small wealth changes, people should behave as if they are risk neutral for small bets. That people would usually be risk neutral for small bets would be the prediction of expected utility theory even if the utility function has such a slope discontinuity, since the probability that wealth is currently at the kink is generally zero. With prospect theory, in contrast, the kink always moves with wealth to stay at the perceived current level of wealth (or the current point of reference); the kink is always relevant.

Samuelson (1963) told a story which he perceived as demonstrating a violation of expected utility theory, and, although it came before Kahneman and Tversky's prospect theory, it illustrates the importance of the kink in the value function. Samuelson reported that he asked a lunch colleague whether he would accept a bet that paid him $200 with a probability of .5 and lost him $100 with a probability of .5. The colleague said he would not take the bet, but that he would take a hundred of them. With 100 such bets, his expected total winnings are $5,000 and he has virtually no chance of losing any money. It seems intuitively compelling to many people that one would readily take the complete set of bets, even if any element of the set is unattractive. Samuelson proved that if his colleague would answer the same way at any wealth level, then he necessarily violates expected utility theory.

Samuelson's colleague is not, however, in violation of prospect theory. When viewing a single bet, the kink in the value function is the dominant consideration. If he were to judge 100 bets sequentially, the kink would always be relevant (the reference point would move with each successive bet) and he would reject all of them. But if he were to judge 100 bets together, the collective outcomes would be far above today's value function kink, and the bet is, by prospect theory, clearly desirable.

The failures to accept many such bets when one considers them individually has been called "myopic loss aversion" by Benartzi and Thaler (1995). They argue that, under estimated values for the magnitude of the kink in the Kahneman–Tversky value function, the "equity premium puzzle" of Mehra and Prescott (1985) can be resolved; see also Siegel and Thaler (1997).

Today, the term "equity premium puzzle," coined by Mehra and Prescott (1985), is widely used to refer to the puzzlingly high historical average returns of stocks relative to bonds.[3] The equity premium is the difference between the historical average return in the stock market and the historical average return on investments in bonds or treasury bills. According to Siegel (1994), the equity premium of U.S. stocks over short-term government bonds has averaged 6.1% a year for the United States for 1926 to 1992, and so one naturally

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[3]Mehra and Prescott did not discover the equity premium. Perhaps that honor should go to Smith (1925), although there must be even earlier antecedents in some forms. Mehra and Prescott's original contribution seems to have been, in the context of present-value investor intertemporal optimizing models, to stress that the amount of risk aversion that would justify the equity premium, given the observed correlation of stocks with consumption, would imply much higher riskless interest rates than we in fact see.

wonders why people invest at all in debt if it is so outperformed by stocks.[4] Those who have tried to reconcile the equity premium with rational investor behavior commonly point out the higher risk that short-run stock market returns show: investors presumably are not fully enticed by the higher average returns of stocks since stocks carry higher risk. But, such riskiness of stocks is not a justification of the equity premium, at least assuming that investors are mostly long term. Most investors ought to be investing over decades, since most of us expect to live for many decades, and to spend the twilight of their lives living off savings. Over long periods of times, it has actually been long-term bonds (whose payout is fixed in nominal terms), not the stocks, that have been more risky in real terms, since the consumer price index has been, despite its low variability from month to month, very variable over long intervals of time, see Siegel (1994). Moreover, stocks appear strictly to dominate bonds: there is no thirty-year period since 1871 in which a broad portfolio of stocks was outperformed either by bonds or treasury bills.[5]

Benartzi and Thaler show (1995) that if people use a one-year horizon to evaluate investments in the stock market, then the high equity premium is explained by myopic loss aversion. Moreover, prospect theory does not suggest that in this case riskless real interest rates need be particularly high. Thus, if we accept prospect theory and that people frame stock market returns as short-term, the equity premium puzzle is solved.

Benartzi and Thaler (1996) demonstrated experimentally that when subjects are asked to allocate their defined contribution pension plans between stocks and fixed incomes, their responses differed sharply depending on how historical returns were presented to them. If they were shown 30 one-year returns, their median allocation to stocks was 40%, but if they were shown 30-year returns their median allocation to stocks was 90%. Thaler, Tversky, Kahneman and Schwartz (1997) shows further experiments confirming this response.

Loss aversion has also been used to explain other macroeconomic phenomena, savings behavior (Bowman, Minehart and Rabin, 1993) and job search behavior (Bryant, 1990).

Regret and Cognitive Dissonance

There is a human tendency to feel the pain of regret at having made errors, even small errors, not putting such errors into a larger perspective. One "kicks oneself" at having done something foolish. If one wishes to avoid the pain of regret, one may alter one's behavior in ways that would in some cases be irrational unless account is taken of the pain of regret.

The pain of regret at having made errors is in some senses embodied in the Kahneman–

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[4]Siegel (1994, p. 20). However, Siegel notes that the U.S. equity premium was only 1.9% per year 1816–70 and 2.8% per year 1871–1925.

[5]Siegel (1994, p. 31). It should be noted that one must push the investor horizon up to a fairly high number, around 30 years, before one finds that historically stocks have always outperformed bonds since 1871; for ten year periods of time one finds that bonds often outperform stocks. There are not many thirty-year periods in stock market history, so this information might be judged as insubstantial. Moreover, Siegel notes that even with a thirty-year period stocks did not always outperform bonds in the U.S. before 1871.

Tversky notion of a kink in the value function at the reference point. There are also other ways of representing how people behave who feel pain of regret. Loomes and Sugden (1982) have suggested that people maximize the expected value of a “modified utility function” which is a function of the utility they achieve from a choice as well as the utility they would have achieved from another choice that was considered. Bell (1982) proposed a similar analysis.

Regret theory may apparently help explain the fact that investors defer selling stocks that have gone down in value and accelerate the selling of stocks that have gone up in value, Shefrin and Statman (1985). Regret theory may be interpreted as implying that investors avoid selling stocks that have gone down in order not to finalize the error they make and not to feel the regret. They sell stocks that have gone up in order that they cannot regret failing to do so before the stock later fell, should it do so. That such behavior exists has been documented using volume of trade data by Ferris, Haugen and Makhija (1988) and Odean (1996b).

Cognitive dissonance is the mental conflict that people experience when they are presented with evidence that their beliefs or assumptions are wrong; as such, cognitive dissonance might be classified as a sort of pain of regret, regret over mistaken beliefs. As with regret theory, the theory of cognitive dissonance (Festinger, 1957) asserts that there is a tendency for people to take actions to reduce cognitive dissonance that would not normally be considered fully rational: the person may avoid the new information or develop contorted arguments to maintain the beliefs or assumptions. There is empirical support that people often make the errors represented by the theory of cognitive dissonance. For example, in a classic study, Erlich, Guttman, Schopenback and Mills (1957) showed that new car purchasers selectively avoid reading, after the purchase is completed, advertisements for car models that they did not choose, and are attracted to advertisements for the car they chose.

McFadden (1974) modelled the effect of cognitive dissonance in terms of a probability of forgetting contrary evidence and showed how this probability will ultimately distort subjective probabilities. Goetzmann and Peles (1993) have argued that the same theory of cognitive dissonance could explain the observed phenomenon that money flows in more rapidly to mutual funds that have performed extremely well than flows out from mutual funds that have performed extremely poorly: investors in losing funds are unwilling to confront the evidence that they made a bad investment by selling their investments.

Anchoring

It is well-known that when people are asked to make quantitative assessments their assessments are influenced by suggestions. An example of this is found in the results survey researchers obtain. These researchers often ask people about their incomes using questionnaires in which respondents are instructed to indicate which of a number of income brackets, shown as choices on the questionnaire, their incomes fall into. It has been shown that the answers people give are influenced by the brackets shown on the questionnaire. The tendency to be influenced by such suggestions is called “anchoring” by psychologists.

In some cases, at least, anchoring may be rational behavior for respondents. They may rationally assume that the deviser of the questionnaire uses some information (in this case, about typical people's incomes) when devising the questionnaire. Not fully remembering their own income, they may rely on the information in the brackets to help them answer better. If the brackets do contain information, then it is rational for subjects to allow themselves to be influenced by the brackets.

While anchoring undoubtedly has an information-response component in many circumstances, it has also been shown that anchoring behavior persists even when information is absent. In one experiment Tversky and Kahneman (1974), subjects were given simple questions whose answers were in percentages, e.g., the percentage of African nations in the United Nations. A wheel of fortune with numbers from 1 to 100 was spun before the subjects. Obviously, the number at which the wheel of fortune stopped had no relevance to the question just asked. Subjects were asked whether their answer was higher or lower than the wheel of fortune number, and then to give their own answer. Respondents' answers were strongly influenced by the "wheel of fortune." For example, the median estimates of the percentage of African countries in the United Nations were 25 and 45 for groups that received 10 and 65, respectively, as starting points (p. 184).

Values in speculative markets, like the stock market, are inherently ambiguous. Who would know what the value of the Dow Jones Industrial Average should be? Is it really "worth" 6,000 today? Or 5,000 or 7,000? or 2,000 or 10,000? There is no agreed-upon economic theory that would answer these questions. In the absence of any better information, past prices (or asking prices or prices of similar objects or other simple comparisons) are likely to be important determinants of prices today.

That anchoring affects valuations, even by experts, was demonstrated by Northcraft and Neale (1987) in the context of real estate valuation. All subjects were taken to a house for sale, asked to inspect the house for up to 20 minutes, and were given a ten-page packet of information about the house and about other houses in the area, giving square footage and characteristics of the properties, and prices of the other properties. The same packet was given to all subjects except that the asking price of the property under consideration and its implied price per square foot were changed between subjects. Subjects were asked for their own opinions of its appraisal value, appropriate listing price, purchase price, and the lowest offer the subject would accept for the house if the subject were the seller. The real estate agents who were given an asking price of $119,900 had a mean predicted appraisal value of $114,204, listing price of $117,745, purchase price of $111,454 and a lowest acceptable offer of $111,136, while the real estate agents who were given an asking price of $149,900 had a mean appraisal value of $128,754, listing price of $130,981, predicted purchase price of $127,318, and a lowest offer of $123,818. The changed asking prices thus swayed their valuations by 11% to 14% of the value of the house. Similar results were found with amateur subjects. While this experiment does not rule out that the effect of the asking price was due to a rational response to the assumed information in the asking price, the effects of asking price are remarkably large, given that so much other information on the house was also given. Moreover, when subjects were asked afterwards to list the items of information that weighed most heavily in their valuations, only 8% of the expert subjects and only 9% of the amateur subjects listed asking price of the property under consideration among the top three items. Note that the valuation problem presented to these subjects is far less difficult or ambiguous than the problem of determining the "correct" value for the stock market, since here they are implicitly being asked to assume that the comparable properties are correctly valued. (See also McFadden, 1974 and Silberman and Klock, 1989.)

One might object that the notion that anchoring on past prices helps determine present price in the stock market might be inconsistent with the low serial correlation of stock price changes, that is with the roughly random-walk behavior of daily or monthly stock prices that has been widely noted.[6] This conclusion is not warranted however. Models of "smart money" (i.e., people who are unusually alert to profit opportunities in financial markets) seeking to exploit serial correlation in price, models which also include ordinary investors, are consistent with the implications that serial correlation is low and yet the anchoring remains important for the level of stock prices (see Shiller, 1984, 1990).

By extension from these experimental results, it is to be presumed that very many economic phenomena are influenced by anchoring. Gruen and Gizycki (1993) used it to explain the widely observed anomaly[7] that forward discounts to not properly explain subsequent exchange rate movements. The anchoring phenomenon would appear relevant to the "sticky prices" that are so talked about by macroeconomists. So long as past prices are taken as suggestions of new prices, the new prices will tend to be close to the past prices. The more ambiguous the value of a commodity, the more important a suggestion is likely to be, and the more important anchoring is likely to be for price determination.

The anchoring phenomenon may help to explain certain international puzzles observed in financial markets. U.S. investors who thought in the late 1980s that Japanese stock price–earnings ratios were outrageously high then may have been influenced by the readily-available anchor of (much lower) U.S. price–earnings ratios. By the mid 1990s, many U.S. investors feel that the Tokyo market is no longer overpriced (see Shiller, Kon-Ya and Tsutsui, 1996), even though price–earnings ratios remain much higher than in the U.S. perhaps because the anchor of the widely-publicized high Tokyo price–earnings ratios of the late 1980s appears to be another anchor.

Anchoring may also be behind certain forms of money illusion. The term money illusion, introduced by Fisher (1928), refers to a human tendency to make inadequate allowance, in economic decisions, for the rate of inflation, and to confuse real and nominal quantities. Shafir, Diamond and Tversky (1997) have shown experimentally that people tend to give different answers to the same hypothetical decision problem depending on whether the problem was presented in a way that stressed nominal quantities or in a way that

[6]The notion that speculative prices approximately describe "random walks" was first proposed by Bachelier (1900, 1964). It became widely associated with the efficient markets hypothesis, the hypothesis that market prices efficiently incorporate all available information, with the work of Fama (1970). For further information on the literature on the random walk and efficient markets theory see also Cootner (1964), Malkiel (1981), and Fama (1991).

[7]For a discussion of the anomaly, see Backus, Foresi and Telmer (1995) and Froot and Thaler (1990).

stressed real quantities. The quantities that were shown in the question (whether nominal or real) may have functioned as anchors.[8]

Mental Compartments

Related to the anchoring and framing phenomena is a human tendency to place particular events into mental compartments based on superficial attributes. Instead of looking at the big picture, as would be implied by expected utility theory, they look at individual small decisions separately.

People may tend to place their investments into arbitrarily separate mental compartments, and react separately to the investments based on which compartment they are in. Shefrin and Statman (1994) have argued that individual investors think naturally in terms of having a "safe" part of their portfolio that is protected from downside risk and a risky part that is designed for a chance of getting rich. Shefrin and Thaler (1988) have argued that people put their sources of income into three categories, current wage and salary income, asset income, and future income, and spend differently out of the present values of these different incomes. For example, people are reluctant to spend out of future income even if it is certain to arrive.

The tendency for people to allow themselves to be influenced by their own mental compartments might explain the observed tendency for stock prices to jump up when the stock is added to the Standard and Poor Stock Index (see Shleifer, 1986). It might also help explain the widely noted "January effect" anomaly. This anomaly, that stock prices tend to go up in January, has been observed in as many as 15 different countries (Gultekin and Gultekin, 1983). The anomaly cannot be explained in terms of effects related to the tax year, since it persists also in Great Britain (whose tax year begins in April) and Australia (whose tax year begins in July), see Thaler (1987). If people view the year end as a time of reckoning and a new year as a new beginning, they may be inclined them to behave differently at the turn of the year, and this may explain the January effect.

A tendency to separate out decisions into separate mental compartments may also be behind the observed tendency for hedgers to tend to hedge specific trades, rather than their overall profit situation. René Stulz (1996, p. 8), in summarizing the results of his research and that of others on the practice of risk management by firms, concludes that:

> It immediately follows from the modern theory of risk management that one should be concerned about factors that affect the present value of future cash flows. This is quite different from much of the current practice of risk management where one is concerned about hedging transaction risk or the risk of transactions expected to occur in the short run.

[8]There appears to be much more to money illusion than just anchoring; people associate nominal quantities with opinions about the economy, anticipated behavior of the government, fairness, and prestige, opinions that are not generally shared by economists, see Shiller (1997a,b).

The Wharton/CIBC Wood Gundy 1995 Survey of Derivatives Usage by U.S. Non-Financial Firms (Bodnar and Marston, 1996) studied 350 firms: 176 firms in the manufacturing sector, 77 firms in the primary products sector, and 97 firms in the service sector. When asked by the Wharton surveyors what was the most important objective of hedging strategy, 49% answered managing "volatility in cashflows," 42% answered managing "volatility in accounting earnings," and only 8% answered managing "the market value of the firm" (1% answered "managing balance sheet accounts and ratios"). Fifty percent of the respondents in the survey reported frequently hedging contractual commitments, but only 8% reported frequently hedging competitive/economic exposure.

It is striking that only 8% reported that their most important objective is the market value of the firm, since maximizing the market value of the firm is, by much financial theory, the ultimate objective of the management of the firm. It is of course hard to know just what people meant by their choices of answers, but there is indeed evidence that firms are driven in their hedging by the objective of hedging specific near-term transactions, and neglect consideration of future transactions or other potential factors that might also pose longer run risks to the firm. In the Wharton study, among respondents hedging foreign currency risks, 50% reported hedging anticipated transactions less than one year off, but only 11% report frequently hedging transactions more than one year off. This discrepancy is striking, since most of the value of the firm (and most of the concerns it has about its market value) must come in future years, not the present year.[9]

Overconfidence, Over- and Under-Reaction and the Representativeness Heuristic

People often tend to show, in experimental settings, excessive confidence about their own judgments. Lichtenstein, Fischhoff and Philips (1977) asked subjects to answer simple factual questions (e.g., "Is Quito the capital of Ecuador?") and then asked them to give the probability that their answer was right: subjects tended to overestimate the probability that they were right, in response to a wide variety of questions.

Such studies have been criticized (see Gigerenzer, 1991) as merely reflecting nothing more than a difference between subjective and frequentist definitions of probability, i.e., critics claimed that individuals were simply reporting a subjective degree of certainty, not the fraction times they are right in such circumstances. However, in reaction to such criticism, Fischhoff, Slovic and Lichtenstein (1977) repeated the experiments asking the

[9]Recent surveys of hedging behavior of firms indicates that despite extensive development of derivative products, actual use of these products for hedging is far from optimal. Of the firms cited in the Wharton/study, only 40.5% reported using derivatives at all. On the other hand, Dolde (1993) surveyed 244 Fortune 500 companies and concluded that over 85% used swaps, forwards, futures or options in managing financial risk. Nance, Smith and Smithson (1993) in a survey of 194 firms reported that 62% used hedging instruments in 1986. These studies concentrated on rather larger companies than did the Wharton study. Overall, these studies may be interpreted as revealing a surprisingly low fraction of respondents who do any hedging, given that firms are composed of many people, any one of whom might be expected to initiate the use of derivatives.

subjects for probability odds that they are right and very clearly explaining what such odds mean, and even asking them to stake money on their answer. The overconfidence phenomenon persisted. Moreover, in cases where the subjects said they were certain they were right, they were in fact right only about 80% of the time: there is no interpretation of subjective probability that could reconcile this result with correct judgments.

A tendency towards overconfidence among ordinary investors seems apparent when one interviews them. One quickly hears what seem to be overconfident statements. But how can it be that people systematically are so overconfident? Why wouldn't people learn from life's experiences to correct their overconfidence?

Obviously, people do learn substantially in circumstances when the consequences of their errors are repeatedly presented to them, and sometimes they even overreact and show too little confidence. But still there seems to be a common bias towards overconfidence. Overconfidence is apparently related to some deep-set psychological phenomena: Ross (1987) argues that much overconfidence is related to a broader difficulty with "situational construal," a difficulty in making adequate allowance for the uncertainty in one's own view of the broad situation, a more global difficulty tied up with multiple mental processes. Overconfidence may also be traced to the "representativeness heuristic," Tversky and Kahneman (1974), a tendency for people to try to categorize events as typical or representative of a well-known class, and then, in making probability estimates, to overstress the importance of such a categorization, disregarding evidence about the underlying probabilities.[10] One consequence of this heuristic is a tendency for people to see patterns in data that is truly random, to feel confident, for example, that a series which is in fact a random walk is not a random walk.[11]

Overconfidence itself does not imply that people overreact (or underreact) to all news. In fact, evidence on the extent of overreaction or underreaction of speculative asset prices to news has been mixed.

There has indeed been evidence of overreaction. The first substantial statistical evidence for what might be called a general market overreaction can be found in the literature on excess volatility of speculative asset prices, Shiller (1979, 1981a,b) and LeRoy and Porter (1981). We showed statistical evidence that speculative asset prices show persistent deviations from the long-term trend implied by the present-value efficient markets model, and then, over horizons of many years, to return to this trend. This pattern of price behavior, it was argued, made aggregate stock prices much more volatile than would be implied by the efficient markets model. It appears as if stock prices overreact to some news, or to their own past values, before investors come to their senses and correct the prices. Our arguments led to a spirited debate about the validity of the efficient markets model in the

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[10]People tend to neglect "base rates," the unconditional probabilities or frequencies of events, see Meehl and Rosen (1955).

[11]Rabin (1996) characterizes this judgment error as a tendency to over-infer the probability distribution from short sequences. Part of overconfidence may be nothing more than simple forgetting of contrary evidence; a tendency to forget is by its very nature not something that one can learn to prevent.

finance literature, a literature that has too many facets to summarize here, except to say that it confirms there are many potential interpretations of any statistical results based on limited data.[12] My own view of the outcome of this debate is that it is quite likely that speculative asset prices tend to be excessively volatile. Certainly, at the very least, one can say that no one has been able to put forth any evidence that there is not excess volatility in speculative asset prices. For an evaluation of this literature, see Shiller (1989), Campbell and Shiller (1988, 1989), West (1988), and Campbell, Lo and MacKinlay (1997, Ch. 7).

Since then, papers by De Bondt and Thaler (1985), Fama and French (1988), Poterba and Summers (1988), and Cutler, Poterba and Summers (1991) have confirmed the excess volatility claims by showing that returns tend to be negatively autocorrelated over horizons of three to five years, that an initial overreaction is gradually corrected. Moreover, Campbell and Shiller (1988, 1989) show that aggregate stock market dividend yields or earnings yields are positively correlated with subsequently observed returns over similar intervals; see also Dreman and Berry (1995).[13] Campbell and Shiller (1998) connect this predictive power to the observed stationarity of these ratios. Since the ratios have no substantial trend over a century and appear mean reverting over much shorter time intervals, the ratio must predict future changes in either the numerator (the dividend or earnings) or the denominator (the price); we showed that it has been unequivocally the denominator, the price, that has restored the ratios to their mean after they depart from it, and not the numerator. La Porta (1996) found that stocks for which analysts projected low earnings growth tended to show upward price jumps on earnings announcement dates, and stocks for which analysts projected high earnings growth tended to show downward price jumps on earnings announcement dates. He interprets this as consistent with a hypothesis that analysts (and the market) excessively extrapolated past earnings movements and only gradually correct their errors as earnings news comes in. The behavior of initial public offerings around announcement dates appears also to indicate some overreaction and later rebound, see Ibbotson and Ritter (1988) and Ritter (1991).

On the other hand, there has also been evidence of what might be called underreaction. Most days when big news breaks have been days of only modest stock market price movements, the big movements tending to come on days when there is little news, see Cutler, Poterba and Summers (1989). Cutler, Poterba and Summers (1991) also found that

[12]There has been some confusion about the sense in which the present-value efficient markets model puts restrictions on the short-run (or high frequency) movements in speculative asset prices. The issues are laid out in Shiller (1979), (appendix). Kleidon (1986) rediscovered the same ideas again, but gave a markedly different interpretation of the implications for tests of market efficiency.

[13]An extensive summary of the literature on serial correlation of US stock index returns is in Campbell, Lo and MacKinlay (1997). Chapter 2 documents the positive serial correlation of returns over short horizons, but concludes that the evidence for negative serial correlation of returns over long horizons is weak. Chapter 7, however, shows evidence that long-horizon returns are negatively correlated with the price-earnings ratio and price-dividend ratio. Recent critics of claims that long-horizon returns can be forecasted include Goetzmann and Jorion (1992), Nelson and Kim (1993) and Kirby (1997). In my view, they succeed in reducing the force of the evidence, but not the conclusion that long-horizon returns are quite probably forecastable.

for a number of indices of returns on major categories of speculative assets there has been a tendency for positive autocorrelation of short-run returns over short horizons, less than a year; see also Jegadeesh and Titman (1993) and Chan, Jegadeesh and Lakonishok (1996).[14] This positive serial correlation in return indices has been interpreted as implying an initial underreaction of prices to news, to be made up gradually later. Bernard and Thomas (1992) found evidence of underreaction of stock prices to changes, from the previous year, in company earnings: prices react with a lag to earnings news; see also Ball and Brown (1968).[15] Irving Fisher (1930, Ch. XXI, pp. 493–94) thought that, because of human error, nominal interest rates tend to underreact to inflation, so that there is a tendency for low real interest rates in periods of high inflation, and high real rates in periods of low inflation. More recent data appear to confirm this behavior of real interest rates, and data on inflationary expectations also bear out Fisher's interpretation that the phenomenon has to do with human error; see De Bondt and Bange (1992) and Shefrin (1997).[16]

Does the fact that securities prices sometimes underreact pose any problems for the psychological theory that people tend to be overconfident? Some observers seem to think that it does. In fact, however, overconfidence and overreaction are quite different phenomena. People simply cannot overreact to everything: if they are overconfident they will make errors, but not in any specified direction in all circumstances. The concepts of overreaction or underreaction, while they may be useful in certain contexts, are not likely to be good psychological foundations on which to organize a general theory of economic behavior.

The fact that both overreaction and underreaction are observed in financial markets has been interpreted by Fama (1997) as evidence that the anomalies from the standpoint of efficient markets theory are just "chance results," and that therefore the theory of market efficiency survives the challenge of its critics. He is right, of course, that both overreaction and underreaction together may sometimes seem a little puzzling. But one is not likely to want to dismiss these as "chance results" if one has an appreciation for the psychological theory that might well bear on these phenomena. In his survey of behavioral finance Fama

[14]Lo and MacKinlay (1988) and Lehmann (1990), however, find evidence of *negative* serial correlation of individual weekly stock returns between successive weeks. As explained by Lo and MacKinlay (1990), weekly returns on portfolios of these same stocks still exhibit positive serial correlation from week to week because the cross-covariances between returns of individual stocks are positive. They conclude that this pattern of cross-covariances is not what one would expect to find based on theories of investor inertia. Lehmann, however, has a different interpretation of the negative week-to-week serial correlation of individual weekly stock returns, that the negative serial correlation reflects nothing more than the behavior of market makers facing order imbalances and asymmetric information.

[15]Firms' management appear acutely aware that earnings growth has a psychological impact on prices, and so attempt to manage earnings accounting to provide a steady growth path. Impressive evidence that they do so is found in Degeorge, Patel and Zeckhauser (1997).

[16]Modigliani and Cohn (1979) argue that public failure to understand the relation of interest rates to inflation has caused the stock market to overreact to nominal interest rate changes.

(1997) makes no more than a couple of oblique references to any literature from the other social sciences. In fact, Fama states that the literature on testing market efficiency has no clearly stated alternative, “the alternative hypothesis is vague, market inefficiency” (p. 1). Of course, if one has little appreciation of these alternative theories then one might well conclude that the efficient markets theory, for all its weaknesses, is the best theory we have. Fama appears to believe that the principal alternative theory is just one of consistent overreaction or underreaction, and says that “since the anomalies literature has not settled on a testable alternative to market efficiency, to get the ball rolling, I assume that reasonable alternatives must predict either over-reaction or under-reaction” (p. 2). The psychological theories reviewed here cannot be reduced to such simple terms, contrary to Fama’s expectations.

Barberis, Shleifer and Vishny (1997) provide a psychological model, involving the representativeness heuristic as well as a principle of conservatism (Edwards, 1968), that offers a reconciliation of the overreaction and underreaction evidence from financial markets; see also Daniel, Hirshleifer and Subrahmanyam (1997) and Wang (1997). More work could be done in understanding when it is that people overreact in financial markets and when it is that they underreact. Understanding these overreaction and underreaction phenomena together appears to be a fertile field for research at the present time. There is neither reason to think that it is easy obtain such an understanding, nor reason to despair that it can ever be done.

Overconfidence may have more clear implications for the volume of trade in financial markets than for any tendency to overreact. If we connect the phenomenon of overconfidence with the phenomenon of anchoring, we see the origins of differences of opinion among investors, and some of the source of the high volume of trade among investors. People may fail to appreciate the extent to which their own opinions are affected by anchoring to cues that randomly influenced them, and take action when there is little reason to do so.

The extent of the volume of trade in financial markets has long appeared to be a puzzle. The annual turnover rate (shares sold divided by all shares outstanding) for New York Stock Exchange Stocks has averaged 18% a year from the 1950s through the 1970s, and has been much higher in certain years. The turnover rate was 73% in 1987 and 67% in 1930. It does not appear to be possible to justify the number of trades in stocks and other speculative assets in terms of the normal life-cycle ins and outs of the market. Theorists have established a “nonspeculation theorem” that states that rational agents who differ from each other only in terms of information and who have no reason to trade in the absence of information will not trade (Milgrom and Stokey, 1982l; Geanakoplos, 1992).

Apparently, many investors do feel that they do have speculative reasons to trade often, and apparently this must have to do with some tendency for each individual to have beliefs that he or she perceives as better than others’ beliefs. It is as if most people think they are above average.

Odean (1996a), in analyzing individual customer accounts at a nationwide discount brokerage house, examined the profits that customers made on trades that were apparently not motivated by liquidity demands, tax loss selling, portfolio rebalancing, or a move to lower-risk securities. On the remaining trades, the returns on the stocks purchased was on average lower, not higher, than on those sold. This appears to be evidence of overconfidence among these investors.

Within the week of the stock market crash of October 19, 1987 I sent out questionnaires to 2,000 wealthy individual investors and 1,000 institutional investors, asking them to recall their thoughts and reasons for action on that day; see Shiller (1987b). There were 605 completed responses from individuals and 284 responses from institutions. One of the questions I asked was: "Did you think at any point on October 19, 1987 that you had a pretty good idea when a rebound was to occur?" Of individual investors, 29.2% said yes, of institutional investors, 28.0% said yes. These numbers seem to be surprisingly high: one wonders why people thought they knew what was going to happen in such an unusual situation. Among those who bought on that day, the numbers were even higher, 47.1% and 47.9% respectively. The next question on the questionnaire was "If yes, what made you think you knew when a rebound was to occur?" Here, there was a conspicuous absence of sensible answers; often the answers referred to "intuition" or "gut feeling." It would appear that the high volume of trade on the day of the stock market crash, as well as the occurrence, duration, and reversal of the crash was in part determined by overconfidence in such intuitive feelings.[17]

If people are not independent of each other in forming overconfident judgments about investments, and if these judgments change collectively through time, then these "noisy" judgments will tend to cause prices of speculative assets to deviate from their true investment value. Then a "contrarian" investment strategy, advocated by Graham and Dodd (1934) and Dreman (1977) among many others, a strategy of investing in assets that are currently out of favor by most investors, ought to be advantageous. Indeed, there is much evidence that such contrarian investment strategy does pay off, see for example, De Bondt and Thaler (1985), Fama and French (1988, 1992), Fama (1991), and Lakonishok, Shleifer and Vishny (1994). That a simple contrarian strategy may be profitable may appear to some to be surprising: one might think that "smart money," by competing with each other to benefit from the profit opportunities, would ultimately have the effect of eliminating any such profit opportunities. But, there are reasons to doubt that such smart money will indeed have this effect; see Shiller (1984), De Long et al. (1990a,b), and Shleifer and Vishny (1996).[18]

[17]See also Case and Shiller (1988) for a similar analysis of recent real estate booms and busts. On the other hand, Garber (1990) analyzes some famous speculative bubbles, including the tulipomania in the 17th century, and concludes that they may have been rational.

[18]Even public expectations of a stock market crash does not prevent the stock market from rising; there is evidence from options prices that the stock market crash of 1987 was in some sense expected before it happened; see Bates (1991, 1995). Lee, Shleifer and Thaler (1991) argue that investor expectations, or rather "sentiment" can be measured by closed-end mutual fund discounts, which vary through time.

The Disjunction Effect

The disjunction effect is a tendency for people to want to wait to make decisions until information is revealed, even if the information is not really important for the decision, and even if they would make the same decision regardless of the information. The disjunction effect is a contradiction to the "sure-thing principle" of rational behavior (Savage, 1954).

Experiments showing the disjunction effect were performed by Tversky and Shafir (1992). They asked their subjects whether they would take one of the bets that Samuelson's lunch colleague, discussed above, had refused a coin toss in which one has equal chances to win $200 or lose $100. Those who took the one bet were then asked whether they then wanted to take another such bet. If they were asked after the outcome of the first bet was known, then it was found that a majority of respondents took the second bet whether or not they had won the first. However, a majority would not take the bet if they had to make the decision before the outcome of the bet was known. This is a puzzling result: if one's decision is the same regardless of the outcome of the first bet, then it would seem that one would make the same decision before knowing the outcome. Tversky and Shafir gave their sense of the possible thought patterns that accompany such behavior: if the outcome of the first bet is known and is good, then subjects think that they have nothing to lose in taking the second, and if the outcome is bad they want to try to recoup their losses. But if the outcome is not known, then they have no clear reason to accept the second bet.

The disjunction effect might help explain changes in the volatility of speculative asset prices or changes in the volume of trade of speculative asset prices at times when information is revealed. Thus, for example, the disjunction effect can in principle explain why there is sometimes low volatility and low volume of trade just before an important announcement is made, and higher volatility or volume of trade after the announcement is made. Shafir and Tversky (1992) give the example of presidential elections, which sometimes induce stock market volatility when the election outcome is known even though many skeptics may doubt that the election outcome has any clear implications for market value.

Gambling Behavior and Speculation

A tendency to gamble, to play games that bring on unnecessary risks, has been found to pervade widely divergent human cultures around the world and appears to be indicative of a basic human trait, Bolen and Boyd (1968). Kallick et al. (1975) estimated that 61% of the adult population in the United States participated in some form of gambling or betting in 1974. They also estimated that 1.1% of men and 0.5% of women are "probably compulsive gamblers," while an additional 2.7% of men and 1% of women are "potential compulsive gamblers." These figures are not trivial, and it is important to keep in mind that compulsive gambling represents only an extreme form of the behavior that is more common.

The tendency for people to gamble has provided a puzzle for the theory of human behavior under uncertainty, since it means that we must accommodate both risk-avoiding behavior (as evidenced by people's willingness to purchase insurance) with an apparent riskloving behavior. Friedman and Savage (1948) proposed that the co-existence of these behaviors might be explained by utility functions that become concave upward in extremely high range, but such an explanation has many problems. For one thing, people who gamble do not appear to be systematically risk seekers in any general sense, instead they are seeking specific forms of entertainment or arousal.[19] Moreover, the gambling urge is compartmentalized in people's lives, it tends to take for each individual only certain forms: people specialize in certain games. The favored forms of gambling tend to be associated with a sort of ego involvement: people may feel that they are especially good at the games they favor or that they are especially lucky with these.

The complexity of human behavior exemplified by the gambling phenomenon has to be taken into account in understanding the etiology of bubbles in speculative markets. Gamblers may have very rational expectations, at some level, for the likely outcome of their gambling, and yet have other feelings that drive their actual behavior. Economists tend to speak of quantitative "expectations" as if these were the only characterization of people's outlooks that mattered. It is my impression, from interviews and survey results, that the same people who are highly emotionally involved with the notion that the stock market will go up may give very sensible, unexciting, forecasts of the market if asked to make quantitative forecasts.

The Irrelevance of History

One particular kind of overconfidence that appears to be common is a tendency to believe that history is irrelevant, not a guide to the future, and that the future must be judged afresh now using intuitive weighing only of the special factors we see now. This kind of overconfidence discourages taking lessons from past statistics; indeed most financial market participants virtually never study historical data for correlations or other such statistics; they take their anchors instead from casual recent observations. Until academic researchers started collecting financial data, most was just thrown away as irrelevant.

One reason that people may think that history is irrelevant is a human tendency toward historical determinism, a tendency to think that historical events should have been known in advance. According to historian Florovsky (1969, p. 364):

> In retrospect we seem to perceive the *logic* of events, which unfold themselves in a regular order, according to a recognizable pattern, with an alleged inner necessity, so that we get the impression that it really could not have happened otherwise.

Fischhoff (1975) attempted to demonstrate this tendency towards historical determinism

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[19]According to the American Psychiatric Association's DSM–IV (1994), "Most individuals with Pathological Gambling say that they are seeking 'action' (an aroused, euphoric state) even more than money. Increasingly larger bets, or greater risks, may be needed to continue to produce the desired level of excitement" (p. 616).

by presenting experimental subjects with incomplete historical stories, stories that are missing the final outcome of the event. The stories were from historical periods remote enough in time that the subjects would almost certainly not know the actual outcome. Subjects were asked to assign probabilities to each of four different possible conclusions to the story (only one of which was the true outcome). There were two groups of subjects, one of which was told that one of the four outcomes had in fact happened. The probability given to the outcomes was on average 10% higher when people were told it was the actual outcome.

Fischhoff's demonstration of a behavior consistent with belief in historical determinism may not demonstrate the full magnitude of such behavior, because it does not capture the effects of social cognition of past events, a cognition that may tend to remember historical facts that are viewed as causing subsequent historical events, or are connected to them, and to forget historical facts that seem not to fit in with subsequent events. It will generally be impossible to demonstrate such phenomena of social cognition in short laboratory experiments.

A human tendency to believe in historical determinism would tend to encourage people to assume that past exigencies (the stock market crash of 1929, the great depression, the world wars, and so on) were probably somewhat known in advance, or, at least, that before these events people had substantial reason to worry that they might happen. There may tend to be a feeling that there is nothing definite on the horizon now, as there presumably was before these past events.[20] It is in this human tendency toward believing history is irrelevant that the equity premium puzzle, discussed above, may have its most important explanation. People may tend just not to think that the past stock market return history itself gives any indication of the future, at least not until they perceive that authorities are in agreement that it does.

According to the representativeness heuristic, discussed above, people may see past return history as relevant to the future only if they see the present circumstances as representative in some details of widely remembered past periods. Thus, for example, the public appears to have made much, just before the stock market crash of 1987, of similarities in that period to the period just before the crash of 1929. Newspapers, including the *Wall Street Journal* on the morning of the stock market crash of October 19, 1987, showed plots of stock prices before October 1929 superimposed on a plot of stock prices before October 1987, suggesting comparisons. In this way, historical events can be remembered and viewed as relevant, but this is not any systematic analysis of past data.

Lack of learning from historical lessons regarding financial and economic uncertainties may explain why many investors show little real interest in diversification around the world and why most investors appear totally uninterested in the correlation of their investments with their labor income, violating with their behavior one of the most fundamental premises of financial theory. Most people do not make true diversification around the world a high priority, and virtually no one is short the company that he or she works for, or is short the

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[20]This feeling can of course be disrupted, if a sudden event calls to mind parallels to a past event, or if the social cognition memorializes and interprets a past event as likely to be repeated.

stock market in one's own country, as would be suggested by economic theory.[21]

A prominent reason that most people appear apathetic about schemes to protect them from price level uncertainty in nominal contracts is that they just do not seem to think that past actual price level movements are any indicator of future uncertainty. In a questionnaire I distributed (1997a) to a random sample from phone books in the U.S.A. and Turkey, the following question was posed:

> We want to know how accurately you think that financial experts in America (Turkey) can predict the price level in 2006, ten years from now. Can you tell us, if these experts think that a "market basket" of goods and services that the typical person buys will cost $1,000 (100 million TL) in 2006, then you think it will probably actually cost:
>
> (Please fill in your lower and upper bounds on the price:)
>
> Between $__________ (TL) and $__________ (TL)

The median ratio between high and low was 4/3 for U.S. respondents and 3/2 for Turkish respondents. Only a few respondents wrote numbers implying double- or triple-digit ratios, even in Turkey. The ratios not far from one that most respondents revealed would seem to suggest excessive confidence in the predictability of price levels. Note that in Turkey the CPI increased three-fold between 1964 and 1974, 31-fold between 1974 and 1984, and 128-fold between 1984 and 1994. But, Turkish respondents appear to connect the price level movements with prior political and social events that may be perceived as having largely predicted the price movements, events that are themselves not likely to be repeated in the same way. While these people have apparently learned to take certain steps to protect themselves from price level uncertainty (such as not investing in long-term nominal bonds), they do not appear to have a well-developed understanding of the potential uncertainty of the Turkish Lira that would allow them to deal systematically with such uncertainty. For example, they have shown relatively little interest in government indexed bonds.

Magical Thinking

B. F. Skinner (1948) in what is now regarded as a classic experiment fed starved experimental pigeons small quantities of food at regular fifteen-second intervals with no dependence whatsoever on the bird's behavior. Even though the feeding was unaffected by their behavior, the birds began to behave as if they had a "superstition" that something in their behavior caused the feeding (see also McFadden, 1974). Each pigeon apparently conditioned itself to exhibit a specific behavior to get the food, and because each bird

[21]Kusko, Poterba and Wilcox (1997) showed, using data on 10,000 401k plan participants in a manufacturing firm, found that barely 20% of participants directed *any* of their own balances into an S&P index fund, while nearly 25% of participants directed *all* of their discretionary balances into a fund invested completely in the own company stock.

exhibited its characteristic behavior so reliably, it was never deconditioned:

> One bird was conditioned to turn counter-clockwise in the cage, making two or three turns between reinforcements. Another repeatedly thrust its head into one of the upper corners of the cage. A third developed a "tossing" response, as if placing its head beneath an invisible bar and lifting it repeatedly.... (1948, p. 168)

Arbitrary behaviors that are so generated are referred to with the term "magical thinking" by psychologists.

A wide variety of economic behaviors are likely to be generated in exactly the same way that the arbitrary behaviors of the pigeons are generated. Thus, for example, firms' investment or management decisions that happened to precede increases in sales or profits may tend to be repeated, and if this happens in a period of rising profits (as when the economy is recovering from a recession) the notion that these decisions were the cause of the sales or profit increase will be reinforced. Because firms are similar to each other and observe each other, the magical thinking may be social, rather than individual, and hence may have aggregate effects.

Roll (1986), with his hubris hypothesis concerning corporate takeovers, argued that managers of bidder firms may become overconfident of their own abilities to judge firms, because of their luck in their first takeovers. This overconfidence can cause them to overbid in subsequent takeover attempts.

The tendency for speculative markets to respond to certain news variables may be generated analogously. The U.S. stock market used often to be buoyed by positive news about the economy, but in recent years it appears to tend to be moved in the opposite direction by such news. This new "perverse" movement pattern for the stock market is sometimes justified in the media by a theory that the good news will cause the Federal Reserve to tighten monetary policy and that then the higher interest rates will lower the stock market. But the whole belief could be the result of a chain of events that was set off by some initial chance movements of the stock market. Because people believe these theories they may then behave so that the stock price does indeed behave as hypothesized, the initial correlations will persist later, and thereby reinforce the belief.

Quasi-Magical Thinking

The term quasi-magical thinking, as defined by Shafir and Tversky (1992), is used to describe situations in which people act as if they erroneously believe that their actions can influence an outcome (as with magical thinking) but in which they in fact do not believe this. It includes acting as if one thinks that one can take actions that will, in effect, undo what is obviously predetermined, or that one can change history.

For example, Quattrone and Tversky (1984) divided subjects into a control and experimental group and then asked people in both groups to see how long they could bear to hold their hands in some ice water. In the experimental group subjects were told that people with strong hearts were better able to endure the ice water. They found that those in the experimental group in fact held their hands in the ice water longer. If indeed, as appears to be the case, those in the experimental group held their hands in the ice water longer to prove that they had strong hearts, then this would be quasi-magical, since no notion was involved that there was any causal link from holding hands in ice water to strengthening the heart.

While this particular experimental outcome might also be explained as the result of a desire for self deception, Shafir and Tversky report as well as other experiments that suggest that people do behave as if they think they can change predetermined conditions. Shafir and Tversky (1992) show, with an experimental variant of Newcomb's Paradox, that people behave as if they can influence the amount of money already placed in a box.

Quasi-magical thinking appears to operate more strongly when outcomes of future events, rather than historical events, are involved. Langer (1975) showed that people place larger bets if invited to bet before a coin is tossed than after (where the outcome has been concealed), as if they think that they can better influence a coin not yet tossed.

It appears likely that such quasi-magical thinking explains certain economic phenomena that would be difficult to explain the basis of strictly rational behavior. Such thinking may explain why people vote, and why shareholders exercise their proxies. In most elections, people must know that the probability that they will decide the election must be astronomically small, and they would thus rationally decide not to vote. Quasi-magical thinking, thinking that in good societies people vote and so if I vote I can increase the likelihood that we have a good society or a good company, might explain such voting. The ability of labor union members or oligopolists to act in concert with their counterparts, despite an incentive to free-ride, or defect, may also be explained by quasi-magical thinking.

The disposition effect (Shefrin and Statman, 1985) referred to above, the tendency for individuals to want to hold losers and sell winners might also be related to quasi-magical thinking, if people feel at some level that holding on to losers can reverse the fact that they have already lost. Public demand for stocks at a time when they are apparently overvalued may be influenced by quasi-magical thinking, a notion that if I hold, then the stocks will continue to rise.

Attention Anomalies and the Availability Heuristic

William James (1890, p. 402) criticized earlier psychologists, who in their theories effectively assumed that the human mind takes account of all sensory input, for taking no note of the phenomenon of selective attention:

> But the moment one thinks of the matter, one sees how false a notion of experience that is which would make it tantamount to the mere presence to the senses of an outward order. Millions of items of the outward order are present to my senses which never properly enter into my experience. Why? Because they have no *interest* for me. *My experience is what I agree to*

attend to. Only those items which I *notice* shape my mind — without selective interest, experience is utter chaos.

The same criticism might equally well be applied to expected utility maximization models in economics, for assuming that people attend to all facts that are necessary for maximization of the assumed objective function (Berger, 1994, elaborates on this point).

Attention is associated with language; the structure of our language invites attention to categories that are represented in the language. Taylor (1989) showed, for example, that certain concepts of "the self" were apparently absent from languages in the time of Augustine. The language shapes our attention to even the most inward of phenomena.

In economics, certain terms were apparently virtually absent from popular discourse fifty or more years ago: gross national product, the money supply, the consumer price index. Now, many economists are wont to model individual attention to these concepts as if they were part of the external reality that is manifest to all normal minds.

Attention may be capricious because it is affected by the "salience" of the object; whether it is easily discerned or not (Taylor and Thompson, 1982) or by the "vividness" of the presentation, whether the presentation has colorful details. Judgments may be affected, according to the "availability heuristic," that is, by the "ease with which instances or associations come to mind" (Tversky and Kahneman, 1974).

Investment fashions and fads, and the resulting volatility of speculative asset prices, appear to be related to the capriciousness of public attention (Shiller, 1984, 1987). Investor attention to categories of investments (stocks versus bonds or real estate, investing abroad versus investing at home) seems to be affected by alternating waves of public attention or inattention. Investor attention to the market at all seems to vary through time, and major crashes in financial markets appear to be phenomena of attention, in which an inordinate amount of public attention is suddenly focussed on the markets.[22]

Economic theories that are most successful are those that take proper account of the limitations and capriciousness of attention. One reason that the hypothesis of no unexploited arbitrage opportunities (a hypothesis that has led to the Black–Scholes (1973) option pricing theory, the Ross (1976) arbitrage pricing theory, and other constructs of finance) has been so successful is that it does not rely on pervasive public attention. The essence of the no-arbitrage assumption, when it is used successfully to produce theories in finance, is that the arbitrage opportunities, were they to ever exist, would be exploited and eliminated even if only a tiny fraction of investors were paying attention to the opportunity.

Culture and Social Contagion

The concept of culture, central to sociology and cultural anthropology ever since the work of Tylor (1871), Durkheim (1893) and Weber (1947), is related to the selective attention that the human mind exhibits. There is a social cognition, reenforced by conversation, ritual and

[22]There is evidence that the stock market crash of 1987 can be viewed in these terms, see Shiller (1989).

symbols, that is unique to each interconnected group of people; to each nation, tribe, or social group. People tend not to remember well facts or ideas that are not given attention in the social cognition, even though a few people may be aware of such facts. If one speaks to groups of people about ideas that are foreign to their culture, one may find that someone in the group will know of the ideas, and yet the ideas have no currency in the group and hence have no influence on their behavior at large.

The array of facts, suppositions, symbols, categories of thought that represent a culture have subtle and far-reaching affects on human behavior. For a classic example, Durkheim (1897), in a careful study of differing suicide rates across countries, found that there was no apparent explanation for these differing rates other than cultural differences.

Cultural anthropologists have used methods of inferring elements of primitive culture by immersing themselves in the society, observing their everyday life, and talking and listening to them nonjudgmentally, letting them direct the conversation. From such learning, for example, Lévy–Strauss (1966, pp. 9–10) wrote persuasively that the customs of primitive people that we may tend to view as inexplicably savage actually arise as a logical consequence of a belief system common to all who belong to the society, a belief system which we can grow to understand only with great difficulty:

> The real question is not whether the touch of a woodpecker's beak does in fact cure toothache. It is rather whether there is a point of view from which a woodpecker's beak and a man's tooth can be seen as 'going together' (the use of this congruity for therapeutic purposes being only one of its possible uses) and whether some initial order can be introduced into the universe by means of these groupings.... The thought we call primitive is founded on this demand for order.

The same methods that cultural anthropologists use to study primitive peoples can also be used to study modern cultures. O'Barr and Conley (1992) studied pension fund managers using personal interviews and cultural anthropological methods. They concluded that each pension fund has its own culture, associated often with a colorful story of the origin of their own organization, akin to the creation myths of primitive peoples. The culture of the pension fund is a belief system about investing strategy and that culture actually drives investment decisions. Cultural factors were found to have great influence because of a widespread desire to displace responsibility for decisions onto the organization, and because of a desire to maintain personal relationships within the organization.[23]

Psychological research that delineates the factors that go into the formation of culture has been undertaken under the rubric of social psychology and attitude change, or under social cognition. There is indeed an enormous volume of research in these areas. For surveys, one may refer to McGuire (1985) for attitude change or Levine and Resnick (1993)

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[23]The psychologist Janis (1972) has documented with case studies how social patterns ("groupthink") within decision making groups can cause even highly intelligent people to make disastrously wrong decisions.

for social cognition.

One difficulty that these researchers have encountered with experimental work is that of disentangling the "rational" reasons for the imitation of others with the purely psychological. Some recent economic literature has indeed shown the subtlety of the informational influences on people's behavior (learning from each other), see Bannerjee (1992), Bikhchandani et al. (1992), Leahy (1994), and Shiller (1995).

A Global Culture

We see many examples of imitation across countries apparently widely separated by both physical and language barriers. Fashions of dress, music, and youthful rebellion, are obvious examples. The convergence of seemingly arbitrary fashions across nations is evidence that something more is at work in producing internationally-similar human behavior than just rational reactions to common information sets relevant to economic fundamentals, see Featherstone (1990).

And yet it will not be an easy matter for us to decide in what avenues global culture exerts its influence (Hannerz, 1990, p. 237):

> There is now a world culture, but we had better make sure that we understand what this means. It is marked by an organization of diversity rather than by a replication of uniformity. No total homogenization of systems of meaning and expression has occurred, nor does it appear likely that there will be one any time soon. But the world has become one network of social relationships, and between its different regions there is a flow of meanings as well as of people and goods.

Sociologists have made it their business to study patterns of influence within cultures, and we ought to be able to learn something about the nature of global culture from their endeavors. For example, one study of patterns of influence regarded as a classic among sociologists is the in-depth study of the town of Rovere by sociologist Robert Merton (1957). After extensive study of the nature of interpersonal influence, he sought meaningful ways to categorize people. He found that it was meaningful to divide people into two broad categories: locals (who follow local news and derive status by their connectedness with others) and cosmopolitans (who orient themselves instead to world news and derive status from without the community). He found that the influence of cosmopolitans on locals transcended both their numbers and their stock of useful information. We must bear this conclusion in mind when deciding how likely it is that incipient cultural trends are pervasive across many different nations.

Reading such sociological studies inclines us to rather different interpretations of globally similar behaviors than might occur naturally to many traditional economists. Why did the real estate markets in many cities around the world rise together into the late 1980s and fall in the early 1990s? (See Goetzmann and Wachter, 1996 and Hendershott, 1997.) Why have the stock markets of the world moved somewhat together? Why did the stock markets of the world show greater tendency to move together after the stock market crash of 1987? (See von Furstenberg and Jeon, 1989 and King, Sentana and Wadhwani, 1994.) If we recognize the global nature of culture, there is no reason to assume that these events have anything to do with genuine information about economic fundamentals.

Concluding Remarks

Since this paper was written in response to an invitation to summarize literature on behavioral theory in finance, it has focussed exclusively on this topic, neglecting the bulk of finance literature. Because of its focus on anomalies and departures from conventional notions of rationality, I worry that the reader of this paper can get a mistaken impression about the place of behavioral theory in finance, and of the importance of conventional theory.

The lesson from the literature surveyed here, and the list of varied behavioral phenomena, is not that "anything can happen" in financial markets. Indeed, while the behavioral theories have much latitude for interpretation, when they are combined with observations about behavior in financial markets, they allow us to develop theories that do have some restrictive implications. Moreover, conventional efficient markets theory is not completely out the window. I could have, had that been the goal of this paper, found very many papers that suggest that markets are impressively efficient in certain respects.

Financial anomalies that intuitive assessments of human nature might lead one to expect to find, or anomalies one hears casually about, often turn out to be tiny, ephemeral, or nonexistent. There is, for example, virtually no Friday the thirteenth effect (Chamberlain et al., 1991; Dyl and Maberly, 1988). Investors apparently aren't that foolish.

Heeding the lessons of the behavioral research surveyed here is not going to be simple and easy for financial researchers. Doing research that is sensitive to lessons from behavioral research does not mean entirely abandoning research in the conventional expected utility framework. The expected utility framework can be a workhorse for some sensible research, if it is used appropriately. It can also be a starting point, a point of comparison from which to frame other theories.

It is critically important for research to maintain an appropriate perspective about human behavior and an awareness of its complexity. When one does produce a model, in whatever tradition, one should do so with a sense of the limits of the model, the reasonableness of its approximations, and the sensibility of its proposed applications.

References

Allais, M. (1953). "Le Comportement de l'Homme Rationnel devant le Risque, Critique des Postulats et Axiomes de l'Ecole Americaine," *Econometrica*, 21:503–546.

American Psychiatric Association (1994). *Diagnostic and Statistical Manual of Mental Disorders, Fourth Edition (DSM–IV).* Washington, DC: American Psychiatric Association.

Bachelier, L.(1964). "Theory of Speculation." In Paul Cootner (ed.), *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press, pp. 17–75. (Originally submitted as doctoral dissertation to the Faculty of Sciences of the Academy of Paris, 1900.)

Backus, D. K., S. Foresi and C. I. Telmer (1995). "Interpreting the Forward Premium Anomaly," *Canadian Journal of Economics*, 28: S108–119.

Ball, R. and P. Brown (1968). "AnEmpirical Examination of Accounting Income Numbers," *Journal of Accounting Research*, 6: 159–178.

Bannerjee, A. V. (1992). "A Simple Model of Herd Behavior," *Quarterly Journal of Economics*, 107(3): 797–817.

Barberis, N., A. Shleifer and R. Vishny (1997). "A Model of Investor Sentiment," reproduced, University of Chicago, presented at the NBER–Sage workshop on Behavioral Economics, Cambridge, MA.

Bates, D. S. (1991). "The Crash of '87: Was It Expected? The Evidence from Options Markets," *Journal of Finance*, 46(3): 1009–1044.

Bates, D. S. (1995). "Post-`87 Crash Fears in S&P Futures Options," reproduced, Wharton School, University of Pennsylvania.

Bell, D. E. (1982). "Regret in Decision Making Under Uncertainty," *Operations Research*, 30(5): 961–981.

Benartzi, S. and R. H. Thaler (1995). "Myopic Loss Aversion and the Equity Premium Puzzle." *Quarterly Journal of Economics*, 110(1): 73–92.

Benartzi, S. and R. H. Thaler (1996). "Risk Aversion or Myopia: The Fallacy of Small Numbers and its Implications for Retirement Saving," reproduced.

Berger, L. A. (1994). "Mutual Understanding, The State of Attention, and the Ground for Interaction in Economic Systems," *Business and Ethics Quarterly*.

Bernard, V. L. and J. K. Thomas (1992). "Evidence that Stock Prices Do Not Fully Reflect the Implications of Current Earnings for Future Earnings," *Journal of Accounting Economics*, 13: 305–340.

Bernard, V. (1992). "Stock Price Reactions to Earnings Announcements." In R. Thaler (ed.), *Advances in Behavioral Finance*. New York: Russell Sage Foundation.

Bikhchandani, S., D. Hirshleifer and I. Welch (1992). "A Theory of Fashion, Social Custom, and Cultural Change," *Journal of Political Economy*, 100(5): 992–1026.

Black, F. and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81: 637–654.

Bodnar, G. and R. Marston (1996). "1995 Survey of Derivatives Usage by US Non-Financial Firms," reproduced, Wharton School, University of Pennsylvania.

Bolen, D. W. and W. H. Boyd (1968). "Gambling and the Gambler: A Review and Preliminary Findings," *Archives of General Psychiatry*, 18(5): 617–29.

Bowman, D., D. Minehart and M. Rabin (1993). "Loss Aversion in a Savings Model," University of California Working Paper in Economics 93–12.

Bryant, R. R. (1990). "Job Search and Information Processing in the Presence of Nonrational Behavior," *Journal of Economic Behavior and Organization*, 14(2): 249–260.

Campbell, J. Y., A. W. Lo and A. C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton, NJ: Princeton University Press.

Campbell, J. Y. and R. J. Shiller (1988). "Stock Prices, Earnings, and Expected Dividends," *Journal of Finance*, 43: 661–676.

Campbell, J. Y. and R. J. Shiller (1989). "The Dividend Price Ratio and Expectations of Future Dividends and Discount Factors," *Review of Financial Studies*, 1: 195–228.

Campbell, J. Y. and R. J. Shiller (1998). "Valuation Ratios and the Long-Term Stock Market Outlook," forthcoming, *Journal of Portfolio Management*.

Case, K. E. and R. J. Shiller (1988). "The Behavior of Home Buyers in Boom and Post-Boom Markets," *New England Economic Review* (Nov./Dec.): 29–46.

Chamberlain, T. W., C. S. Cheung, and C. C. Y. Chang (1991). "The Friday the Thirteenth Effect," *Quarterly Journal of Business and Economics*, 30(2): 111–117.

Chan, L., N. Jegadeesh, and J. Lakonishok (1996). "Momentum Strategies," NBER Working Paper 5375, forthcoming, *Journal of Finance*.

Cootner, P. H. (1964). *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press.

Cutler, D. M., J. M.Poterba, and L. H. Summers (1989). "What Moves Stock Prices?" *Journal of Portfolio Management*, 15(3): 4–12.

Cutler, D. M., J. M.Poterba, and L. H. Summers (1991). "Speculative Dynamics," *Review of Economic Studies*, 58(3): 529–546.

Daniel, K., D. Hirshleifer, and A. Subrahmanyam (1997). "A Theory of Overconfidence, Self-Attribution, and Security Market Over- and Underreaction," unpublished paper, presented at the NBER–Sage workshop on Behavioral Economics, Cambridge, MA.

De Bondt, W. F. and M. M. Bange (1992). "Inflation Forecast Errors and Time Variation in Term Premia," *Journal of Finance and Quantitative Analysis*, 24: 479–496.

De Bondt, W. F. and R. H. Thaler (1985). "Does the Stock Market Overreact?" *Journal of Finance*, 40: 793–805.

De Bondt, W. and R. H. Thaler (1996). "Financial Decision-Making in Markets and Firms: A Behavioral Perspective." In *Handbook in Operations Research and Management Science*, Vol. 9 North–Holland.

De Long, J. B., A. Shleifer, L. Summers, and R. J. Waldman (1990a). "Noise Trader Risk in Financial Markets," *Journal of Political Economy*, 98: 703–38.

De Long, J. B., A. Shleifer, L. Summers, and R. J. Waldman (1990b). "Positive Feedback Investment Strategies and Destabilizing Rational Speculation," *Journal of Finance*, 45(2): 379–395.

Degeorge, F., J. Patel, and R. Zeckhauser (1997). "Earnings Manipulations to Exceed Thresholds," unpublished paper, HEC School of Management, presented at the NBER–Sage Workshop on Behavioral Finance, Cambridge, MA.

Dolde, W. (1993). "The Trajectory of Corporate Financial Risk Management," *Journal of Applied Corporate Finance*, 6:33–41.

Dreman, D. (1977). *Psychology and the Stock Market: Why the Pros Go Wrong and How to Profit*. New York: Warner Books.

Dreman, D. and M. Berry (1977). "Investor Overreaction and the Low P/E Effect," unpublished paper, Dreman Foundation, presented at the NBER–Sage Workshop in Behavioral Finance, Cambridge, MA.

Durkheim, É. (1893). "Représentations individuelles et représentations collectives," *Revue de Métaphysique et de Morale*, 6: 273–302.

Durkheim, É. (1897). *Le Suicide*.

Dyl, E. A. and E. D. Maberly (1988). "The Anomaly That Isn't There: A Comment on Friday the Thirteenth," *Journal of Finance*, 43(5): 1285–1286.

Edwards, W. (1968). "Conservatism in Human Information Processing." In B. Kleinmutz (ed.), *Formal Representation of Human Judgment*. New York: John Wiley & Sons.

Eisner, R. and R. H. Strotz (1961). "Flight Insurance and the Theory of Choice," *Journal of Political Economy*, 69: 355–368.

Erlich, D., P. Guttman, P. Schopenbach, and J. Mills (1957). "Postdecision Exposure to Relevant Information," *Journal of Abnormal and Social Psychology*, 54: 98–102.

Fama, E. F. (1970). "Efficient Capital Markets: A Review of Empirical Work," *Journal of Finance*, 25: 383–417.

Fama, E. F. (1991). "Efficient Capital Markets II," *Journal of Finance*, 46(5): 1575–1617.

Fama, E.F. (1992). “The Cross-Section of Expected Stock Returns,” *Journal of Finance*, 47: 427–465.
Fama, E. F. (1997). “Market Efficiency, Long-Term Returns, and Behavioral Finance,” Center for Research in Security Prices Working Paper 448, University of Chicago.
Fama, E. F. and K. R. French (1988). “Permanent and Temporary Components of Stock Returns,” *Journal of Political Economy*, 96: 246–273.
Featherstone, M. (1990). *Global Culture: Nationalism, Globalization and Modernity*. London: Sage Publications.
Ferris, S. P., R. A. Haugen and A. K. Makhija (1988). “Predicting Contemporary Volume with Historic Volume at Differential Price Levels: Evidence Supporting the Disposition Effect,” *Journal of Finance*, 43(3): 677–697.
Festinger, L. (1957). A Theory of Cognitive Dissonance., Stanford, CA: Stanford University Press.
Fisher, I. (1928). *The Money Illusion*. New York: Adelphi.
Fisher, I. (1930). *The Theory of Interest*. New York: MacMillan.
Fischhoff, B. (1975). “Hindsight is not Foresight: The Effect of Outcome Knowledge on Judgment Under Uncertainty,” *Journal of Experimental Psychology: Human Perception and Performance*, 1: 288–299.
Fischhoff, B., P. Slovic and S. Lichtenstein (1977). “Knowing With Uncertainty: The Appropriateness of Extreme Confidence,” *Journal of Experimental Psychology: Human Perception and Performance*, 3: 552–564.
Florovsky, G. (1969). “The Study of the Past.” In R. H. Nash (ed.), *Ideas of History*, Vol. 2. New York: Dutton.
Fortune, P. (1996). “Anomalies in Option Pricing: The Black–Scholes Model Revisited,” *New England Economic Review*, March/April, 17–40.
Friedman, M. and L. J. Savage (1948). “The Utility Analysis of Choices Involving Risk,” *Journal of Political Economy*, 56: 279–304.
Froot, K. and R. Thaler (1990). “Anomalies: Foreign Exchange,” *Journal of Economic Perspectives*, 4(3): 179–192.
Garber, P. (1990). “Famous First Bubbles,” *Journal of Economic Perspectives*, 42(2): 35–54.
Geanakoplos, J. (1992). “Common Knowledge,” *Journal of Economic Perspectives*, 6(4): 53–82.
Gertler, M. (1994). “Monetary Policy, Business Cycles, and the Behavior of Small Manufacturing Firms,” *Quarterly Journal of Economics*, 109(2): 309–340.
Gibson, G. R. (1889). *The Stock Markets of London, Paris and New York*. New York: G. P. Putnam’s Sons.
Gigerenzer, G. (1991). “How to Make Cognitive Illusion Disappear: Beyond ‘Heuristics and Biases’,” *European Review of Social Psychology*, 2: 83–115.
Goetzmann, W. N. and P. Jorion (1993). “Testing the Predictive Power of Dividend Yields,” *Journal of Finance*,48: 63.
Goetzmann, W. N. and N. Peles (1993). “Cognitive Dissonance and Mutual Fund Investors,” reproduced, Yale School of Management.
Goetzmann, W. N. and S. M. Wachter (1996). “The Global Real Estate Crash: Evidence from an International Database,” reproduced, Yale University.
Graham, B. and D. L. Dodd (1934). *Security Analysis*. New York: McGraw Hill.
Gruen, D. K. and M. C. Gizycki (1993). “Explaining Forward Discount Bias: Is It Anchoring?” Princeton University Woodrow Wilson School Discussion Paper in Economics 164.
Gultekin, M. and N. B. Gultekin (1983). “Stock Market Seasonality: International Evidence,” *Journal of Financial Economics*, 12: 469–481.

Hannerz, U. (1990). "Cosmopolitans and Locals in World Culture," *Theory, Culture and Society*, 7: 237–51.

Harawini, G., and D. B. Keim (1995). "On the Predictability of Common Stock Returns: Worldwide Evidence," In R. Jarrow, M. Maksiomovic and W. T. Ziemba (eds.), *Finance*, Handbooks in Operations Research and Management Science. Amsterdam: North–Holland, Volume 9.

Hendershott, P. H. (1997). "Systematic Valuation Errors and Property Cycles: A Clinical Study of the Sydney Office Market," reproduced, Ohio State University.

Ibbotson, R. and J. R. Ritter (1988). "Initial Public Offerings," *Journal of Applied Corporate Finance*, 1: 37–45.

Jackwerth, J. C. and M. Rubinstein (1995). "Recovering Probability Distributions from Contemporaneous Security Prices," reproduced, Haas School of Business, University of California, Berkeley.

James, W. (1890). *Principles of Psychology*. New York: Dover Publications (reprinted 1950).

Janis, I. (1972). *Victims of Groupthink*. Houston: Boston.

Jegadeesh, N. and S. Titman (1993). "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, 48(1): 65–91.

Kahneman, D. and A. Tversky (1979). "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, 47: 263–291.

Kahneman, D., P. Slovic, and A. Tversky (1974). *Judgement Under Uncertainty: Heuristics and Biases*. Cambridge, England: Cambridge University Press.

Kallick, M., D. Suits, T. Dielman, and J. Hybels (1975). *A Survey of American Gambling Attitudes and Behavior*. Ann Arbor: Survey Research Center, Institute for Social Research, University of Michigan.

King, M., E. Sentana, and S. Wadhwani (1994). "Volatility and Links Between National Stock Markets," *Econometrica*, 62(4): 901–934.

Kirby, C. (1997). "Measuring the Predictable Variation in Stock and Bond Returns," *Review of Financial Studies*, 10: 579–630.

Kusko, A. L., J. M. Poterba, and D. Wilcox (1997). "Employee Decision with Respect to 401(k) Plans: Evidence from Individual-Level Data." Forthcoming in O. S. Mitchell and S. J. Schieber (eds.), *Living with Defined Contribution Pension Plans: Remaking Responsibility for Retirement*. Philadelphia: University of Pennsylvania Press.

Lakonishok, J., A. Shleifer, and R. W. Vishny (1994). "Contrarian Investment, Extrapolation and Risk," *The Journal of Finance*, 49(5): 1541–1578.

Langer, E. J. (1975). "The Illusion of Control," *Journal of Personality and Social Psychology*, 32: 311–328.

La Porta, R. (1996). "Expectations and the Cross-Section of Stock Returns," *Journal of Finance*, 51: 1715–1742.

Leahy, J. (1994). "Miracle on Sixth Avenue," reproduced, Harvard University.

Lee, C., A. Shleifer, and R. Thaler (1991). "Investor Sentiment and the Closed-End Fund Puzzle," *Journal of Finance*, 46: 75–109.

Lehmann, B. N. (1990). "Fads, Martingales and Market Efficiency," *Quarterly Journal of Economics*, 105(1): 1–28.

Lehmann, B. N. (1991). "Asset Pricing and Intrinsic Values: A Review Essay," *Journal of Monetary Economics*, 28: 485–500.

LeRoy, S. F., and R. D. Porter (1981). "Stock Price Volatility: A Test Based on Implied Variance Bounds," *Econometrica*, 49: 97–113.

Levine, J. and L. B. Resnick (1993). "Social Foundations of Cognition," *Annual Review of Psychology*, 44: 585–612.

Lévy–Strauss, C. (1966). *The Savage Mind*. Chicago, IL: The University of Chicago Press.
Lo, A. W. and A. C. MacKinlay (1988). "Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test," *Review of Financial Studies*, 1: 41–66.
Lo, A. W. and A. C. MacKinlay (1990). "When Are Contrarian Profits Due to Stock Market Overreaction?" *Review of Financial Studies*, 3: 175–208.
Loomes, G. and R. Sugden (1982). "Regret Theory: An Alternative Theory of Rational Choice Under Uncertainty," *The Economic Journal*, 92: 805–824.
Mackay, D. (1841). *Memoirs of Extraordinary Popular Delusions*. London: Bentley.
Malkiel, B. (1981). *A Random Walk Down Wall Street*, 2nd ed. New York: Norton.
Mayhew, S. (1995). "Implied Volatility," *Financial Analysts' Journal*, 51(4): 8–20.
McFadden, D. (1974). "On Some Facets of Betting." In M. Balch et al. (eds.), *Essays on Economic Behavior Under Uncertainty*. North–Holland, pp. 99–122.
McGuire, W. J. (1985). "Attitudes and Attitude Change." In G. Lindzey and E. Aronson (eds.), *Handbook of Social Psychology*. Reading, MA: Addison Wesley, pp.233–346.
Meehl, P. and A. Rosen (1955). "Antecedent Probability and the Efficiency of Psychometric Signs, Patterns, and Cutting Scores," *Psychological Bulletin*, 52: 194–215.
Mehra, R. and E. C. Prescott (1985). "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15: 145–162.
Merton, R. K. (1957). *Social Theory and Social Structure*. Glencoe, IL: Free Press.
Milgrom, P. and N. Stokey (1982). "Information, Trade, and Common Knowledge," *Econometrica*, 49: 219–222.
Modigliani, F. and R. Cohn (1979). "Inflation, Rational Valuation, and the Market," *Financial Analysts Journal*, 35: 24–44.
Nance, D. R., C. W. Smith, and C. W. Smithson (1993). "On the Determinants of Corporate Hedging," *Journal of Finance*, 48: 267–284.
Nelson, C. and M. Kim (1993). "Predictable Stock Returns: The Role of Small Sample Bias," *Journal of Finance*, 48(2): 641–661.
Northcraft, G. B. and M. A. Neale (1987). "Experts, Amateurs, and Real Estate: An Anchoring-and-Adjustment Perspective on Property Pricing Decisions," *Organizational Behavior and Human Decision Processes*, 39: 84–97.
O'Barr, W. M. and J. M. Conley (1992). *Fortune and Folly: The Wealth and Power of Institutional Investing*. Homewood, IL: Irwin.
Odean, T. (1996a). "Are Investors Reluctant to Realize Their Losses?" unpublished paper, University of California, Berkeley.
Odean, T. (1996b). "Why Do Investors Trade Too Much?" unpublished paper, University of California, Berkeley.
Poterba, J. M. and L. H. Summers (1988). "Mean Reversion in Stock Prices: Evidence and Implications," *Journal of Financial Economics*, 22(1): 27–59.
Quattrone, G. A. and A. Tversky (1984). "Causal versus Diagnostic Contingencies: On Self Deception and on the Voter's Illusion," *Journal of Personality and Social Psychology*, 46(2): 237–248.
Rabin, M. (1996). "Psychology and Economics," *Journal of Economic Literature*.
Ritter, Jay R. (1991). "The Long-Run Performance of Initial Public Offerings," *Journal of Finance*, 46: 3–28.
Roberts, H. V. (1967). "Statistical versus Clinical Prediction of the Stock Market," unpublished paper presented to the Seminar on the Analysis of Security Prices, University of Chicago.
Roll, R. (1986). "The Hubris Hypothesis of Corporate Takeovers," *Journal of Business*, 59(2): 541–566.

Ross, L. (1987). "The Problem of Construal in Social Inference and Social Psychology." In N. Grunberg, R. E. Nisbett and J. Singer (eds.), *A Distinctive Approach to Psychological Research: The Influence of Stanley Schachter*. Hillsdale, NJ: Erlbaum.

Ross, S. A. (1976). "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13: 341–360.

Samuelson, P. A. (1963). "Risk and Uncertainty: A Fallacy of Large Numbers," *Scientia*, 98(4–5): 108–113.

Savage, L. J. (1954). "The Sure-Thing Principle." In Leonard J. Savage, *The Foundations of Statistics*. New York: John Wiley, pp. 21–26.

Shafir, E., P. Diamond and A. Tversky (1997). "On Money Illusion," *Quarterly Journal of Economics*, 92: 341–374.

Shafir, E. and A. Tversky (1992). "Thinking Through Uncertainty: Nonconsequential Reasoning and Choice," *Cognitive Psychology*, 24: 449–474.

Shefrin, H. (1997). "Erroneous Investor Beliefs: Implications for the Term Structure of Interest Rates,' unpublished paper, Santa Clara University.

Shefrin, H. and M. Statman (1985). "The Disposition to Sell Winners Too Early and Ride Losers Too Long," *Journal of Finance*, 40: 777–790.

Shefrin, H. and M. Statman (1994). "Behavioral Portfolio Theory," unpublished paper, Santa Clara University.

Shefrin, H. and R. H. Thaler (1988). "The Behavioral Life-Cycle Hypothesis," *Economic Inquiry*. Reprinted in R. H. Thaler, *Quasi Rational Economics*.

Shiller, R. J. (1979). "The Volatility of Long Term Interest Rates and Expectations Models of the Term Structure," *Journal of Political Economy*, 87: 1190–1219.

Shiller, R. J. (1981a). "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review*, 71(3): 421–436.

Shiller, R. J. (1981b). "The Use of Volatility Measures in Assessing Market Efficiency," *Journal of Finance*, 36: 291-304.

Shiller, R. J. (1984). "Stock Prices and Social Dynamics," *Brookings Papers on Economic Activity*, II, pp. 457–98.

Shiller, R. J. (1987a). "Fashions, Fads and Bubbles in Financial Markets." In Jack Coffee (ed.), *Knights, Raiders and Targets: The Impact of the Hostile Takeover*. Oxford, England: Oxford University Press.

Shiller, R. J. (1987b). "Investor Behavior in the October 1987 Stock Market Crash: Survey Evidence," National Bureau of Economic Research Working Paper 2446. (Reprinted in Robert Shiller, *Market Volatility*, 1989.)

Shiller, R. J. (1989). *Market Volatility*. Cambridge, MA: MIT Press.

Shiller, R. J. (1990). "Market Volatility and Investor Behavior," *American Economic Review*, 80(2): 58–62.

Shiller, R. J. (1995). "Conversation, Information, and Herd Behavior," *American Economic Review*, 85(2): 181–185.

Shiller, R. J. (1997a). "Public Resistance to Indexation: A Puzzle," *Brookings Papers on Economic Activity*, I, pp. 159–228.

Shiller, R. J. (1997b). "Why Do People Dislike Inflation?" In C. Romer and D. Romer (eds.), *Reducing Inflation: Motivation and Strategy*. National Bureau of Economic Research and University of Chicago Press, pp. 13–65.

Shiller, R. J., F. Kon-Ya, Y. Tsutsui (1996). "Why Did the Nikkei Crash? Expanding the Scope of Expectations Data Collection," *Review of Economics and Statistics*, 78: 156–164.

Shleifer, A. (1986). "Do Demand Curves for Stocks Slope Down?" *Journal of Finance*, 41: 579–589.

Shleifer, A. and L. Summers (1990). "The Noise Trader Approach to Finance," *Journal of Econometrics*, 4(2): 19–23.

Shleifer, A. and R. Vishny (1996). "The Limits of Arbitrage," NBER Working Paper 5167, forthcoming, *Journal of Finance*.

Siegel, J. J. (1994). *Stocks for the Long Run*. New York: Irwin.

Siegel, J. J. and R. H. Thaler (1997). "Anomalies: The Equity Premium Puzzle," *Journal of Economic Perspectives*, 11(1): 191–200.

Silberman, J. and M. Klock (1989). "The Behavior of Respondents in Contingent Valuation: Evidence on Starting Bids," *Journal of Behavioral Economics*, 18: 51–60.

Skinner, B. F. (1948). "Superstition in the Pigeon," *Journal of Experimental Psychology*, 38:168–72. (Reprinted in same journal (1992), 121(3):273–274.)

Slovic, P. (1972). "Psychological Study of Human Judgment: Implications for Investment Decision Making Under Uncertainty," *Journal of Finance*, 27(4): 779–799.

Smith, E. L. (1925). *Common Stocks as Long-Term Investments*. New York: MacMillan.

Stulz, R.M. (1996). "Rethinking Risk Management," *Journal of Applied Corporate Finance*, 9(5): 8–24.

Taylor, C. (1989). *Sources of the Self: The Making of Modern Identity*. Cambridge: University Press.

Taylor, S. E. and S. C. Thompson (1982). "Stalking the Elusive Vividness Effect," *Psychological Review*, 89: 155–181.

Thaler, R. H. (1987). "The January Effect," *Journal of Economic Perspectives*, 1(1): 197–201.

Thaler, R. H. (1987). "Seasonal Movements in Security Prices II: Weekend, Holiday, Turn of the Month and Intraday Effects," *Journal of Economic Perspectives*, 1(1): 169–177.

Thaler, R. H., A. Tversky, D. Kahneman and A. Schwartz (1997). "The Effect of Myopia and Loss Aversion on Risk Taking: An Experimental Test," *Quarterly Journal of Economics*, 112(2): 647–661.

Turner, J.C. (1991). *Social Influence*. Buckingham, England: Open University Press.

Tversky, A. and D. Kahneman (1974). "Judgment Under Uncertainty: Heuristics and Biases," *Science*, 185: 1124–1131.

Tversky, A. and D. Kahneman (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, 5: 297–323.

Tversky, A. and E. Shafir (1992). "The Disjunction Effect in Choice Under Uncertainty," *Psychological Science*, 3(5): 305–309.

Tylor, E. B. (1871). *Primitive Culture*.

von Furstenberg, G. M. and B. N. Jeon (1989). "Relations Among Stock Markets Around the World," *Brookings Papers on Economic Activity*, I, 125–180.

Wang, F. A. (1997). "Overconfidence, Delegated Fund Management, and Survival," unpublished paper, Columbia University, presented at the NBER–Sage Workshop on Behavioral Finance, Cambridge, MA.

Weber, M. (1947). *The Theory of Social and Economic Organization*, edited by T. Parsons. Glencoe, IL: The Free Press.

West, K. D. (1988). "Bubbles, Fads, and Stock Price Volatility: A Partial Evaluation," *Journal of Finance*, 43: 639–655.

Zimmerman, J. L. (1983). "Taxes and Firm Size," *Journal of Accounting and Economics*, 5: 119–149.

Dynamic Financial Analysis

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Dynamic Financial Analysis

- General ideas
- DFA and solvency testing
- Identifying sources of stochastic behaviour
- Strengths, weaknesses and limitations of DFA
- DFA in action

Idea

For analyzing the financial effects of different strategies for insurance companies over a given time horizon there are two primary techniques in use today:

- *Scenario testing* projects results under specific scenarios in the future. The disadvantage of this deterministic approach is the fact that only a few arbitrary scenarios are tested in order to decide how good a strategy is.

- Stochastic simulation, better known as *Dynamic Financial Analysis (DFA)*.
  Here many different scenarios are generated stochastically with the aim of giving information about the distribution of some important variables, like surplus or loss ratio.

Fixing the Time Period

- We would like to model over as long a time period as possible in order to see the long-term effects of a chosen strategy.
- Simulated values get more and more unreliable the longer this time period is.
- A compromise must be made in order to fix the length of the simulated time period.

What Does DFA Stand for?

- *Dynamic* means stochastic or variable, as opposed to static or fixed.
- *Financial* reflects the fact that not only the underwriting business is simulated but rather the total of all assets and liabilities.
- *Analysis* is defined as an examination of the whole complex, its elements and their interrelationships.

Which Risks Should be Modelled?

- Asset risk:
  - How will assets develop?
- Liability risk:
  - Which liabilities will be incurred?
  - When will they be incurred?
  - How big are they?
- Interrelation between both sides:
  - How do these risks depend on each other?
- It is neither possible nor appropriate to model all sources of risk: It can be dangerous to place confidence in a detailed, but perhaps inappropriate model. It is often better to use a simple model that captures only the key features.

Aim of DFA

DFA gives the opportunity to compare the effects of different strategies before applying them to reality.

It does not necessarily give an optimal solution but leaves the decision of selecting a strategy to management.

So DFA serves as a decision tool that requires a good understanding of insurance business and some analytical/actuarial skills to be successfully implemented.

Applications of DFA Models

Before using a DFA model, management has to choose a financial or economic measure which should be analyzed.

The most common concept is the *efficient frontier concept*:

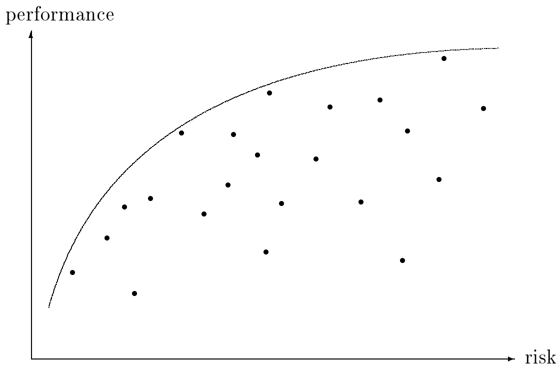
1. Choose a measure for performance,

   e.g. expected surplus.

2. Choose a measure for risk, e.g.
   - ruin probability,
   - quantiles (VaR) of distribution of surplus,
   - conditional expected loss.

3. Compare different strategies by plotting the measured risk and the measured performance.

Comparing Strategies with Respect to Performance and Risk



Efficient Frontier

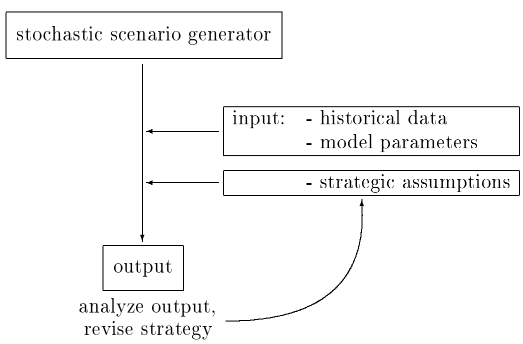
Link Between DFA and Solvency Testing

A better known concept than DFA is *solvency testing*, which deals with one central question:

Does the company have enough capital compared to the level of risk to which it is exposed, i.e. does the company have enough capital to keep the probability of ruin below a given level for the risks taken?

DFA gives us an estimate for the distribution of the surplus. A negative surplus is equivalent to the company becoming insolvent. Therefore DFA can also help answer the question of survival/ruin that is asked in solvency testing.

Main Structure of a DFA Model



Which Variables are Generated Stochastically?

An important step in the process of building an appropriate model is to identify the most important variables, and the sources of stochastic behaviour.

There are many possible ways of doing this.

A reasonable approach is the one implemented in Dynamo: Several different risk categories are selected and each is modelled with the help of a stochastic generator.

- Non-catastrophe losses
- Catastrophes
- Interest rates
- Stock returns
- Business cycles
- Payment patterns

Non-Catastrophe Losses for Each LOB

Aging phenomenon: The loss ratio – i.e. the ratio of losses divided by earned premiums – decreases when the age of policy increases. Therefore it might prove useful to divide insurance business into three classes, as done in Dynamo:

- New business (superscript 0)
- Renewal business – first annual (superscript 1)
- Renewal business – second annual and subsequent (superscript 2)

For every class we can simulate

- Number of losses ($j = 0, 1, 2$)

$$N_t^j \sim \text{NB, Pois, Bin}, \ldots$$

- Mean severity $X_t^j = \frac{\sum_{i=1}^{N_t^j} X_t^j(i)}{N_t^j}$

$$X_t^j \sim \text{Gamma, GPD}, \ldots$$

- Losses in year t $$\sum_{j=0}^{2} N_t^j X_t^j$$

Catastrophes

- Number of catastrophes

$$N_t \sim \text{NB, Pois, Bin}, \ldots$$
$$N_1, N_2, \ldots \text{ i.i.d.}$$

- Severity of an individual catastrophe $i = 1, \ldots, N_t$.

$$X_t(i) \sim \text{lognormal, Pareto, GPD}, \ldots$$
$$X_t(1), \ldots, X_t(N_t) \text{ i.i.d.}$$

- Total severity is divided up among LOBs affected by event.

$$X_{t,k}(i) = a_{t,k}(i)\, X_t(i), \quad \begin{aligned} k &= 1, \ldots, l, \\ l &= \#\ \text{LOBs}, \\ \textstyle\sum_{k=1}^{l} & a_{t,k}(i) = 1. \end{aligned}$$

- Catastrophe losses in year t

$$\sum_{k=1}^{l} b_{t,k} \left(\sum_{i=1}^{N_t} X_{t,k}(i) \right),$$
$$b_{t,k} = \text{market share of the company.}$$

Interest Rate Generator

- Interest rates r_t (financial assets)

  CIR: $dr_t = a\,(b - r_t)\,dt + s\,\sqrt{r_t}\,dZ_t$,
  Z_t = a standard Brownian motion.

  Yearly discretization:
  - $r_t = r_{t-1} + a\,(b - r_{t-1}) + s\,\sqrt{|r_{t-1}|}\,Z_t$,
  - $r_t = r_{t-1} + a\,(b - r_{t-1}) + s\,\sqrt{r_{t-1}{}^{+}}\,Z_t$,
  - $r_t = \left(r_{t-1} + a\,(b - r_{t-1}) + s\,\sqrt{r_{t-1}}\,Z_t\right)^{+}$,

  $$Z_t \sim \mathcal{N}(0,1), \ Z_1, Z_2, \ldots \text{ i.i.d.}$$

- Long term interest rates $R_{t,T}$

$$R_{t,T} = \frac{r_t\, B_T - \ln\, A_T}{T},$$

where

$$A_T = \left(\frac{2G\, e^{(a+G)\,T/2}}{(a+G)\,(e^{GT}-1) + 2G} \right)^{2ab/s^2},$$
$$B_T = \frac{2(e^{GT}-1)}{(a+G)\,(e^{GT}-1) + 2G},$$
$$G = \sqrt{a^2 + 2s^2}.$$

- Return on stock portfolio r_t^S

  CAPM:

$$\mathbb{E}[r_t^S | R_{t,1}] = (e^{R_{t,1}} - 1) + \beta_t \Big(\mathbb{E}[r_t^M | R_{t,1}] - (e^{R_{t,1}} - 1) \Big),$$

where

$$\mathbb{E}[r_t^M | R_{t,1}] = a^M + b^M\,(e^{R_{t,1}} - 1),$$
$$\beta_t = \frac{\mathrm{Cov}(r_t^S,\, r_t^M)}{\mathrm{var}(r_t^M)},$$
$$e^{R_{t,1}} - 1 = \text{risk-free return.}$$

Assuming a lognormal distribution for $1 + r_t^S$ leads to

$$1 + r_t^S \sim \text{lognormal}\,(\mu_t, \sigma^2),$$

with μ_t chosen to yield

$$m_t = e^{\mu_t + \frac{\sigma^2}{2}},$$

where

$$m_t = 1 + \mathbb{E}[r_t^S | R_{t,1}],$$
$$\sigma^2 = \text{estimated variance of logarithmic historical values.}$$

- Inflation i_t (loss payments)

  $$i_t = a^I + b^I\, r_t + \sigma^I\, \epsilon_t^I,$$
  $$\epsilon_t^I \sim \mathcal{N}(0,1),\ \epsilon_1^I, \epsilon_2^I, \dots \text{ i.i.d.}$$

- Impact of i_t on each LOB
  - Impact on mean number of losses:

    A reasonable model is
    $$\mathbb{E}[N_t^j] = (1+\delta_t^N)\,\mathbb{E}[N_{t-1}^j],$$
    $$\text{var}[N_t^j] = (1+\delta_t^N)^2\,\text{var}[N_{t-1}^j],$$
    where
    $$\delta_t^N = \max(a^N + b^N\, i_t + \sigma^N\, \epsilon_t^N, -1),$$
    $$\epsilon_t^N \sim \mathcal{N}(0,1),\ \epsilon_1^N, \epsilon_2^N, \dots \text{ i.i.d.}$$
  - Impact on mean loss severity:

    A reasonable model is
    $$\mathbb{E}[X_t^j] = (1+\delta_t^X)\,\mathbb{E}[X_{t-1}^j],$$
    $$\text{var}[X_t^j] = \frac{(1+\delta_t^X)^2}{1+\delta_t^N}\,\text{var}[X_{t-1}^j],$$
    $$\mathbb{E}[X_t(i)] = (1+\delta_t^X)\,\mathbb{E}[X_{t-1}(i)],$$
    $$\text{var}[X_t(i)] = (1+\delta_t^X)^2\,\text{var}[X_{t-1}(i)],$$
    where
    $$\delta_t^X = \max(a^X + b^X\, i_t + \sigma^X\, \epsilon_t^X, -1),$$
    $$\epsilon_t^X \sim \mathcal{N}(0,1),\ \epsilon_1^X, \epsilon_2^X, \dots \text{ i.i.d.}$$

Business Cycles by LOB

Is there strong competition among insurance companies in this LOB? Is there a general recession?

We can use a homogeneous Markov chain model where we classify each LOB for every year into one of the following states

1 Weak competition

2 Average competition

3 Strong competition

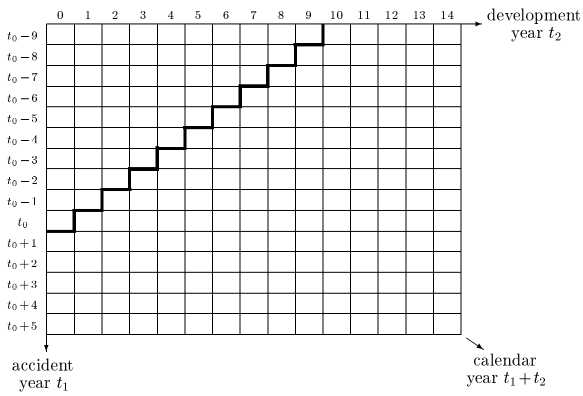
When the company writes l LOBs, there are 3^l states of the world. Because business cycles of different LOBs are strongly correlated, only few of the 3^l states are attainable. So we have to model $L \ll 3^l$ states.

Transition probabilities p_{ij}, $i, j \in \{1, \dots, L\}$ from one year to the next are equal for every year. (Markov chain is homogeneous.)

Main effect of business cycles: The weaker the competition, the higher the premiums.

Payment patterns

When are losses paid?



Paid losses in the upper triangle bounded by the solid line are known, while those in the lower triangle must be simulated.

To model percentages of paid losses we can use for example beta distributions.

Critical Appraisal of DFA:

Strengths of DFA

Compared to scenario testing where only a few arbitrary and possibly unrepresentative scenarios are considered, DFA gives better information on the effects of chosen strategies, because DFA simulates dynamically many different scenarios.

Because of the large number of simulations a DFA model can run, it gives us information not only on behaviour under ordinary circumstances, but also when extremal events occur. Of course the stochastic generators must be sufficiently flexible to generate occasional extreme values.

Weaknesses of DFA

Because reality is complex, it's not possible to model all sources of risk. We have to restrict attention to some key risk factors. So in a DFA model there is not only the randomness by reason of the inherent variability, but also the uncertainty caused by incomplete knowledge.

Generally DFA overestimates probability of ruin since it does not take into consideration that an insurance company has the opportunity to make additional capital available – e.g. by issuing stocks – when it runs the risk of ruin.

Limitations of DFA

DFA does not provide an optimal strategy. It serves as a decision tool that helps management compare different strategies. When a DFA model is used without enough actuarial knowledge, it is only a black box of limited utility.

Because reality can never be represented perfectly, we should of course always be cautious, and never rely completely upon the output produced by a DFA model.

DFA in Action

Model assumptions are:

- Time horizon: 10 years.
- Performance measure: expected surplus.
- Risk measure: ruin probability.
- Only 1 LOB.
- New business and renewal business are not modelled separately.
- Number of non-catastrophe losses $\sim$ NB (154, 0.025).
- Mean severity of non-catastrophe losses $\sim$ Gamma (9.091, 242), inflation-adjusted.
- Number of catastrophes $\sim$ Pois (18).
- Severity of individual catastrophes $\sim$ lognormal (13, 1.5^2), inflation-adjusted.
- Market share: 5%.
- Written premiums in the last year: 20 million.
- Expenses: 28.5% of written premiums.
- Optional excess of loss reinsurance with deductible 500 000 (inflation-adjusted), and cover ∞.
- Premiums for reinsurance: 175 000 p.a. (inflation-adjusted).

- For interest rates we use the discretization $r_t = r_{t-1} + a\,(b - r_{t-1}) + s\,\sqrt{|r_{t-1}|}\,Z_t$.
- Parameters for interest rate generator: $a = 0.25,\ b = 5\%,\ s = 0.1,\ r_1 = 2\%$.
- Parameters for generating return on stock portfolio: $a^M = 4\%,\ b^M = 0.5,\ \beta_t \equiv 0.5,\ \sigma = 0.15$.
- Parameters for modelling inflation: $a^I = 0\%,\ b^I = 0.75,\ \sigma^I = 0.025$.
- No impact of inflation on the number of claims for the modelled LOB.
- Parameters for modelling the impact of inflation on the severity of claims for the modelled LOB: $a^X = 3.5\%,\ b^X = 0.5,\ \sigma^X = 0.02$.
- Business cycles: 1 = weak, 2 = average, 3 = strong. State in year 0: 1 (weak). Transition probabilities: $p_{11} = 60\%,\ p_{12} = 25\%,\ p_{13} = 15\%,$ $p_{21} = 25\%,\ p_{22} = 55\%,\ p_{23} = 20\%,$ $p_{31} = 10\%,\ p_{32} = 25\%,\ p_{33} = 65\%.$
- Payment patterns are deterministic.

- All liquidity is reinvested. There are only two investment possibilities:
  1) buy a risk-free bond with maturity one year,
  2) buy an equity portfolio with a fixed beta.
- Market valuation: assets and liabilities are stated at market value, i.e. assets are stated at their current market values, liabilities are discounted at the appropriate term spot rate determined by the model.
- No transaction costs.
- No taxes.
- No dividends paid.
- Initial surplus: 12 million.

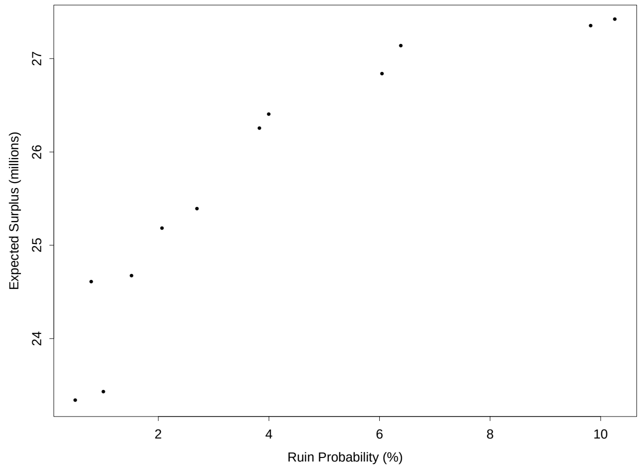
In this model one can choose:

- How many simulations should be run.
- Whether reinsurance should be purchased or not.
- How the liquidity is divided between bond and portfolio.

Example with 10 000 Runs

Expected surplus & ruin probabilities for twelve different strategies:

	with reinsurance	without reinsurance
100 % bonds 0 % stocks	23.33 mio. 0.50 %	23.42 mio. 1.01 %
50 % bonds 50 % stocks	25.17 mio. 2.07 %	25.38 mio. 2.70 %
0 % bonds 100 % stocks	27.34 mio. 9.82 %	27.41 mio. 10.26 %
$\leq$ 5 mio. bonds rest stocks	26.83 mio. 6.05 %	27.13 mio. 6.39 %
$\leq$10 mio. bonds rest stocks	26.25 mio. 3.83 %	26.40 mio. 4.00 %
$\leq$20 mio. bonds rest stocks	24.60 mio. 0.79 %	24.66 mio. 1.52 %



Enterprise Risk Management

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Introduction

Enterprise Risk Management is a relatively new term that is quickly becoming viewed as the ultimate approach to risk management. Consultants are advertising their ability to perform enterprise risk management. Auditors are examining how to incorporate enterprise risk management approaches into company audits.[1] Presentations are being made on this topic at many actuarial, risk management and other insurance meetings.[2] Seminars devoted to this topic are being conducted to explain the process, provide examples of applications and discuss advances in the field. Papers on enterprise risk management are beginning to appear in journals and books on the topic are starting to be published.[3] Some universities are even starting to offer courses titled enterprise risk management. It appears that a new field of risk management is opening up, one requiring new and specialized expertise, one that will make other forms of risk management incomplete and less attractive. This paper will explain what enterprise risk management is, why it has developed so quickly, how it differs from traditional risk management, what new skills are involved in this process and what advantages and opportunities this approach offers compared to prior techniques.

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[1] See the Institute of Internal Auditors website for an extensive list of references and discussion of enterprise risk management.

[2] See the CAS website, and particularly the presentations by Friedel, Kawamoto, Miccolis, and Miccolis and Shah.

[3] See Davenport and Bradley (2000), Deloach and Temple (2000), Doherty (2000), Guthrie, et al (1999), Lam (2000) and Shimpi (1999).

Definition of Enterprise Risk Management

Enterprise risk management is, in essence, the latest name for an overall risk management approach to business risks. Precursors to this term include corporate risk management, business risk management, holistic risk management, strategic risk management and integrated risk management. Although each of these terms has a slightly different focus, in part fostered by the risk elements that were of primary concern to organizations when each term first emerged, the general concepts are quite similar.

According to the Casualty Actuarial Society (CAS), enterprise risk management is defined as:

> "The process by which organizations in all industries assess, control, exploit, finance and monitor risks from all sources for the purpose of increasing the organization's short and long term value to its stakeholders."

The CAS then proceeds to enumerate the types of risk subject to enterprise risk management as hazard, financial, operational and strategic. Hazard risks are those risks that have traditionally been addressed by insurers, including fire, theft, windstorm, liability, business interruption, pollution, health and pensions. Financial risks cover potential losses due to changes in financial markets, including interest rates, foreign exchange rates, commodity prices, liquidity risks and credit risk. Operational risks cover a wide variety of situations, including customer satisfaction, product development, product failure, trademark protection, corporate leadership, information technology, management fraud and information risk. Strategic risks include such factors as completion, customer preferences, technological innovation and regulatory or political impediments. Although there can be disagreement over which category would apply to a specific instance, the primary point is that enterprise risk management considers all types of risk an organization faces.

A common thread of enterprise risk management is that the overall risks of the organization are managed in aggregate, rather than independently. Risk is also viewed as a potential profit opportunity, rather than as something simply to be minimized or eliminated. The level of decision making under enterprise risk management is also shifted, from the insurance risk manager, who would generally seek to control risk, to the chief executive officer, or board of directors, who would be willing to embrace profitable risk opportunities (Kawamoto, 2001).

Basically, though, enterprise risk management simply represents a return to the original roots of risk management, a field that was first developed in the 1950s by a group of innovative insurance professors. The first risk management text, presciently titled *Risk Management and the Business Enterprise*, was published in 1963, after six years of development, by Robert I. Mehr and Bob Hedges. As initially introduced in this text, the objective of risk management is, "to maximize the productive efficiency of the enterprise." The basic premise of this text was that risks should be managed in a comprehensive manner, and not simply insured.

The initial focus of risk management was on what is now termed hazard risk. This specialty area developed its own terminology and techniques for addressing risk. Financial risks began to be addressed much later, and by a separate business segment of most organizations. This field also developed its own terminology and techniques for addressing risk, independently of those used in traditional risk management. Each specialty area also developed different methods for reporting the risks the organization faced within each area. Since the hazard risk manager and the financial risk manager both generally reported to a common position, frequently the treasurer or chief financial officer of the firm, the different, and separate, approaches to dealing with risk created a problem. Potentially, each area could be expending resources to deal with a risk that, in aggregate, would cancel out within the firm. Also, the tolerance for risk applied in each area could be vastly different between hazard risks and financial risks. These discrepancies provided the impetus for developing a common terminology and common techniques for dealing with risk. In addition, this common approach could then be applied to other risks, such as operational and strategic risks, that could adversely affect the organization. This common approach to dealing with all risks that a firm faces is the heart of enterprise risk management, and represents an encompassing application of Mehr and Hedges objective," to maximize the productive efficiency of the enterprise."

Historical Development

Risk management has been practiced for thousands of years.[4] One can imagine a proto-risk manager burning a fire at night to keep wild animals away. Early lenders must have quickly learned to reduce the risk of loan defaults by limiting the amount loaned to any one individual and by restricting loans to those considered most likely to repay them. Individuals and firms could manage the risk of fire through the choice of building materials and safety practices, or after the introduction of fire insurance in 1667, by shifting it to an insurer. However, it wasn't until the 1960s that the field was formally named, principles developed and guidelines established. Robert Mehr and

[4] For an excellent overview of the treatment of risk through the ages, see Bernstein (1996).

Bob Hedges, widely acclaimed as the fathers of risk management, enumerated the following steps for the risk management process:

1. Identifying loss exposures
2. Measuring loss exposures
3. Evaluating the different methods for handling risk
    - Risk assumption
    - Risk transfer
    - Risk reduction
4. Selecting a method
5. Monitoring results

Initially, the risk management process focused on what has been termed "pure risks." Pure risks are those in which there is either a loss or no loss. Either something bad happens, or it doesn't. The states of possible outcomes in a pure risk situation do not allow for any outcome more favorable than the current position.

A typical example of a pure risk is owning a house. Your house may burn down, be hit by an earthquake or be infested by insects. If none of these, or other, unfavorable developments occur, then you are in the no loss position. This is no better than where you started, but no worse either.

The other classification of risk is "speculative risk." In a speculative risk, there is the possibility of a gain. For example, investing in the stock market generates the possibility of a loss (the stock could go down in value), the possibility that the value would not change (the stock price remains where you bought it), and the possibility of a gain (the stock price could increase).

Traditional risk management has focused on pure risks for several reasons. First, the field of risk management was developed by individuals who taught or worked in the insurance field, so the focus was on risks that insurers would be willing to write. In fact, some risk managers job duties are limited to buying insurance, an unfortunate limitation since many other options are readily available and should be explored. Another reason for the focus on pure risks is that in many cases these represented the most serious short term threats to the financial position of an organization at the time this field was founded. A fire could quickly put a firm out of business. Efforts to reduce the likelihood of a fire occurring, or to minimize the damage a fire would cause, or to establish a contingency plan to keep the business going in the event of a fire, or to purchase an insurance policy to compensate the owners for the damages caused by a fire, were easily seen to be beneficial to the firm. Finally, there were simply not a lot of reasons or options for dealing with financial risks such as interest rate changes, foreign exchange rate movements or equity market fluctuations, when this field was first developing.

At the time the field of risk management first emerged, interest rates were stable, foreign exchange rates were intentionally maintained within narrow bands and inflation was not yet a concern to most corporations. Thus, financial risks were not a major issue for most businesses. Indeed, the field of finance was primarily institutional at the time. Although Markowitz had proposed portfolio theory (Markowitz, 1952), the Capital Asset Pricing Model had not yet been developed. The mathematics for quantifying financial risk were not sufficient to put these risks in the same framework as most pure risks. The primary risks of the time were hazard risks: the risk of fire, windstorm or other property damage, or liability. Environmental risks had not yet developed into significant losses. Pensions were, at this point, neither guaranteed nor regulated.

Given the primary risks facing businesses were hazard risks, the initial focus of risk management was on these types of risks. Risks were quantified, the evaluation of different methods of dealing with risk was advanced and standardized, and an extensive terminology for managing risk was developed. Such terms as maximum possible loss (the largest loss that could occur) and maximum probable loss (the largest loss that is likely to occur) were introduced to help define risk exposure. Probability and statistical analysis were used to estimate the range of likely losses and the effect of adopting steps to mitigate these risks.

Risk managers did their job quite effectively. Firms almost universally handled their hazard risk in an appropriate manner. When they didn't, such as the MGM Grand Hotel that found it was not adequately insured for liability coverage after a major fire, new methods of handling risk, in this case retroactive insurance, were developed (Smith and Witt, 1985). Rarely did companies face financial ruin as a result of failure to manage their hazard risks effectively.

Beginning in the 1970s, financial risk became an important source of uncertainty for firms and, shortly thereafter, tools for handling financial risk were developed. These new tools allowed financial risks to be managed in a similar fashion to the ways that pure risks had been managed for decades. In 1972 the major developed countries ended the Bretton Woods agreement which had kept exchange rates stable for three decades. The result of ending the Bretton Woods agreement was to introduce instability in exchange rates. As foreign exchange rates varied, the balance sheets and operating results of corporations engaging in international trade began to fluctuate. This instability affected the performance of many firms. Also during the 1970s, oil prices began to rise as the Organization of Petroleum Exporting Countries (OPEC) developed agreements to reduce production to raise prices. Later in the same decade, a policy shift by the U. S. Federal Reserve to focus on fighting inflation (a result of oil price increases) instead of stabilizing interest rates led to a rapid rise, and increasing volatility, of interest rates in the United States, and had a spillover effect in other nations as well. Thus, volatility in foreign exchange rates, prices and interest rates caused financial risk to become an important concern for institutions.

Although financial risk had become a major concern for institutions by the early 1980s, organizations did not begin to apply the standard risk management tools and techniques to this area. The reasons for this failure were based on the artificial categorization of risk into pure risk and speculative risk (D'Arcy, 1999). Since fixed income assets, investments denominated in foreign currency and operating results that were affected by inflation or foreign exchange rates all had the possibility of a gain, they represented speculative risk. Risk managers had built a wall around their specialty, called pure risk, within which they operated. When a new risk area emerged, they did not expand to incorporate it into their domain. To do so would have required learning about financial instruments and moving away from the type of risks commonly covered by insurance. This would have been a bold move, but one that the innovative thinkers who developed risk management would have espoused. This failure was costly to organizations, and to the risk management field. With the emergence of enterprise risk management, traditional risk managers will be pushed into a wider arena of risk analysis, one that incorporates financial risk management and other forms of risk analysis. Thus, the refusal to expand into financial risks did not prevent risk managers from having to learn about financial risk management, it simply delayed it by a few decades.

A Primer in Financial Risk Management

The basic tools of financial risk management are forwards, futures, swaps and options (Smithson, 1998). These contracts are all termed derivatives, since their values are derived from some other instrument's value. Forwards are contracts entered into today in which the exchange will take place at some future date. The terms of the contract, the price, the date and the specific characteristics of the underlying asset, are all determined when the contract is established, but no money changes hands when the contract is initiated. At the specified date, each party is obligated to consummate the transaction. Since each forward contract is individually negotiated between the two parties, there is considerable flexibility regarding the terms of the contract. However, since forwards are contracts between the two parties, the risk of failure to perform exists, in the same manner that credit risk is a factor in any loan. In financial markets, this risk is termed counterparty risk. Also, since the contracts are specialized agreements between two parties, the contract is not liquid and can be very hard to terminate prior to the specified date if conditions were to change for one or both of the parties.

Futures contracts were developed to address the credit risk and liquidity concerns of forward contracts. Similar to forwards, futures are entered into today for an exchange that will take place at some future date. The terms of the contract are determined when the contract is entered into and no money changes hands when the contract is initiated. However, there are several significant differences between forward and futures. First, a clearinghouse (a firm that guarantees the performance of the parties in an exchange-traded derivatives transaction - Hull, 2000) serves as an intermediary to the contract. Each party is contracting with the clearinghouse, not with the other party. Thus, the risk of nonperformance is significantly reduced. Next, in order to reduce the risk of default, several financial requirements are introduced. Each party must post collateral, termed margin, with its broker. The amount of the margin that must be posted initially is determined for each futures contract (initial margin). Also, each day futures contracts are "marked-to-market" with cash payments flowing from one party to the other based on changes in the value of the futures contract. Thus, if the price of a futures contract increases by $500, then the party that is short the contract (has sold the asset) pays $500 to the party that is long the contract (has bought the asset). These funds come out of, and flow into, the respective margin accounts. If the margin account, falls below a predetermined value (maintenance margin), then a deposit must be made into the margin account to restore it to the initial margin level.

Swaps are agreements between two parties to exchange a series of cash flows based on a predetermined arrangement. Early swaps were based on exchanging a series of payments based on different currencies. For example, one company would pay a predetermined sum in Korean won and the other party would pay in US dollars each quarter for several years. Often the value of the exchanges would be netted (the respective values of each payment would be determined, and one party would pay the counterparty the difference in values). The most common swap today is an interest rate swap in which one party pays a fixed interest rate and the other pays a floating interest rate based on a set index such as the London Interbank Offer Rate (LIBOR). However, swaps can also be based on commodity prices or equity values. Similar to forwards and futures, swaps do not involve a payment by either party went the transaction is initiated.

The final basic tool of financial risk management is an option. An option provides the right, but not the obligation, to engage in a financial transaction at a predetermined price in the future. The owner of the option has the choice about consummating the transaction. The seller of the option is required to fulfill the contract if the buyer chooses. Since an option represents one sided risk, there is an initial cost to purchasing an option, which is termed the option premium. Options can be based on equities, bonds, interest rates, commodities, foreign exchange rates, or any other financial variable. A call option provides the right to buy the underlying asset at the predetermined price; a put option provides the right to sell the underlying asset. Although all options have these general characteristics, many specialized forms of options have been generated to produce a wide variety of different payoffs.

Introduction of Financial Risk Management

Forwards, futures and options had all been traded based on non-financial assets long before they were adapted to deal with financial risk. Swaps were not introduced until 1981, when the first currency swap was announced (Smithson, 1998). However, it did not take long after financial risk began to affect institutions for a wide array of financial risk management products to be generated to help corporations deal with financial risk. Foreign exchange futures were first offered in May, 1972. Interest rate futures began trading in October, 1975. Options on U.S. Treasury bonds were introduced in October, 1982. Options on foreign exchange rates were introduced in

December, 1982. Additional futures, swaps and options, as well as combination products, quickly followed. These tools allowed financial institutions and other corporations to manage financial risk in the much the same fashion that they used for pure risks.

Unfortunately, these tools were not always used wisely or effectively. Since financial risk management was generally not handled by the traditional risk management department, many of the standards for managing risk were not followed in this area. In 1994 alone, due to an unexpected rise in interest rates, the following losses from derivatives occurred (Smithson, 1998):

> Codelco, Chile's national copper trading company, lost $207 million
> Gibson Greetings lost $20 million
> Procter and Gamble lost $157 million
> Mead lost $7 million
> Air Products lost $60 million
> Federal Paper lost $19 million
> Caterpillar lost $13 million

Even more serious losses from the misuse of derivatives include (Jorion, 2001, Holton, 1996):

> Barings Bank went bankrupt in 1995 as a result of $1.3 billion in losses in futures and options trading based on the Nikkei 225 and Japanese bonds
> Metallgelsellschaft lost $1.3 billion on oil futures contracts
> Orange County lost $1.8 billion in 1994 from leveraged interest rate contracts
> Daiwa lost $1.1 billion from unauthorized derivatives trading
> Sumitomo lost $1.8 billion from concealed trading in copper and derivatives on copper by the head trader

In many cases, these losses occurred due to the failure to follow common risk management practices, such as not having transactions verified by an independent authority, not setting limits to potential losses or failure to understand the risks to which the organization was exposed. Managers and boards of directors were, in some cases, reluctant to question individuals who were providing, or at least reporting, impressive profits in a new area of financial transactions, and were willing to provide authority to these individuals without adequate oversight. The fear was that the normal level of oversight, if exercised in these areas, would drive a person with extraordinary talent away from their firm. Thus, they were lured into risk areas they neither understood nor would have accepted.

Imagine the approach that would have been taken if a traditional risk manager, newly hired by a firm, claimed to be able to provide insurance coverage through a self-funding strategy at half the price that the current providers were charging. What if this risk manager wanted to take control of the funds for managing risks and wanted to be the person in charge of handling, and reporting, all monetary transactions involving this fund, but would not provide details about the fund to the company? Despite the apparent cost savings, I doubt that any firm would be foolish enough to disregard its oversight process in this situation, or to provide this person with performance bonuses based on the apparent cost savings. Traditional risk management has developed a series of checks and balances to prevent such obvious abuses. Financial risk management did not initially have this level of expertise. One reason for this failure is because traditional risk managers abdicated the area of speculative risk, exposing many organizations to disastrous losses.

The basic rule of risk taking, whether it is hazard risk, financial risk or any other form of risk, is that if you do not fully understand a risk, you do not engage in it, regardless of what profits are claimed or reported. This basic rule is, unfortunately, violated by individuals consistently. Promises of impressive returns entice many individual investors to participate in fraudulent investment schemes. Unfortunately, many corporations fell into this trap as well.

The losses of the mid-1990s led organizations to realize the importance of financial risk management. The financial instruments that were developed to deal with financial risk were complex, and often only understood by those in the financial areas of the firm. Thus, the use of these tools to manage financial risk was generally not coordinated with the approach used to manage other risks. This lack of coordination resulted in a number of problems, including the development of a different terminology from that used in traditional risk management, different measures of risk and different goals. For example, traditional risk managers frequently focus on the probable maximum loss, the largest loss that could reasonably be expected to occur. If that loss exceeds the ability of the firm to cope with, then steps are taken to manage that risk, by transferring some of the risk to other parties, by reducing loss severity through loss control steps or other standard practices. Instead of adopting this approach, financial risk managers developed a measure termed the Value-at-Risk (VaR). This value indicates the loss that the firm would expect to have occur over the selected time interval (for example, daily) the selected percentage of the time. Thus, the daily VaR at the 1% level is the loss that can be expected to occur once every 100 days. This is not the largest loss that is likely to occur, so it does not provide the same level of information as probable maximum loss. The daily VaR at the 5% level, which is expected to occur once every 20 days, is smaller than the 1% value. VaR indicates what losses to expect, not what losses could occur. Even the time frame is different, as the traditional risk manager is likely dealing with loss probabilities over an annual basis, or over the term of an insurance contract, while VaR is often based on daily or weekly price movements.

Another difference between hazard risk and financial risk is the degree of independence among separate elements. In hazard risk management, risks are frequently independent of each other. Thus, the calculation of the number of accidents that a pool of vehicles is likely to be involved in during a year is determined by assuming that each accident is independent of every other accident. Financial risks, on the other hand, are not considered to be independent. In many cases, the correlation between different financial transactions forms the basis of the risk management strategy. Financial risk management considers the relationships among different financial variables to construct hedges. For example, a firm exposed to long term interest rate risk might use futures on short term instruments, due to the high correlation between short and long term interest rates, to hedge their interest rate exposure. Financial risk management approaches can lead to difficulty when the historical relationships between financial variables shifts. For example, the hedge fund Long Term Capital Management lost 92 percent its value (approximately $4.5 billion) in 1998 when historical patterns between variables, including yields on U.S. and Russian bonds, changed significantly.

Thus, the Board of Directors and other managers that are determining the overall risk management strategy of the firm are likely to receive different types of information on financial risk and on hazard risk. The risks are different, the terminology is different and the measures of risk are different. This makes the task of coordinating the firm's overall exposure to risk more difficult. In addition to desiring a common approach to hazard and financial risks, these decision makers have also envisioned incorporating other forms of risk, including strategic and operational, into the same approach. It is this vision that has led to the creation of enterprise risk management.

Other Factors Leading to Enterprise Risk Management

A number of other factors have also contributed to the development of enterprise risk management. Recent advances in computing power provide the powerful modeling tools necessary to perform sophisticated risk analysis for hazard risks, such as catastrophes, for financial risks, such as interest rate movements, and for other risks. Also, the availability of extensive data bases of financial and other information allows users to examine historical information to determine trends, correlations and other relationships among variables that is essential to enterprise risk management.

Insurers are also developing an expertise in, and a focus on, financial risk management. Some insurers are beginning to provide policies that coordinate financial and pure risk. One insurer has offered a policy that provides protection against foreign currency losses within it insurance coverage (Banham, 1999). Another insurer provided protection for a utility in which the amount of coverage is a function of rainfall, which affect utility income (Taylor, 2001).

Insurers are beginning to utilize the financial markets themselves through the securitization on insurance risk. Several types of insurance securitization have been developed (ISO, 1999). The first was the use of exchange traded derivatives. Both futures and options on catastrophe risk have been traded on the Chicago Board of

Trade. Trading in futures began in 1992 based on an index of catastrophe losses paid by a number of insurers reporting to ISO. In 1995 the index was changed to catastrophe losses reported by Property Claim Services, and trading in options was instigated. Although neither of these instruments is traded currently, their existence provided an impetus for insurers to learn about financial risk management tools and encouraged subsequent development of other approaches. The second approach is through contingent capital. One form of this is termed a Cat-E-Put, or catastrophe-equity-put. Under this contract, an insurer purchases a contract under which the counterparty agrees to purchase equity in the firm, at a predetermined price, in the event of a catastrophe as defined in the contract. This is, essentially, a put option that is triggered by a catastrophe. A third type of securitization is termed risk capital, in which an insurer, through an intermediary, issues debt on which the repayment of interest and principal is dependent on catastrophe loss experience. The debt is not fully repaid if a certain level of catastrophic losses occur. As a result of these innovations, insurers have been able to tap the capital markets to help spread catastrophic losses. The successes in this area are encouraging additional growth into the financial risk management field.

Insurers and risk managers have a significant role to play in the field of financial risk management. From the point of view of the firm, the risk of a fire that costs the firm $1 million has the same impact on the firm's financial position as a loss in its bond portfolio of $1 million. Protection is available against both of these risks. A coordinated approach to an organization's risk would be preferable to a segmented approach.

After the shocks of mismanaged financial risks, the failed investments in interest rate derivatives, Nikkei 225 stock index futures, and the later success that financial risk management has had in reducing such exposure, corporations have begun to question whether other risks can be handled in a similar, integrated approach.

The Skills Required for Enterprise Risk Management

Although enterprise risk management represents a return to the roots of risk management, in order to be involved with enterprise risk management, traditional risk managers will need to obtain some additional skills. The starting point is to learn the terminology of finance and financial risk management. Due to their importance as potential investments and the growing use of this form of financing, often involving insurance guarantees, the role of asset backed securities should be given special attention. Although new instruments for financial risk management are constantly being generated, they can generally be broken down into their basic components of forwards, futures, swaps and options to be more easily understood. Traditional risk managers also need to learn about VaR in order to engage any comprehensive risk management process. Knowledge of portfolio theory as a method for dealing with correlated risks is also critical. Simulation and modeling are also important aspects of enterprise risk management. The ability to locate, and exploit natural hedges, those conditions that affect different aspects of an organization in offsetting ways, is vital as well. For example, telephone companies have a natural hedge against major disasters (Molnar, 2000). When a disaster strikes, the company will suffer a loss to its property, but the higher volume of telephone traffic that typically follows a major disaster will help offset this loss. However, the basic approach of identifying, measuring, evaluating, selecting and monitoring risk remains the same. The primary challenge to traditional risk managers is to examine all risks that an organization faces, and not just focus on those that are insurable.

Since enterprise risk management involves so many different aspects of an organization's operations, and integrates a wide variety of different types of risks, no one person is likely to have the expertise necessary to handle this entire role. In most cases, a team approach is used, with the team drawing on the skills and expertise of a number of different areas, including traditional risk management, financial risk management, management information systems, auditing, planning and line operations. The use of a team approach, though, does not allow traditional risk managers to remain focused only on hazard risk. In order for the team to be effective, each area will have to understand the risks, the language and the approach of the other areas. Also, the team leader will need to have a basic understanding of all the steps involved in the entire process and the methodology used by each area.

In assessing the potential losses an organization could experience, many items not covered under hazard risk or financial risk emerge. The company could suffer a significant loss if the chief executive officer were to step down and an adequate replacement could not be found. If the reputation of one of the company's key products is tarnished by a serious loss (Firestone tires, for example), the company could incur significant monetary losses. If the firm is found liable for underpaying taxes by losing a tax dispute, the required payment could be extremely large. A labor dispute could severely impact a firm's operations. A failed merger could have repercussions that puts the firm into a worse financial position than it was in before the negotiations commenced.

Although these risks are both present and significant, the ability to quantify such exposures is far less sophisticated than the approach that can be used for most hazard and financial risks. The lack of data and the difficulty in predicting the likelihood of a loss or the financial impact if a loss were to occur make it hard to quantify many risks a firm faces.

One feature of enterprise risk management is the consideration of offsetting risks within a firm. Catastrophe losses are one example. A major hurricane increases the losses of an insurer, but after most disasters people are more likely to purchase insurance against future catastrophes. Thus, future earnings increase, which can offset, on an enterprise risk management approach, the increase in losses the firm has to pay.

The steps of enterprise risk management are quite familiar to traditional risk managers. Shawna Ackerman, a consultant at MHL/Paratus Consulting, lists these steps as (Ackerman, 2001):

Identify the question(s)
Identify risks
Risk measurements
Formulate strategies to limit risk
Implement strategies
Monitor results
And repeat…

Another consulting firm lists the steps as (ARI 2001):

Identify risk on an enterprise basis
Measure it
Formulate strategies and tactics to limit or leverage it
Execute those strategies and tactics

Monitor process

The steps of enterprise risk management are the same, expect for minor changes in wording, as those first enumerated by Mehr and Hedges in 1963. Enterprise risk management is risk management applied to the entire organization. The basic approach, the goals and the focus of enterprise risk management are the same as those that have worked so effectively for traditional risk managers since the field was first developed.

Conclusion

The impetus for enterprise risk management arose when the traditional risk manager and the financial risk manager began reporting to the same individual in a corporation, commonly the treasurer or chief financial officer. Each risk management specialty had its own terminology, its own methodology and its own focus. However, each dealt with risk the firm was facing. It quickly became apparent that a common approach to risk management would be preferable to an individual approach and an integrated approach preferable to a separatist approach. The evident success of first hazard risk management and later financial risk management has encouraged managers to try to include these and other forms of risk in an overall risk management strategy. Whether this approach succeeds will depend on the ability of those involved in the separate risk categories to develop an integrated approach and extend it to other areas of risk. This is not truly a new form of risk management, it is simply a recognition that risk management means total risk management, not some subset of risks. The new focus on the concept of enterprise risk management provides an opportunity for risk managers to apply their well established and successful approaches to risk on a broader and more vital scale than previously. This is an excellent opportunity to advance the science of risk management.

References

Ackerman, Shawna. 2001. The Enterprise in Enterprise Risk Management. *Casualty Actuarial Society Enterprise Risk Management Seminar.*

ARI Risk Management Consultants. 2001. Enterprise Risk Management: The Intersection of Risk and Strategy. http://www.riskadviser.net/Cases/case.htm

Banham, Russ. 1999. Understanding the Skepticism about Enterprise Risk Management. *CFO Magazine*. April 1, 1999.

Bernstein, Peter L. 1996. *Against the Gods: The Remarkable Story of Risk*. John Wiley and Sons, Inc. New York.

Casualty Actuarial Society Websites:
http://www.casact.org/research/ermsurv.htm
http://www.casact.org/CONEDUC/specsem/erm/2001/handouts/handouts.htm

D'Arcy, Stephen P. 1999. Don't Focus on the Tail: Study the Whole Dog! *Risk Management and Insurance Review*. 2(2):iv-xiv.

Davenport, Edgar W. and L. Michelle Bradley. 2000. Enterprise Risk Management: A Consultative Perspective. *Casualty Actuarial Society Discussion Paper Program* p. 23-42.

Deloach, James and Nick Temple. 2000. *Enterprise-Wide Risk Management: Strategies for Linking Risk and Opportunity.* Financial Times Management.

Doherty, Neil A. 2000. *Integrated Risk Management.* McGraw-Hill New York.

Friedel, Wolfgang F. 2001. Enterprise Risk Management - Fad or Fact? *Casualty Actuarial Society Enterprise Risk Management Seminar.*

Guthrie, Vernon H., David A. Walker and Bert N. Macesker. 1999. Enterprise Risk Management. 17th International System Safety Conference. ABS Group Inc. Risk & Reliability Division and United States Coast Guard Research and Development Center (vguthrie@abs-group.com; dawalker@abs-group.com; bmacesker@rdc.uscg.mil).

Holton, Glyn A. 1996. Enterprise Risk Management. *Contingency Analysis*. (http://www.contingencyanalysis.com/_frame/frameerm.htm)

Hull, John C. 2000. *Options, Futures, and Other Derivatives* (Fourth Edition). Prentice Hall. Upper Saddle River, NJ.

Insurance Services Office. 1999. *Financing Catastrophe Risk: Capital Market Solutions*.

Institute of Internal Auditors. 2001. Risk Management Readings (http://www.theiia.org/ecm/guide-ia.cfm?doc_id=1604)

Jorion, Philippe. 2001. *Value at Risk* (Second Edition) McGraw-Hill New York.

Kawamoto, Brian. 2001. Issues in Enterprise Risk Management: From Theory to Application. Casualty Actuarial Society Spring Meeting.

Lam, James. 2000. Enterprise-Wide Risk Management and the Role of the Chief Risk Officer. *Erisk* March 25, 2000. Erisk.com

Markowitz, Harry M. 1952. Portfolio Selection. *Journal of Finance* 7:77-91.

Mehr, Robert I. and Bob A. Hedges. 1963. *Risk Management in the Business Enterprise*. Richard D. Irwin, Inc. Homewood, IL

Miccolis, Jerry and Samir Shah. 2000. Enterprise Risk Management: An Analytic Approach. Tillinghast - Towers Perrin Monograph.

Molnar, Michele. 2000. More Companies Embrace Enterprise Risk Management. *Office.com.*

Shimpi, Prakash A. 1999. *Integrating Corporate Risk Management.* Swiss Re New Markets

Smith, Michael L. and Robert C. Witt. 1985. An Economic Analysis of Retroactive Liability Insurance. *Journal of Risk and Insurance* 52:379-401.

Smithson, Charles W. 1998. *Managing Financial Risk: A Guide to Derivative Products, Financial Engineering, and Value Maximization* (Third Edition) McGraw-Hill New York.

Taylor, Gary. 2001. New Developments in Enterprise Risk Management in the Energy Industry With a Specific Focus on the Weather Risk Management Market. Casualty Actuarial Society Spring Meeting.

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Enterprise Risk Management

An Analytic Approach

A Tillinghast – Towers Perrin Monograph

Foreword

Business Risk Management…Holistic Risk Management…Strategic Risk Management…Enterprise Risk Management. Whatever you choose to call it, the management of risk is undergoing fundamental change within leading organizations. Worldwide, they are moving away from the "silo-by-silo" approach to manage risk more comprehensively and coherently.

This heightened interest in Enterprise Risk Management (ERM) has been fueled in part by external factors. In just the last few years, industry and government regulatory bodies, as well as institutional investors, have turned to scrutinizing companies' risk management policies and procedures. In more and more countries and industries, boards of directors are now required to review and report on the adequacy of the risk management processes in the organizations they govern.

And internally, company managers are touting the benefits of an enterprise-wide approach to risk management. These benefits include:

- reducing the cost of capital by managing volatility
- exploiting natural hedges and portfolio effects
- focusing management attention on risks that matter by expressing disparate risks in a common language
- identifying those risks to exploit for competitive advantage
- protecting and enhancing shareholder value.

ERM is actually a straightforward process. And, in most cases, the requisite intellectual capital and business practices needed to carry out ERM already exist within the company. But an accurate, useful ERM process is based on sound analytics. Without valid measurements, managing risk is effective and efficient only by chance.

In the following pages, we hope to add analytical rigor to the public discourse on ERM. Drawing from our client experiences, we offer a rational, scientific approach — one grounded in sound principles and practical realities.

"Risk," by definition and by nature, cannot be eliminated. Nor do leading organizations wish it gone. Rather, they want to manage the factors that influence risk so that they can pursue strategic advantage. How to identify and manage these factors is the subject of this monograph.

It is our intention to periodically update this document. We would be most interested in readers' comments and suggestions.

Contents

I Introduction

Purpose of this monograph

Pressure to adopt ERM has increased from both internal and external forces. Although optional in most cases, a formalized risk management culture and its benefits have gained recognition and have fueled interest in the process.

With this monograph, we intend to add analytical rigor to the public discourse on ERM by presenting a scientific approach grounded in sound business principles and practical realities.

In this document, we will:

- define the ERM process
- discuss what motivates organizations to adopt ERM
- describe our conceptual ERM framework and outline the process steps
- detail a comprehensive, analytic approach to ERM
- discuss methods by which organizations implement ERM.

Definition and objective of ERM

We define ERM as follows:

ERM is a rigorous approach to assessing and addressing the risks from all sources that threaten the achievement of an organization's strategic objectives. In addition, ERM identifies those risks that represent corresponding opportunities to exploit for competitive advantage.

ERM's objective — to enhance shareholder* value — is achieved through:

- improving capital efficiency
  - providing an objective basis for allocating resources
  - reducing expenditures on immaterial risks
  - exploiting natural hedges and portfolio effects
- supporting informed decision making
  - uncovering areas of high-potential adverse impact on drivers of share value
  - identifying and exploiting areas of "risk-based advantage"
- building investor confidence
  - establishing a process to stabilize results by protecting them from disturbances
  - demonstrating proactive risk stewardship.

Motivation for considering ERM

External pressures

Some organizations adopt ERM in response to direct and indirect pressure from corporate governance bodies and institutional investors:

- In Canada, the Dey report, commissioned by the Toronto Stock Exchange and released in December 1994, requires companies to report on the adequacy of internal control. Following that, the clarifying report produced by the Canadian Institute of Chartered Accountants, "Guidance on Control" (CoCo report, November 1995), specifies that internal control should include the processes of risk assessment and risk management. While these reports have not forced Canadian-listed companies to initiate an ERM process, they do create public pressure and a strong moral obligation to do so. In actuality, many companies have responded by creating ERM processes.
- In the United Kingdom, the London Stock Exchange has adopted a set of principles — the Combined Code — that consolidates previous reports on corporate governance by the Cadbury, Greenbury and Hampel committees.

* In this monograph, the emphasis is on shareholders rather than the broader category of stakeholders (which also includes customers, suppliers, employees, lenders, communities, etc.). Though some observers prefer to define the scope of ERM to include the interests of all stakeholders, we believe this is not pragmatic at the current evolutionary state of ERM and would result in too diffuse a focus. While shareholder value is not directly relevant to some organizations (e.g., privately held and nonprofit entities), the concepts and approaches developed in this monograph clearly apply to those organizations.

This code, effective for all accounting periods ending on or after December 23, 2000 (and with a lesser requirement for accounting periods ending on or after December 23, 1999), makes directors responsible for establishing a sound system of internal control, reviewing its effectiveness and reporting their findings to shareholders. This review should cover all controls, including operational and compliance controls and risk management. The Turnbull Committee issued guidelines in September 1999 regarding the reporting requirement for nonfinancial controls.

- Australia and New Zealand have a common set of risk management standards. Their 1995 standards call for a formalized system of risk management and for reporting to the organization's management on the performance of the risk management system. While not binding, these standards create a benchmark for sound management practices that includes an ERM system.
- In Germany, a mandatory bill — the Kon TraG — became law in 1998. Aimed at giving shareholders more information and control, and increasing the accountability of the directors, it includes a requirement that the management board establish supervisory systems for risk management and internal revision. In addition, it calls for reporting on these systems to the supervisory board. Further, auditors appointed by the supervisory board must examine implementation of risk management and internal revision.
- In the Netherlands, the Peters report in 1997 made 40 recommendations on corporate governance, including a recommendation that the management board submit an annual report to the supervisory board on a corporation's objectives, strategy, related risks and control systems. At present, these recommendations are not mandatory.
- In the U.S., the SEC requires a statement on opportunities and risks for mergers, divestitures and acquisitions. It also requires that companies describe distinctive characteristics that may have a material impact on future financial performance within 10-K and 10-Q statements. Several factors broaden the requirement to report on the risks to the organization, leading to setting in place an enterprise-wide approach to risk management:
  - The report, "Internal Control — An Integrated Framework," produced by the Committee of the Sponsoring Organizations of the Treadway Commission (COSO), favors a broad approach to internal control to provide reasonable assurance of the achievement of an entity's objectives. Issued in September 1992, it was amended in May 1994. While COSO does not require corporations to report on their process of internal control, it does set out a framework for ERM within an organization.
  - In September 1994, the AICPA produced its analysis, "Improving Business Reporting — A Customer Focus" (the Jenkins report), in which it recommends that reporting on opportunities and risks be improved to include discussion of all risks/opportunities that:

    — are current

    — are of serious concern

    — have an impact on earnings or cash flow

    — are specific or unique

    — have been identified and considered by management.

    The report also recommends moving toward consistent international reporting standards, which may include disclosures on risk as is required in other countries.

Institutional investors, such as Calpers, have begun to push for stronger corporate governance and to question companies about their corporate governance procedures — including their management of risk.

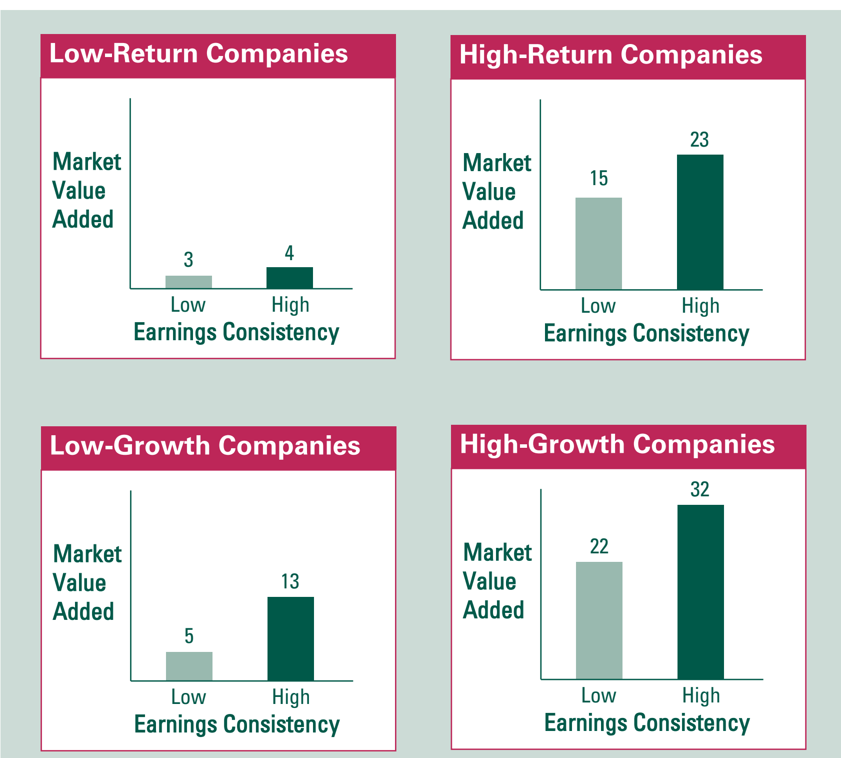
Internal reasons

Other organizations simply see ERM as good business. For example:

- The Board of Directors at a large utility mandated an integrated approach to risk management throughout the organization. They introduced the process in a business unit that was manageable in size, represented a microcosm of the risks faced by the parent and did not have entrenched risk management systems. This same unit was the focus of the parent's strategy for seeking international growth — a strategy that would take the organization into unfamiliar territory — and had no established process for managing the attendant risks in a comprehensive way.

- The CFO of a manufacturing company with an uninterrupted 40-year history of earnings growth embarked on ERM. This step followed the company's philosophy of "identifying and fixing things before they become problems." The movement was spurred by the company's rapid growth, increasing complexity, expansion into new areas and the heightened scrutiny that accompanied its recent initial public offering.
- A large retail company's new Treasurer, with the support of the CFO, wanted to "assess the feasibility of taking a broader approach to risk management in developing the organization's future strategy." As part of this effort, she hoped to "evaluate our hazard risk and financial risk programs and strategies, to identify alternative methods of organizing and managing these exposures on a collective basis."
- The Chairman of the Finance Committee of the Board at a manufacturing company complained about reports from Internal Audit that repeatedly focused on immaterial risks. His concern led to formation of a cross-functional Risk Mitigation Team to identify and report on processes to deal with risks within an ERM framework. The team now reports directly to the finance committee on a quarterly basis.

FIGURE 1



Companies with higher earnings consistency tend to have much higher stock valuations than their similarly situated competitors. Details and definitions are presented in Appendix A.

These organizations view systematic anticipation of material threats to their strategic plans as integral to executing those plans and operating their businesses. They seek to eliminate the inefficiencies built into managing risk within individual "silos." And they appreciate that their cost of capital can be reduced through managing volatility.

Some observers argue that investors do not put a premium on an organization's attempt to manage volatility. These observers maintain that investors can presumably achieve this result more efficiently by diversifying the holdings in their own portfolio. They argue further that investors do not appreciate, and do not reward, an organization that spends its resources on risk management to smooth results on investors' behalf.

Our research into the link between performance consistency and market valuation, however, indicates otherwise. We found that consistency of earnings explains a high degree of difference in share value (specifically, "market value added") among companies within an industry. This is true even after allowing for other influences such as growth and return (see *Figure 1* and Appendix A). Investors assign a higher value, all else equal, to organizations whose earnings are more consistent than those of their peers. This clearly reduces the cost of capital for these organizations.

In summary, organizations can use ERM to enhance the drivers of share value: growth, return on capital, consistency of earnings and quality of management. ERM can identify and manage serious threats to growth and return while identifying risks that represent opportunities to exploit for above-average growth and return. Achieving earnings consistency is, of course, a central goal of ERM. And institutional investors increasingly define management quality to include enterprise-wide risk stewardship.

II Framework for ERM

Company information and procedures already in place can make the ERM process efficient and effective. Our conceptual framework for ERM consists of four elements.

Assessing risk

Risk assessment focuses on risk as a threat as well as an opportunity. In the case of risk-as-threat, assessment includes identification, prioritization and classification of risk factors for subsequent "defensive" response. In the case of risk-as-opportunity, it includes profiling risk-based opportunities for subsequent "offensive" treatment.

Shaping risk

This "defensive track" includes risk quantification/modeling, mitigation and financing.

Exploiting risk

This "offensive track" includes analysis, development and execution of plans to exploit certain risks for competitive advantage.

Keeping ahead

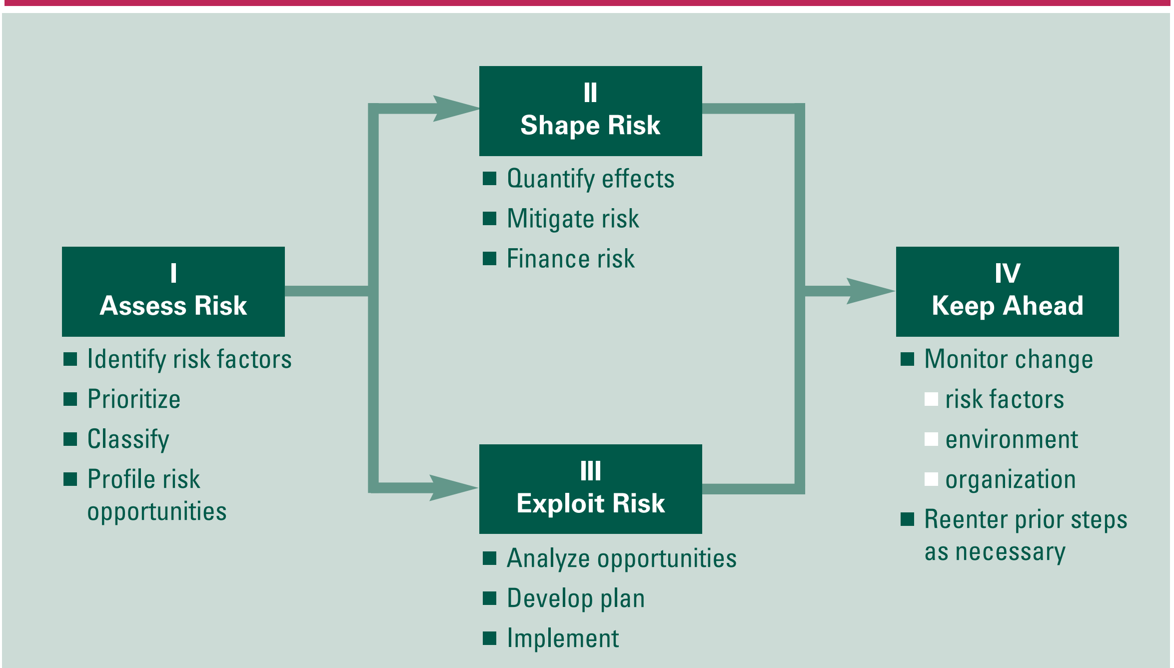
The nature of risk, the environment in which it operates, and the organization itself change with time. The situation requires continual monitoring and course corrections.

The chapters that follow provide a fuller description of the above elements (outlined in *Figure 2*).

The larger part of the discussion in this monograph is on the first two elements — risk assessment and risk shaping — as these create the foundation for the remaining elements. Accordingly, there will be more focus on the defensive track of ERM.

FIGURE 2

The Conceptual Approach to ERM



The conceptual approach to ERM is straightforward.

III A Rational Approach to Assessing Risk

Overview

We approach risk assessment believing that managing risk effectively requires measuring risk accurately — and that accurate risk measurement requires well-formulated risk modeling. Such measuring and modeling:

- allow senior management to see a compelling demonstration of the "portfolio effect," i.e., the fact that independent and/or favorably correlated risks tend to offset each other without the organization having to invest in explicit hedges
- promote the proper allocation of capital resources to risks that really matter
- permit sizing of investments in risk remediation
- provide an objective framework for systematic risk monitoring.

Do all risks that face an organization need modeling? And isn't model-building on this scale daunting?

The answer to the first question is: "No." Methods to prioritize risk factors can screen for those that require modeling. These methods are qualitative; we focus on these later in this chapter.

The answer to the second question is: "Not typically." These models often have been built and exist in some form somewhere in the organization. This will be the focus of Chapter IV.

Before we discuss the steps in risk assessment, we should distinguish risks from the risk factors underlying them. Here we focus on the negative side of risk — as a threat, not as an opportunity. In this context, risk is the possibility that something will prevent — directly or indirectly — the achievement of business objectives. Risk factors are the events or conditions that give rise to risk. Loss of market share is a risk; lack of preparedness for the entry of new competitors is a risk factor. Risk is not something that can be directly managed or controlled. Risk factors, however — the causes of risk — can be. Therefore, managing risk, and particularly assessing risk, requires focusing on its causes rather than its manifestations.

STEP 1
Identify risk factors

In this initial step, a wide net is cast to capture all risk factors that potentially affect achieving business objectives. Risk factors arise from many sources — financial, operational, political/regulatory or hazards. The key characteristic of each is that it can prevent the organization from meeting its goals. In fact, if a risk factor does not have this potential, it is not truly a risk factor under an enterprise-wide interpretation of risk. Thus, the first "screen" through which a candidate risk factor must pass is materiality.

In identifying risk factors, we favor a qualitative approach — gathering material from interviews with experts and reviewing documents. The interviews typically span the organization's:

- Senior management
- Operations management
- Corporate staff, including:
  - Finance
  - Treasury
  - Legal
  - Audit
  - Strategic Planning
  - Human Resources
  - Risk Management
  - Safety
  - Environmental.

These interviews solicit informed opinion on:

- how the business works, and the way components of the business — the interviewees' realms of responsibility — mesh
- key performance indicators used to manage the business and its components
- tolerable variation in key performance indicators over relevant time horizons
- events or conditions that cause variations beyond the risk tolerances, and the probable frequency and possible maximum effect of these.

Often we find it helpful to supplement internal interviews with interviews among the organization's external partners, their counterparties (banks, insurers, brokers), analysts, customers, and — on occasion — competitors.

We also review the organization's strategic plans, business plans, financial reports, analyst reports and risk stewardship reports.

From all these data and information, a picture emerges of the organization's:

- corporate culture
- objectives
- forms of capital (human, financial, market and infrastructure)
- business processes (which convert the capital into cash flows)
- control environment
- roles and responsibilities
- key performance measures
- risk tolerance levels
- capacity and readiness for change
- preliminary list of risk factors.

Importantly, this approach starts with the business, not a checklist of risks — far different from an audit-type approach. In other words, this approach goes from the top down and not the bottom up. Such an organic method is strongly preferable because preconceived checklists of risk factors are usually incomplete. Further, the most crucial risk factors are usually unique to each organization and its culture. This alone makes generic checklists far less relevant than a business-first approach.

STEP 2
Prioritize risk factors

The resulting list of risk factors (typically several dozen long at this stage) is not yet useful or actionable, although each factor has passed the materiality screen. It now requires prioritizing.

In Step 1 (Identify risk factors), we compiled information on each risk factor's likelihood, frequency, predictability and potential effect on the organization's key performance indicators. We also examined the quality of the process, systems and cultural controls in place to mitigate these factors. At this stage, the information is subjective, but quite sufficient. Now, the objective is to cull the list of these factors into a manageable number for senior management. The attributes of each factor can be combined in an overall score that, when combined with subjective judgment on the timing and duration of the financial impact, can be expressed as a "net present value" score. In the example in *Figure 3,* this "NPV" score is on a scale of 1 (low) to 5 (high). Once scores are assigned, we can sort the risk factors from low to high and produce a prioritized list.

A team of risk management experts typically does this evaluation and scoring. They often collaborate with representatives of management. In addition, we find a follow-up questionnaire or focus group(s) extremely helpful for cross-validation purposes. In these, the interviewees view the collective results of the identification step — the full list of risk factors, the consensus view on key performance indicators and risk tolerances, etc. Then, with this richer context and some facilitation, they can prioritize risks. We compare the results of this exercise with those from the independent prioritization conducted by the expert team, and the differences are reconciled.

The number of risk factors that will ultimately pass through the prioritization screen is often known before the process begins. Given the demands on senior management, expecting them to concentrate on a dozen or more "top priority" risk factors is unrealistic. Generally, six or less is manageable, but this depends on the organization. Also, natural breakpoints in the prioritized list and strategic links among the risk factors can influence the ultimate number. The short list should, however, contain items deserving of consideration at the highest levels of the organization — factors that should influence the strategic plan and the affected business plans, alter the day-to-day priorities of business unit managers and affect the behavior of the rank and file.

STEP 3

Classify risk factors

Still, any list of risk factors, however short and prioritized, is a sterile device. Organizing this information to clearly indicate what type of risk-shaping action is necessary comes next.

We have used several classification schemes in our work, some more detailed than others, each tailored to the client organization. One general scheme that may have nearly universal relevance is described below (see *Figure 4*). Additional refinements can be added as appropriate.

In this scheme, high-priority risk factors are of two types. One is characterized by the fact that the environment in which they arise is familiar to the organization, and the skills to remedy those risk factors are already in-house. However, for some reason, these risk factors had not been given the attention they deserve. We label these "manageable risk factors." Other risk factors arise because the organization enters unfamiliar

FIGURE 3

When Prioritizing Risk Factors...

...subjective scoring is appropriate at this stage

Risk Factors	Likelihood	Severity	Quality of Controls	Aggregate "NPV" Score (1-5)
A. Strategy				
Informal planning, process and communications allow surprises	H	H	L	4.5
Market share and earning objectives are not aligned	H	L	L	3.0
⋮				
B. Growth				
Infrastructure is increasingly strained, will be difficult to retain culture and values with the changes that growth demands	H	H	L	4.5
Increased size creates more opportunity for mistakes	M	L	M	2.0
⋮				
C. Company Reputation				
Pressure to make numbers may prompt behavior that will impair company's credibility with financial markets	M	H	H	3.5
Adverse publicity (e.g., business practices, ethics) can affect image across multiple brands	L	H	H	2.5
⋮				
D. Human Resources				
⋮				
J. Systems				
⋮				

Risk factors can be prioritized using a subjective process.

FIGURE 4

When Classifying Risk Factors...

...use a scheme that implies action

"Manageable" Risk Factors	"Strategic" Risk Factors
■ Known environment	■ Unfamiliar territory
■ Capabilities and resources on hand to address	■ Capabilities or resources may not be in place
■ Fell between the cracks?	■ Major change in market or business
Just get on with it	Requires allocation of capital or shift in strategic direction

Proper classification clearly implies the appropriate risk-shaping action.

business territory (due, perhaps, to a major acquisition, a powerful new competitor or a significant change in customer buying patterns), or the organization lacks the skills necessary to respond. These are considered "strategic risk factors" and may require significant capital outlay and/or a major change in strategic direction.

Manageable risk factors in our experience include:

- "The R&D division is not keeping pace with the demand for new products."
- "Contingency planning is weak in the critical production facilities."
- "Mid-level employees are dissatisfied with their opportunities for advancement."

Strategic risk factors we have encountered include:

- "The share value is dependent on continuing uninterrupted earnings growth; this growth must come from top-line revenue growth; and opportunities for top-line growth are limited without branching out of the organization's product line and/or niche market."
- "Needed infrastructure changes clash with the current success formula and culture."

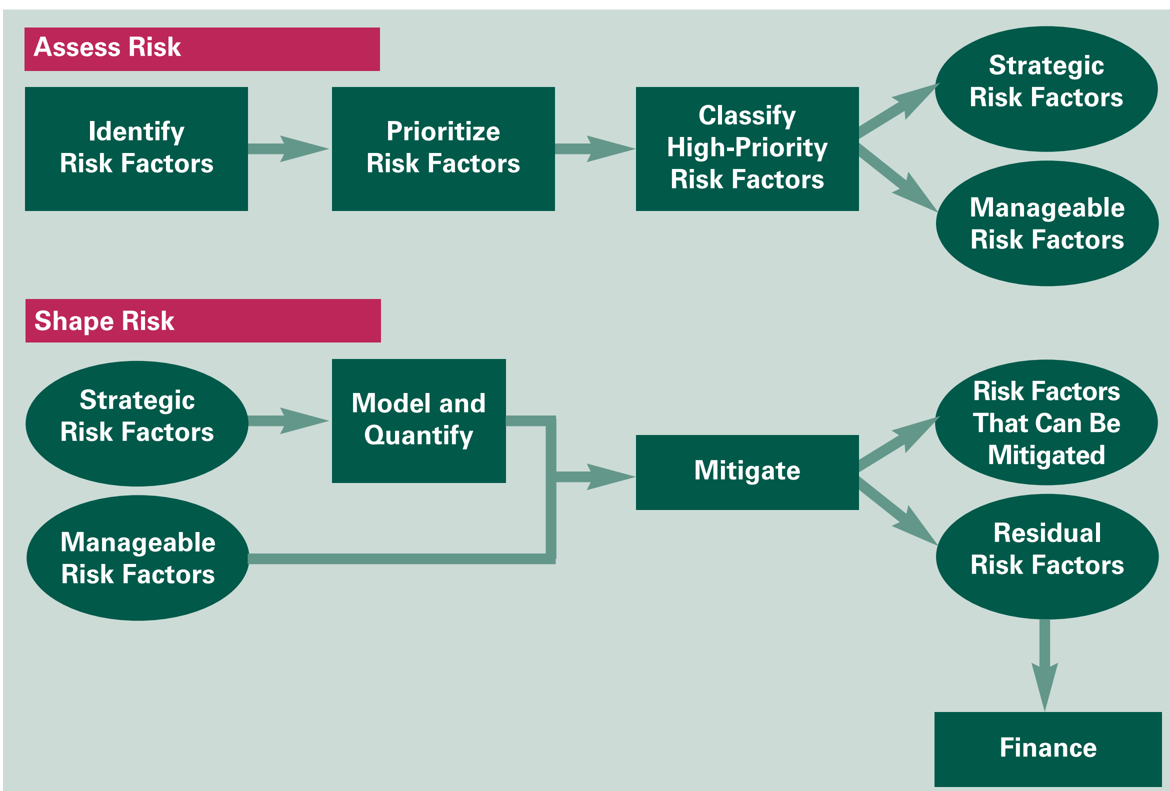
The proper response to manageable risk factors is to "just get on with it" — in other words, deal with them. The relevant skills already exist; they just need to be refocused on these high-priority items. Strategic risks, however, require greater analysis; this is covered in Chapter IV.

Recap… and segue

The steps described above are illustrated below (*Figure 5*). This graphic also illustrates the follow-on steps — the risk-shaping steps — that are the subject of the next chapter. The graphic demonstrates that not all risk factors need to be quantified and modeled, nor do all risk factors need to be financed. Risk factors needing quantification are those that pass through the "triple screen" — they are material, high-priority *and* strategic. Risk factors that need to be financed pass through the first two screens and cannot be fully mitigated through other means.

Underlying our approach to risk shaping — described in Chapter IV — is the premise that modeling, quantifying and formulating the strategy for mitigation and financing can be carried out simultaneously.

FIGURE 5



Triple screening in risk assessment creates efficiency in risk shaping.

IV A Scientific Approach to Shaping Risk

Overview

In this section, we will describe our approach to shaping risk and provide illustrations of its application. The approach to risk shaping relies heavily on Operations Research methods such as applied probability and statistics, stochastic simulation and portfolio optimization. To our knowledge, no organization has implemented this approach in its entirety as of the date of this publication, although we know of several that use portions of it in their incremental pursuit of ERM. (In Chapter VI, we describe how some of these organizations have gotten started.)

The Four Steps in Our Approach

Model the Various Sources of Risk → **Link Risk Sources to Financial Measures** → **Develop Portfolio of Risk Remediation Strategies** → **Optimize Investment Across Portfolio of Strategies**

In the first step, each source of risk is modeled as a probability distribution, and the correlation among the risk sources is determined. These probability distributions are typically expressed in terms of different operational and financial measures. The second step links these disparate distributions to a common financial measure (e.g., Free Cash Flow) through a stochastic financial model. These two steps represent the bulk of the analytical effort. At this stage, we have a holistic financial model of the business that can be used to:

- measure the volatility of the financial metric(s) under current operating conditions
- analyze the impact of risk management decisions through "what-if" scenarios.

The third step involves developing risk remediation strategies to be evaluated using the stochastic financial model. This basket of strategies represents a portfolio of risk management investment choices. In the final step, the ERM budget is allocated optimally across these strategies using portfolio optimization methods. Each step is described in greater detail below.

To illustrate this approach, we will introduce a hypothetical company (let's call it HypoCom) facing a broad array of strategic risks and show how the company would implement this approach in shaping these risks. Assume that HypoCom is a manufacturing company and has the following profile:

- Sells its product to retailers in the United States and Europe — with limited competition
- Has production plants in France, Mexico and Indonesia that deliver products to retailers through HypoCom's own distribution network
- Faces the following risks in the next fiscal year:
  - fire at a warehouse
  - volatility in the price of the raw materials used in the production process
  - possible employee union strike at the plant in France
  - possible new competitor entering the market.

While a real company, similar to HypoCom, would face many risks, we have limited their number here for the sake of simplicity. Please note, however, that the risks were selected to span those that are traditionally considered within the domain of risk management (hazard and commodity price risks) and those that are not (operational and competitor risks).

Again, to keep the example simple, we assume a one-year time horizon. At the end of this section, however, we discuss extending these steps to a more typical multi-period decision horizon.

STEP 1

Model various risk factors individually

Generate probability distributions

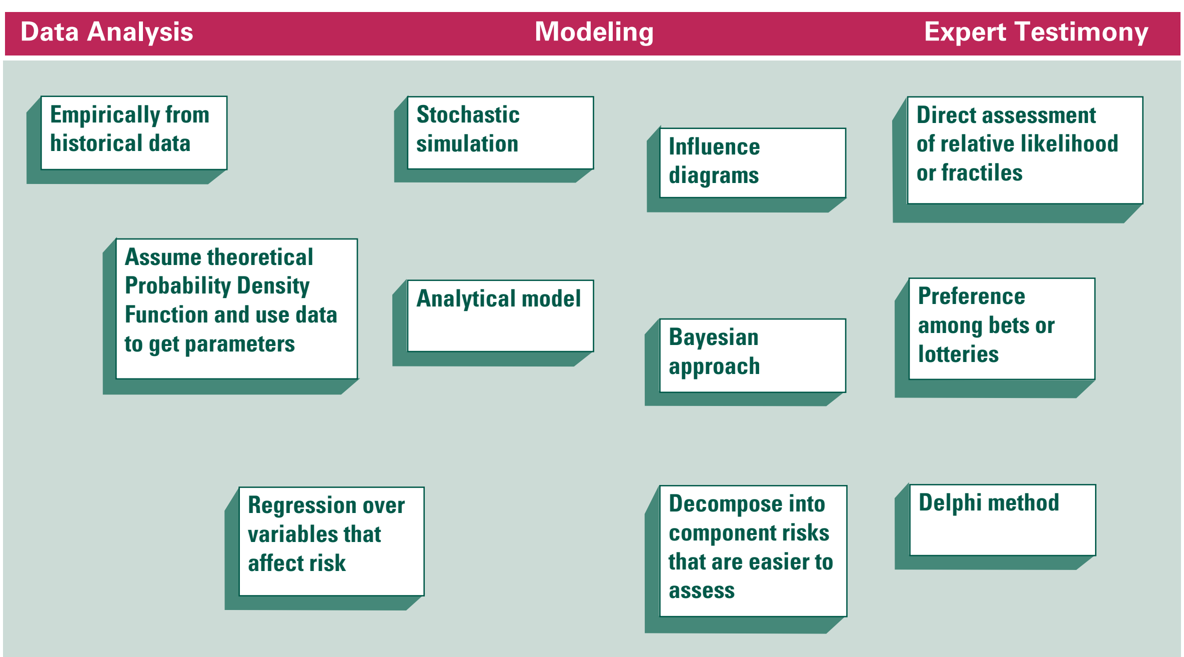
In Chapter III we outlined the approach for identifying which risk factors need to be modeled. Each risk factor contains uncertainty about how, when and to what degree it will manifest itself. This uncertainty is represented as a probability distribution. No one approach for developing probability distributions can be used for all the risks that an enterprise faces.

Risks that fall within the traditional domain of risk management — for instance, insurable risks or risks that can be hedged in the financial markets — are typically modeled using statistical methods that rely on the availability of historical data. However, when the domain is extended to enterprise-wide risks, it is unlikely that enough historical data exist to employ the same methods. Here, it is more likely that assessment of the uncertainty will be based entirely on expert testimony. Also, some risk sources will have to be modeled based on historical data combined with assumptions set by experts. Extending risk management to enterprise-wide risks suggests a continuum of methods for developing probability distributions. Such a continuum ranges from relying entirely on data to relying on expert testimony.

Figure 6 identifies methods for assessing probability distributions along this continuum. Readers of this monograph are likely to be familiar with methods based primarily on historical data (leftmost section of Figure 6). Therefore, instead of describing them, we have included references to source documents at the end of this monograph. At the opposite end of the continuum, there are formal methods developed and used by decision and risk analysts to elicit expert testimony for assessing uncertainty. We have provided brief descriptions of some of these in Appendix B. In the middle of the continuum, stochastic simulation modeling predominates for combining historical data and assumptions set through expert testimony. We will use this method to model the risk associated with an employee union strike at the HypoCom production plant in France.

(continued on page 16)

FIGURE 6



A continuum of methods for developing probability distributions ranges from those relying on data to those that rely on expert testimony. The positions of the methods identified above suggest which to use depending on the availability of data.

HypoCom – developing probability distributions for the four risks

Risk 1

Fire

A fire at a plant or warehouse can result in direct and indirect loss of sales volume. Direct losses result from destruction of inventory and work in progress. Indirect losses result from a prolonged interruption of production, through loss of short-term sales and perhaps through loss of market share. These risks have been insurable for a long time. Reliable methods exist for measuring the frequency and severity of losses based on review of historical data and business interruption worksheets. We will assume that for HypoCom, the frequency distribution is negative binomial and the severity distribution is lognormal (see references in Chapter VII for descriptions of these distributions).

Risk 2

Volatility in price of raw materials

Historical price data for commodities can be obtained from HypoCom's own purchase data or through financial markets if the commodity is traded on a futures exchange. Given the availability of data, several methods exist for developing the probability distribution. These are:

- Use empirical distribution
- Assume lognormal distribution using the sample mean and standard deviation
- Assume a stochastic process (e.g., jump diffusion) and use simulation to generate distribution of price movement.

An example of a stochastic process is the Schwartz-Smith two-factor model for the behavior of commodity prices (Schwartz & Smith 1999). The two-factor approach models both the uncertainty in the long-term trend and the short-term deviation from that trend.

For the sake of this example, we will assume that HypoCom faces a lognormally distributed price with a 2% standard deviation from the current price.

Risk 3

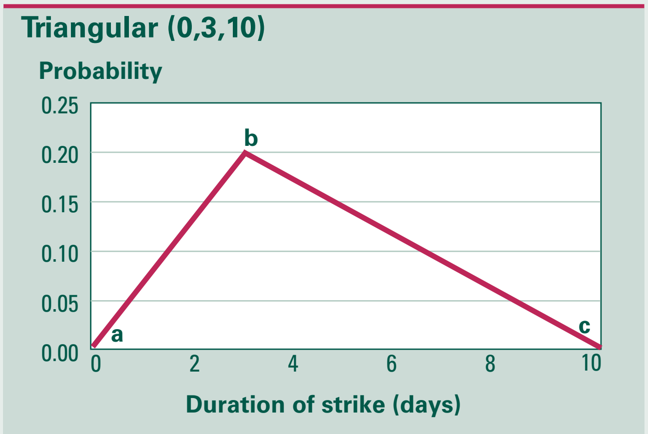
Employee union strike

An employee strike at the plant in France results in losses in sales volume. HypoCom services its European and U.S. markets from production at three plants (France, Mexico and Indonesia). This strike would result in a temporary shutdown of the plant in France. If the other two plants have capacity to increase production quickly enough to satisfy all demand, then there is little risk of loss in sales. But if all three plants are already running at high utilization (a more likely scenario), then the loss of one plant would result in longer lead times to market — the time from order placement to delivery. The strike would then affect HypoCom's ability to satisfy orders and lead-time commitments or expectations; this would result in a short-term loss of sales or possibly market share.

The probability distribution for the sales volume loss can be developed in three steps. First, determine the probability distribution for the length of the strike. It's quite likely that development of this distribution will have to be based almost entirely on expert testimony. As illustrated in Figure 6, there are several methods for assessing probabilities based on expert testimony: the Delphi method, eliciting preferences among bets or lotteries, and directly assessing relative likelihood or fractiles (see Appendix B for details on these methods). The labor relations manager(s) at HypoCom can be interviewed using one of these methods to determine the probability distribution for the length of the strike. For example, the result may be a triangular distribution as illustrated in *Figure 7*.

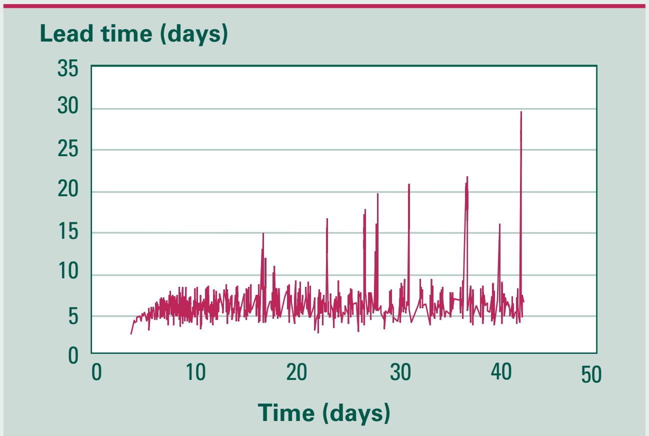
Second, develop a distribution on lead times conditioned on the length of the strike. We have developed a discrete-event stochastic simulation model of HypoCom's distribution network, using graphical, animated simulation software called ProModel®. The simulation modeled stochastic arrival of demand based on

FIGURE 7



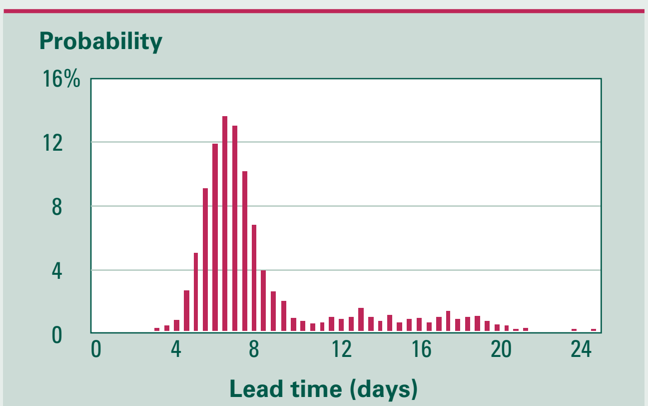
Triangular probability distribution with parameters minimum, mode and maximum (a, b and c, respectively). The expected value is (a+b+c)/3 and the standard deviation is $(a^2 + b^2 + c^2 - ab - bc - ac)/18$. This distribution is used often as a rough model when there is little historical data.

FIGURE 8



The chart shows the impact of a strike on lead times from one of the simulation runs. The strike starts on the 20th day and can last anywhere from 1 to 10 days, based on the probability distribution in Figure 7. You can see that the impact of the strike is felt long after the strike is over.

FIGURE 9



Discrete probability mass distribution generated from the lead-time data in Figure 8. The extended tail toward longer lead times is a consequence of an employee strike.

historical data, production rates at each of the plants and the logistics of distribution from the plant to regional distribution centers and then to retailers. It incorporated a distribution policy of supplying those distribution centers with the greatest backlog of orders. Inputs to this model are typically easy to get; in fact, many organizations already have a stochastic supply chain model used to optimize the logistics of their distribution network. The effect of the strike was simulated by shutting production at the plant in France and recording the increase in lead times. The chart of individual lead times in *Figure 8* is an output from a simulation run.

We usually run simulations a statistically valid number of times to attain a high level of confidence in the results. An empirical distribution of lead times based on these simulated data is shown in *Figure 9.*

Finally, determine the loss in sales conditioned on the increase in the lead times. With information in hand on the increase in the lead times, the sales and marketing managers at HypoCom would assess the effect on sales. One of the probability assessment methods for expert testimony described in Appendix B would be used here. The assessment would reflect contractual agreements with retailers as well as lead-time expectations and the competitive environment. So the final distribution on the decrease in the number of sales may be represented by a triangular distribution with parameters min. = 0, most likely = 4 million, max. = 10 million.

Risk 4
New competitor

Expert testimony provides the entire basis for the assessment of uncertainty associated with a new competitor. This process entails interviewing sales and marketing managers of HypoCom either individually or as a group. Any method described in Appendix B could be used here.

Here we develop a probability distribution on how new competition affects sales volume loss. It is helpful to dissect risk events into conditional causal events. For HypoCom, the causal events are illustrated in *Figure 10.*

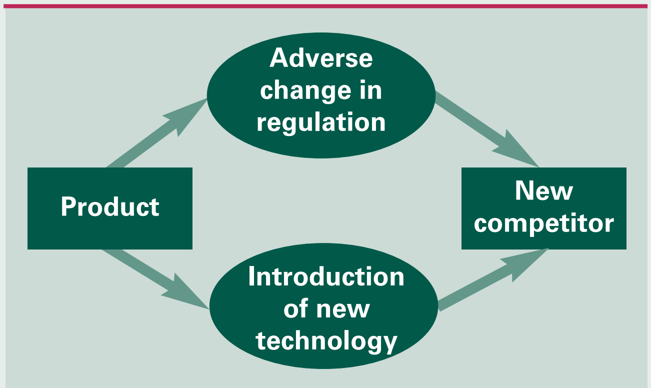
The probability of loss in sales volume due to competition, P(C), can be decomposed into:

$$P(C) = \Sigma_i P(C_i \mid R_i, T_i) P(R_i, T_i)$$

where i is the product index, $P(R_i, T_i)$ is the joint probability of an adverse change in regulation (R_i) and introduction of new technology (T_i) and $P(C_i \mid R_i, T_i)$ is the conditional probability of a loss in sales volume for product i due to new competition. If regulatory changes and introduction of new technology are not highly correlated, then $P(R_i, T_i)$ can be decomposed into the product of $P(R_i)$ and $P(T_i)$.

Instead of assessing P(C) directly, it is easier to ask different experts to assess the

FIGURE 10



Given the product, the possibility for change in regulation or introduction of new technology could influence the loss in sales due to competition.

conditional and joint probabilities. Company lobbyists are interviewed to assess the probability of adverse regulation for a specific product, $P(R_i)$, using one of two methods: preference among bets or judgment of relative likelihood (see Appendix B).

Managers of the Research and Development function are interviewed to assess the probability of introduction of new technology, $P(T_i)$. Finally, sales and marketing managers are interviewed to assess the probability of a new competitor, given the state of new regulation and technology, $P(C_i \mid R_i, T_i)$. Of course, experts may be interviewed as a group using the Delphi method (see Appendix B) instead of separately. This process is applied over all products of interest and the results summed according to the formula indicated above.

Determine correlation among risk sources

It is not enough to develop probability distributions on individual risk sources. One primary benefit of managing risks on an enterprise-wide basis is being able to take advantage of natural hedges and to explicitly reflect correlation among risks. Therefore, it is necessary to develop a matrix of correlation coefficients among pairs of risks that would be used in the next step to link the individual risk sources to a common financial measure.

It is unlikely that relevant data will exist to develop correlation among risks that span an enterprise. Thus, it is likely that this will have to be developed based on professional judgment and expert testimony. In some cases, it may be easier to develop correlations between risks implicitly by analyzing their correlation with a common linking variable. This process also ensures that a correlation matrix is internally consistent.

FIGURE 11

	Fire	Commodity Price	Union Strike	New Competitor
Fire	1.0	0.0	0.2	0.0
Commodity Price	0.0	1.0	0.0	-0.5
Union Strike	0.2	0.0	1.0	0.7
New Competitor	0.0	-0.5	0.7	1.0

Correlations among risks are modeled using correlation coefficients among risk pairs. For example, the risk due to commodity price fluctuations is negatively correlated with a new competitor entering the market.

For HypoCom, we would expect a negative correlation between the commodity price movements and a new competitor entering the market. If the commodity price increases, it creates a greater barrier to entry into the market for a new competitor and vice versa. However, a union strike is probably positively correlated with competition. Finally, there may be some slight correlation between a union strike and the incidence of fire.

It is unlikely that correlations would be determined with a high degree of precision. Rather, it is more likely that they could be judged in fuzzy terms such as high, medium or low. These terms suggest some natural ranges for correlation coefficients such as: high correlation = .70 to .80, medium correlation = .45 to .55, low correlation = .20 to .30. Within these ranges, there should be little sensitivity on the results. The inclusion of correlations should have a significant impact on the results, but the error within these ranges should have little impact. Using these as guides, a Correlation Coefficient Matrix can be developed for HypoCom as shown in *Figure 11*.

STEP 2

Link risk factors to common financial measures

Select financial metrics

The prior step provides a set of probability distributions representing enterprise-wide risks. Note that the probability distributions were expressed in terms of different units. We modeled the union strike as a probability distribution on lead time and then sales volume. Commodity price risk was modeled in terms of the price of raw materials. Other risks would be modeled in terms of the operational and financial measures that they directly affect. In this step, all these risks are combined and linked to one financial measure.

Managers of different organizations vary in their preference and propensity for the financial measures by which they manage the business. The financial measure will also vary depending on the objectives and goals of the organization. Above all, it is important that there is general agreement on the financial measure selected. For this document, we will use Free Cash Flow (FCF) to capture the impact of risk on both the income statement and balance sheet.

Develop a financial model to link risks to financial metric

Once a financial measure is selected, we can then model the aggregate impact of the sources of risk on the financial measure. We can construct a pro forma FCF model by decomposing each element in the calculation of FCF into its constituent metrics. See *Figure 12* for an illustration of this. The elements should be broken down to the level of the operational and financial measures used for modeling the individual risks in Step 1.

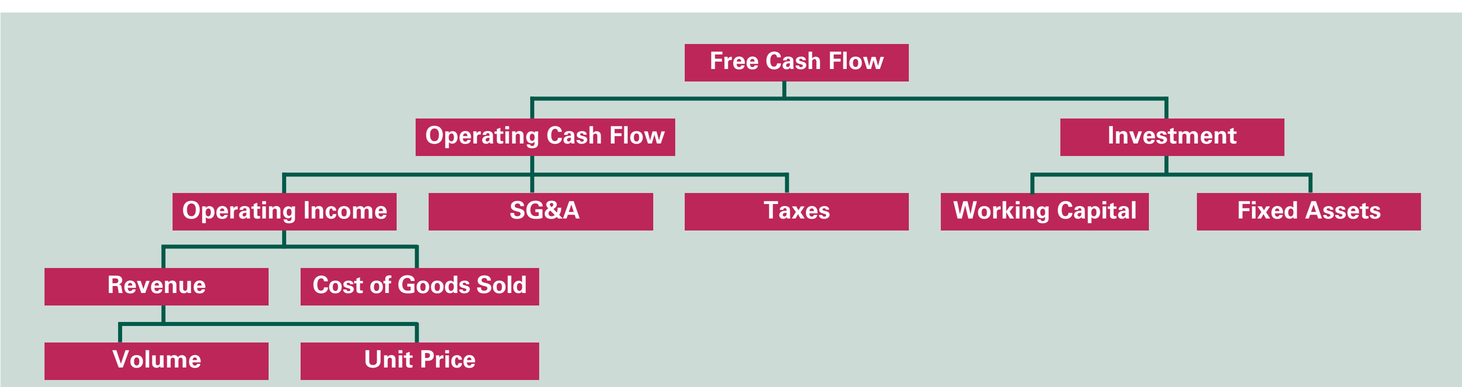
Some elements of the FCF model may be stochastic without consideration of the risks from Step 1. For example, there is some inherent uncertainty in product demand and price as well as cost of goods sold. These measures may fluctuate based on supply and demand economics. These inherent uncertainties are included in the base FCF model. The probability distributions from Step 1 are then added to the corresponding elements of the model. Finally, the Correlation Coefficient Matrix (from Step 1) is added to the model to reflect the interaction among the sources of risk. The resulting stochastic pro forma financial model links all the risks to FCF, the financial measure by which the risk remediation strategies will be evaluated in the next two steps.

Measure current level of enterprise risk before mitigation strategies

Before proceeding to risk remediation strategies, however, it is worth taking note of the value of the model thus far. At this point, we have a financial model that can be used to determine the current level of volatility in FCF. This information by itself would be extremely valuable in budgeting and financial planning. This analysis helps move managers' thinking away from the one-dimensional certainty of typical budgets and toward the range of possible outcomes and managing probable rather than definite outcomes.

(continued on page 21)

FIGURE 12



Free Cash Flow is decomposed into its elements: Operating Cash Flow and Change in Investment, which are further decomposed. Each element is broken down into its constituents until all operational and financial measures used for the distributions in Step 1 are isolated.

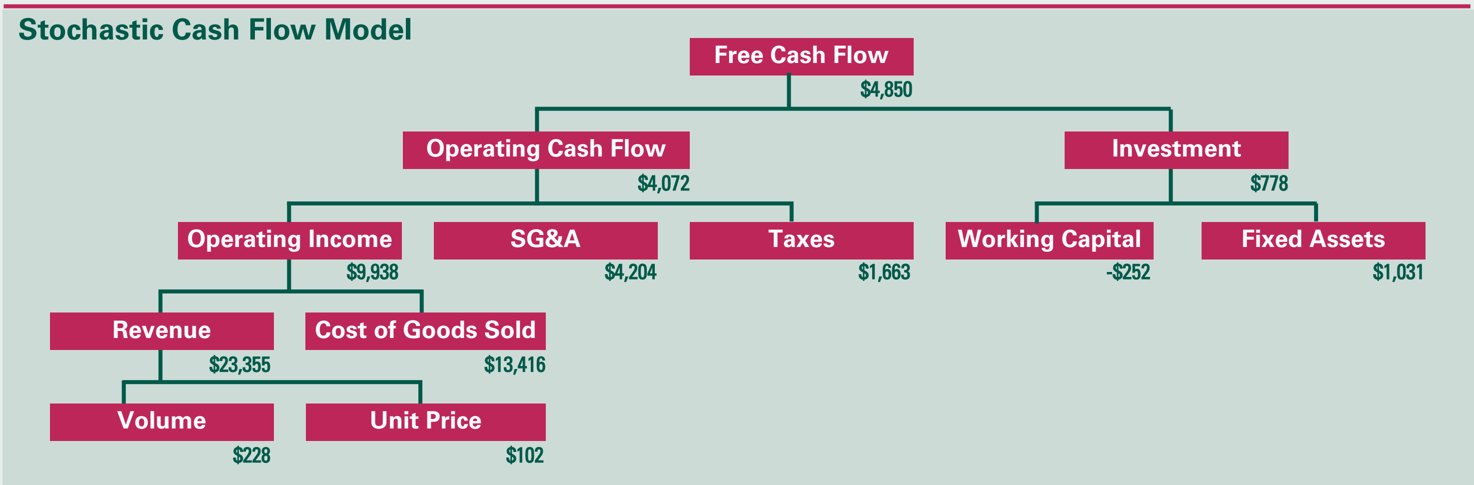
For HypoCom

We developed an FCF model (see *Figure 13*). This model includes inherent uncertainty in volume, price and cost of goods sold. It also includes a correlation of -0.7 between volume and price, and a correlation of +0.5 between price and cost of goods sold before inclusion of the four risks from Step 1.

The fire risk effect on FCF was modeled by layering on the probability of loss in Volume developed in Step 1 (see *Figure 14A*). Also, an adjustment was made to Working Capital and Fixed Assets to reflect loss of inventory and the investment in rebuilding the plant destroyed by fire. The size of this adjustment was a function of the loss in Volume (i.e., the magnitude of the loss due to fire). The other risks were incorporated similarly — as shown in *Figures 14B, 14C* and *14D.*

(continued on page 20)

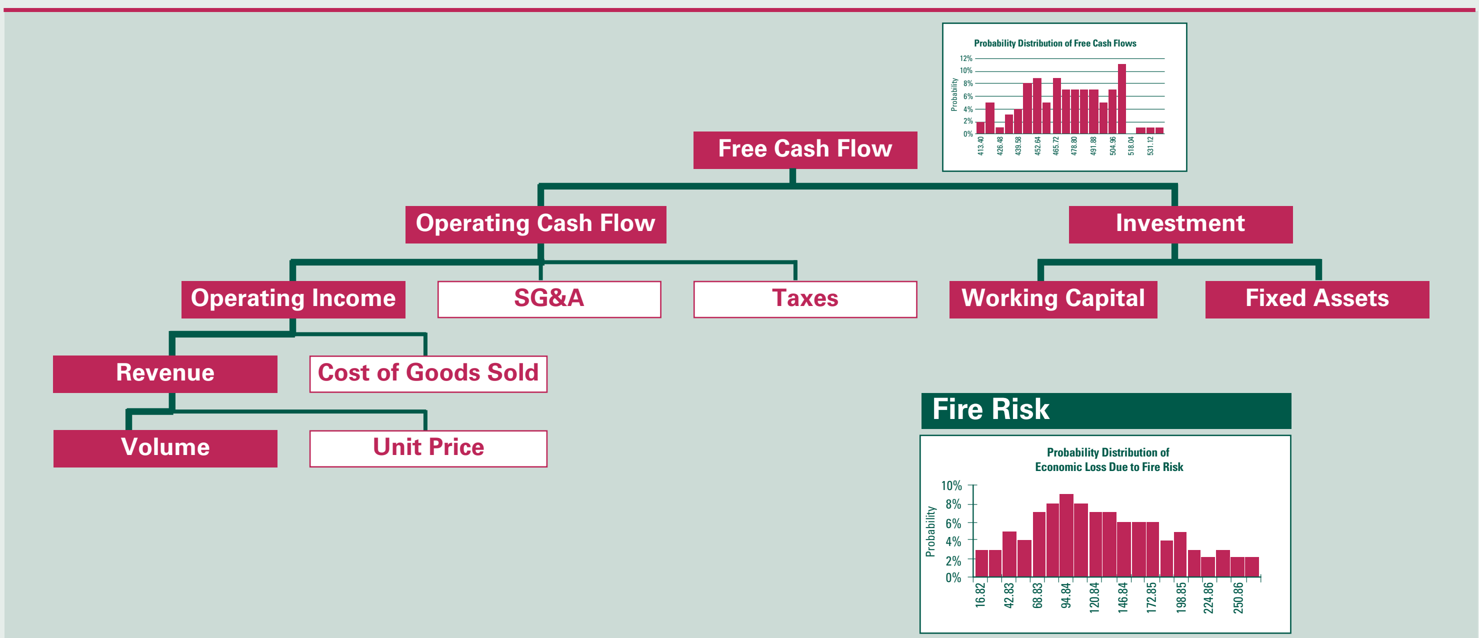
FIGURE 13



Stochastic Free Cash Flow for HypoCom. Volume, Unit Price and Cost of Goods Sold are represented as random variables with specified probability distributions and correlations.

Risk profiles are linked...

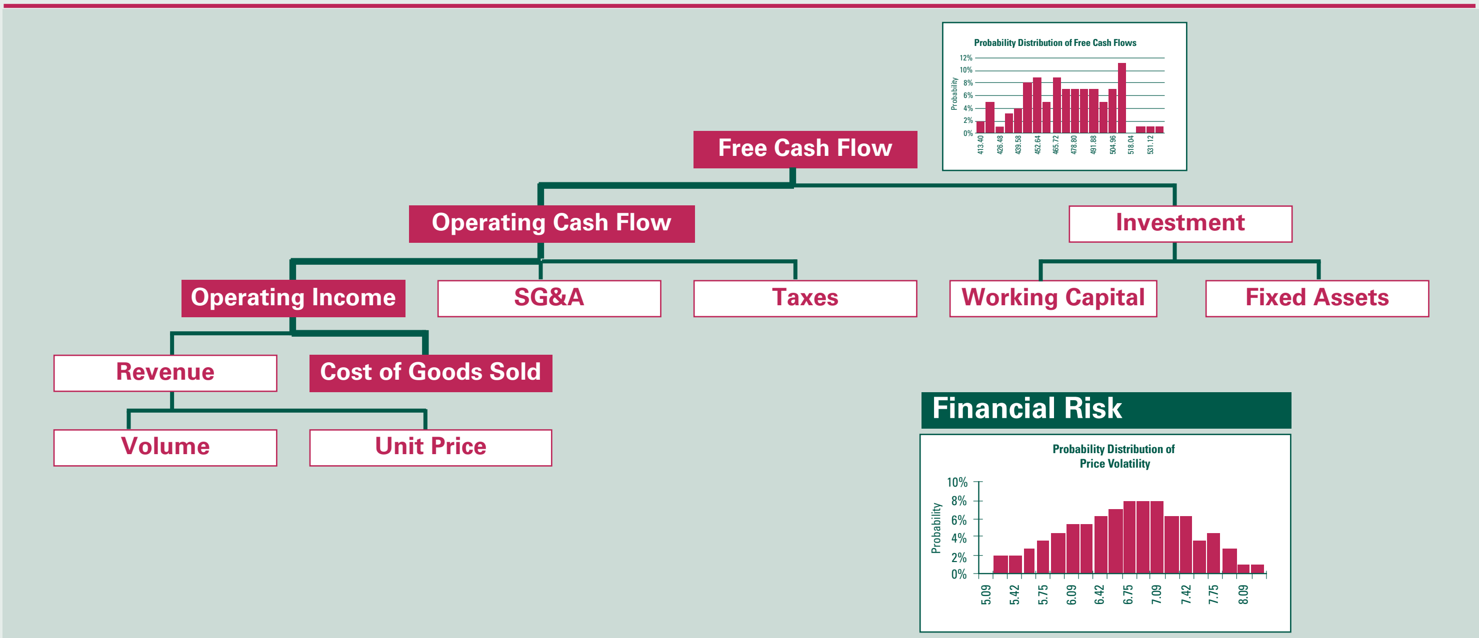
FIGURE 14A



The probability distribution for fire risk is linked to FCF through its effect on sales volume, working capital and fixed assets.

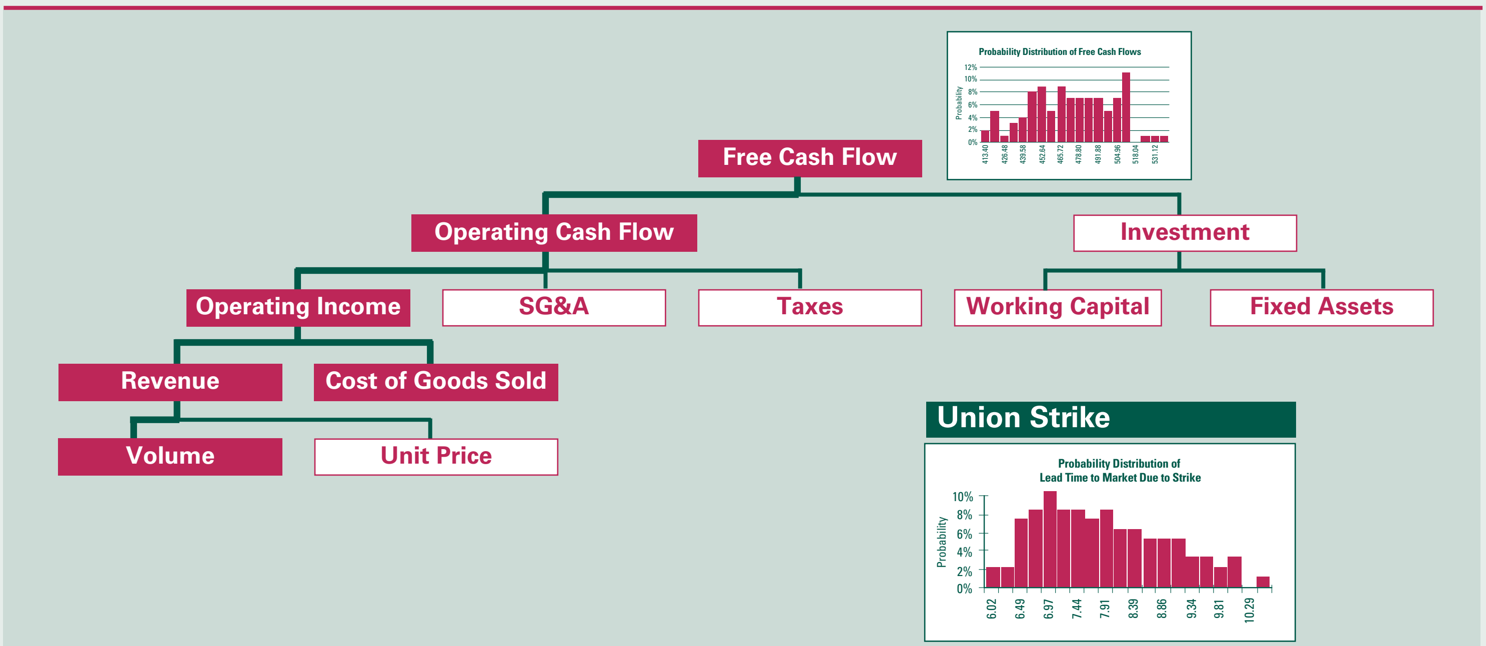
Risk profiles are linked... (cont'd)

FIGURE 14B



The probability distribution for commodity price risk is linked to FCF through its effect on cost of goods sold.

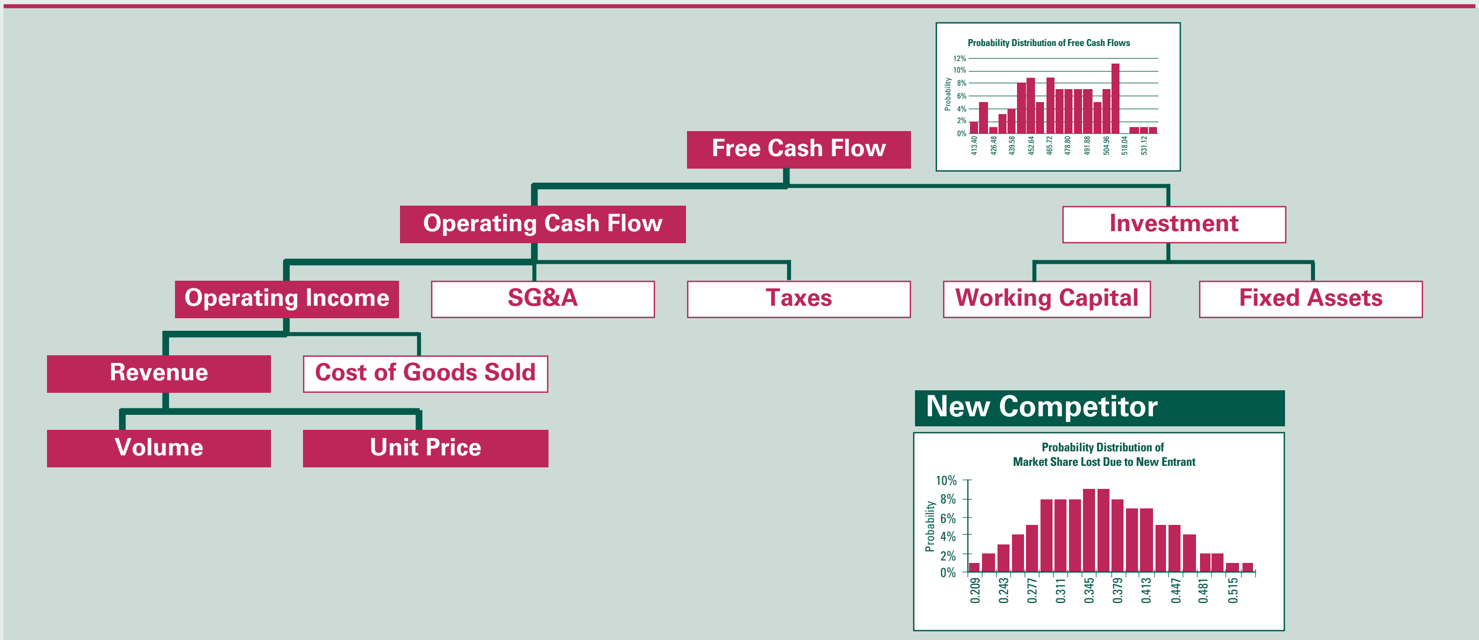
FIGURE 14C



The probability distribution for risk due to a union strike is linked to FCF through its effect on sales volume and cost of goods sold.

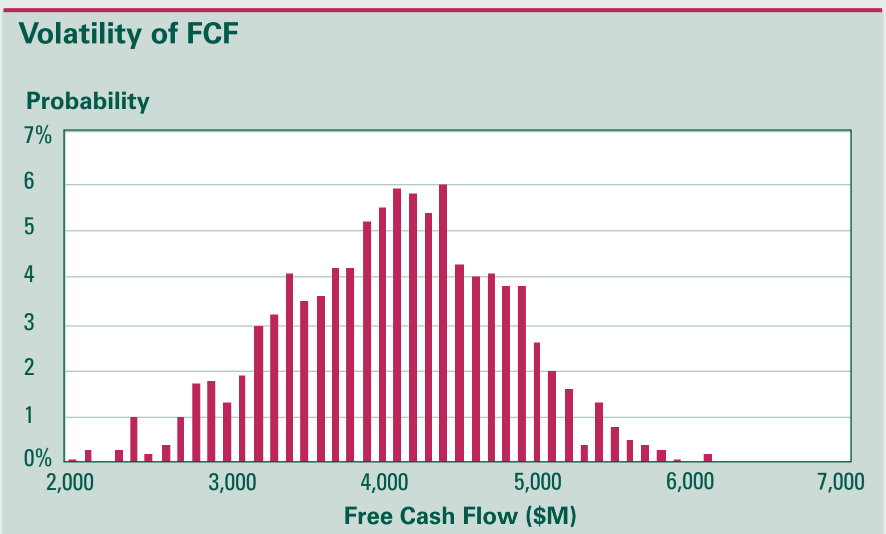
Risk profiles are linked... (cont'd)

FIGURE 14D



The probability distribution for new competitor risk is linked to FCF through its effect on sales volume and unit price.

FIGURE 15



Volatility of Free Cash Flow for HypoCom. This reflects the aggregate impact of all four risks without inclusion of any remediation strategies.

The size of the FCF model and the number of risks modeled for HypoCom were kept small to simplify describing our approach. This way, we could construct this model in MS Excel™ and run simulations using @RISK™ software. However, in practice, models are built using specialized, industrial simulation and optimization software. The aggregate impact of all four risks on FCF is shown as a probability distribution in *Figure 15.*

STEP 3

Set up a portfolio of risk remediation strategies

The steps in the analysis thus far have produced information on the current level of risk for Free Cash Flow or any other financial measure selected for this analysis. Steps 3 and 4 outline a course of action to mitigate the current level of risk based on management's risk preferences. In Step 3, a portfolio of risk remediation strategies is developed as follows.

Identify risk remediation strategies

With a measure of riskiness of the FCF established, we can now determine how to reduce this risk. We can consult domain experts on strategies for mitigating each source of risk. This is a collaborative brainstorming effort among internal and external experts on the topic. Strategies are not restricted to financial remediation through insurance or financial derivatives; in fact, for many business risks, it may be impossible to find either insurance or a hedge in the financial markets. All the risk remediation strategies together constitute a portfolio of investment choices. To determine the optimal allocation of investment, the cost and benefit of each combination of strategies must be calculated.

Model effect of each strategy on financial metric

Each strategy aims to shape the risk on FCF to suit the risk preferences of management and shareholders. Shaping the risk means altering the shape of the probability distribution for FCF. At least three meaningful ways exist to shape the probability distribution:

- Shift the first moment of the distribution, i.e., increase the expected value of FCF.
- Shift the second moment of the distribution, i.e., decrease the deviations from the expected value of FCF.
- Reduce the tail of the distribution on the down side, i.e., reduce the worst-case scenario of Cash Flow-at-Risk (CFaR). This is a Value-at-Risk (VaR) type measure that is commonly used in financial risk management. For FCF, this means increasing the 5th percentile FCF so that there is less than 5% probability of FCF falling below some threshold value.

Each risk remediation strategy will affect the probability distribution of FCF in at least one of the three ways enumerated above. Thus, the measure by which the strategies should be evaluated will be a function of these three measures — described in greater detail in Step 4.

The FCF model from Step 3 measures the effect of each combination of strategies on the distribution of FCF. Simulations are run for each possible portfolio or combination of strategies and the resulting probability distribution of FCF is recorded for use in the next step.

Keep in mind that remediation strategies focused on mitigating the effect of one risk source may create a new source(s) of risk. For example, hedging in the financial markets may create counterparty risks. These unintended sources of risks should be incorporated into the financial model if they are deemed significant.

There is typically a cost associated with implementing each strategy, which can be measured directly. The cost may vary depending on the degree to which the strategy is undertaken. For example, various levels of insurance can be purchased, each with a different premium.

For HypoCom

Strategies for mitigating each risk appear in *Figure 16.* Note that for risks falling in the traditional domain of risk management — namely, fire risk and commodity price volatility — the strategies are also conventional, i.e., insurance and financial hedging, respectively. For mitigating the risk due to a union strike, however, there are several alternatives:

- build up inventory
- contract with third parties to provide a supply of products
- satisfy some or all union demands.

Like most manufacturing companies, HypoCom's distribution centers and plants optimize their inventory and production policies to minimize cost. However, the company did this without considering the impact of a union strike. As noted above, one alternative is to build up inventory beyond optimal levels; this would certainly mitigate the strike's impact. If there is no strike, however, the buildup of inventory beyond optimal levels creates a holding cost that can be calculated directly.

Similarly, each strategy alternative listed in Figure 16 has a cost that can be measured directly. The benefit of each strategy is determined through simulations using the FCF model. There are three alternative strategies each for mitigating fire risk, commodity price risk and union strike risk. Loss of sales due to new competition has only two possible strategies in our illustration. (Note that in each case, one of the alternatives is a default "do nothing" strategy.)

Altogether, there are 54 (3 x 3 x 3 x 2) possible combinations or portfolio strategies. Each of the 54 possible portfolios was evaluated by running simulations using the FCF model and recording the resulting probability distribution on FCF. The cost/benefit information for each portfolio produced in this step will be used in the next step to determine the optimal portfolio.

FIGURE 16

	Classification of Remediation Strategies		
	Insure	**Hedge in Financial Markets**	**Mitigate Through Business Activity**
Fire	■ Full range of loss ■ Catastrophic loss		
Commodity Price Volatility		■ Upside hedge ■ Full hedge	■ Acquire supplier of commodity
Union Strike			■ Build up inventory ■ Contract with third parties for product
New Competitor			■ Reduce price

Portfolio of risk remediation strategy alternatives for HypoCom. For each risk, there is also the default strategy of "do nothing."

STEP 4

Optimize investment across remediation strategies

This step takes the results from the prior steps to determine the optimal allocation of investment to the risk management portfolio. To do this, we must formulate the decision as a portfolio optimization problem and solve it using optimization technology. The following will describe how to formulate and solve this portfolio optimization problem.

Identify optimization objective(s)

To compare portfolios of different combinations of strategies for risk remediation, first determine the criteria for the comparison. In optimization terms, this is called the objective function.

As indicated in Step 3, the risk remediation strategies alter risk in at least three meaningful ways:

- increase the expected value of FCF
- decrease the deviation from the expected value of FCF
- increase the 5th percentile of FCF distribution (CFaR) so that there is less than 5% probability of FCF falling below some threshold value.

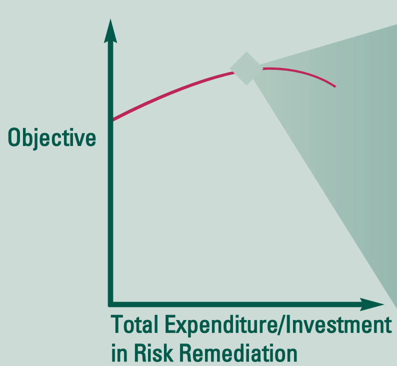
Therefore, one possibility is to use a weighted combination of these three measures as the objective function for comparing portfolios. The weightings would reflect the risk preferences of the decision-makers (who may be representing shareholder interest).

An alternative is to use expected utility of FCF as the objective function. First, a utility function must be developed that captures management's risk preferences for FCF. Development of a utility function is well documented in standard texts on decision analysis, two of which are included in the References (von Winterfeldt & Edwards 1986, Clemen 1996). The utility function is applied to the distribution of FCF to produce a distribution of utility or utiles. The expected value of this distribution is the expected utility. The relative preferences over the three measures of risk used in the prior method are captured in the shape of the utility function. One advantage of this method is that it easily extends to a multi-period objective using multi-attribute utility theory. This is explained further in a later section on multi-period risk management.

Either method can be used to develop the objective function of the portfolio optimization problem. The objective is to find the portfolio of strategies that maximizes this function.

Note that this method recognizes that management teams often differ in their risk preferences. We know that some companies are more aggressive than others in taking on strategic risks as a way of competing. Thus, the objective

FIGURE 17



	Insure	Hedge in Financial Markets	Mitigate Through Business Activity	Total
Fire	35%			35%
Increase in Commodity Price		10%		10%
Union Strike			25% Build up inventory 5% Contract with third parties	30%
New Competitor			25% Reduce price	25%
Total	35%	10%	55%	100%

The efficient frontier is a plot of all the portfolios that maximize the objective function given a fixed level of total risk remediation investment. Each point represents a unique allocation of the investment across the portfolio of strategies.

must be tailored to the unique risk preferences of the management team.

Identify constraints to optimization

Optimization may include some constraints on the optimum portfolio of strategies. A typical constraint may be a limit on the cost of implementing the portfolio of risk management strategies. There may also be constraints on the minimum/maximum level of insurance purchased, use of financial hedging, and/or the level of risk mitigated through business activity. Constraints on the downside risks to FCF may also be preferred. The constraints narrow the range of portfolios over which the objective function is maximized. Therefore, constraints have the effect of lowering the maximum value of the objective function.

Develop an efficient frontier of remediation strategies

The portfolio optimization problem as formulated above can be solved using optimization technology. Given a constraint on the size of the risk management budget, the optimization algorithms will determine the allocation of this budget to the alternative strategies that maximizes the objective function. This process can be repeated for varying levels of risk management budget. Plotting the results with the level of the risk management budget on the x-axis and the maximum value of the objective function on the y-axis produces a graph of the efficient frontier. The efficient frontier represents all the portfolios of strategies that constitute the optimal allocation of the risk management budget (see *Figure 17*).

For HypoCom

As mentioned at the end of Step 3, all 54 possible portfolios of strategies were simulated and the probability distribution of FCF was recorded. This information was then used to develop the objective function and the efficient frontier.

The objective function was based on a weighted combination of the three risk measures as follows:

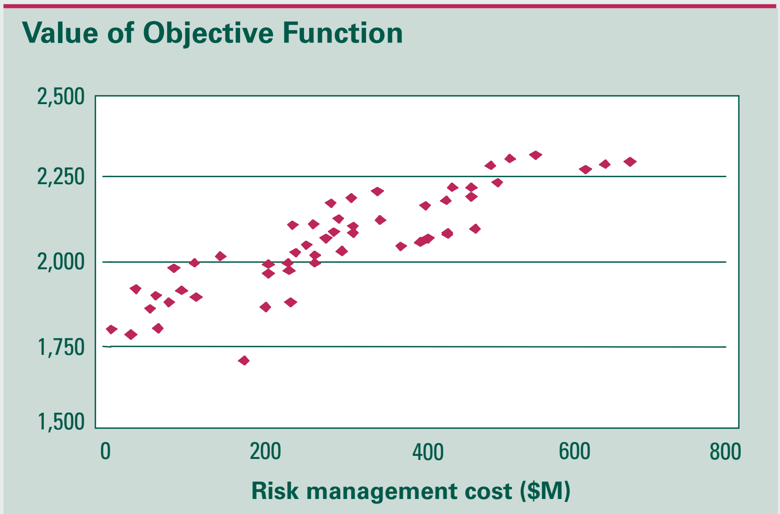
.40 * Expected FCF

+ .30 * Length of 90% confidence range of FCF

+ .30 * Value of FCF that has less than 5% probability of occurring.

Each of the 54 simulation runs produced a probability distribution of FCF. The objective function value was determined by applying the above formula to each of the runs. The results were plotted as an efficient frontier (see *Figure 18*).

FIGURE 18



Efficient frontier for HypoCom. Connecting all the points on top edge of the plot will produce an efficient frontier. Each point on the efficient frontier represents an optimum portfolio of strategies given the risk management cost. Portfolio points within the efficient frontier are suboptimal and should not be chosen.

Extension to multi-period risk shaping

Although the approach described above was based on a one-year decision horizon, in practice, most companies prefer a multi-year optimization analysis due to the strategic nature of this allocation. Fortunately, the method easily extends to a multi-year model.

In essence, all model variables and parameters are indexed by time (e.g., years). Thus, in Step 1, the probability distributions are developed for each time period in the investment horizon. Similarly, linking individual risks to a common financial measure involves indexing the probability distribution of FCF by year. Thus, the riskiness of FCF may vary from year to year.

The evolution of risk over time is typically modeled using a scenario generation system. A scenario generator uses stochastic differential equations (SDEs) to generate thousands of possible paths that a variable may follow over time. An SDE typically expresses a change in the value of a variable (e.g., interest rate) over a small time period as the sum of a predictable change and an unpredictable change. The predictable change is typically a deterministic function of the current value of the variable, but can also be a function of other variables with which there is correlation. The unpredictable effect is represented as a random variable with a specified probability distribution. An SDE is used iteratively to produce a scenario of how a variable can change over time. Typically, the scenario generator will model several correlated variables together to develop scenarios that are internally consistent. These scenarios are then fed into a financial model to develop stochastic forecasts of financial metrics over time. (Please refer to Section VII, "References and Recommended Reading," for papers and texts that describe scenario generation and stochastic differential equations.)

The risk remediation strategies in Step 3 may involve phased implementation of the strategy or there may be a time lag between incurring the cost for a strategy and its impact on the volatility of cash flow. In particular, the time lag may extend to more than a year.

Finally, in Step 4, the objective function based on expected utility can be extended to a weighted sum of the expected utility for each year in the time horizon. The weights applied to each year's expected utility can be determined by applying methods based on multi-attribute utility theory. Furthermore, budget constraints may vary over time.

In the multi-year time horizon, the output of the analysis is a path of risk remediation investments over the time horizon rather than separate optimum portfolios and efficient frontiers — as in the single-year case. Dynamic programming determines the optimum path of investments in risk remediation strategies.

Recap

In summary, the four-step analytical process for managing risk across an enterprise includes:

- quantifying each risk source by applying the appropriate tool and method for developing a probability distribution
- linking all the risk sources to a common financial metric
- developing a portfolio of strategies to mitigate each risk
- selecting the optimal portfolio of strategies.

The first two steps represent the bulk of the analytical effort and provide crucial information on the underlying dynamics of the enterprise. Different tools and methods (see Figure 6) for probability assessment will quantify the risk source and develop correlation among risk sources, depending on the relative availability of relevant data and domain experts. Aggregating these risks by linking them to a common financial metric provides an assessment of the overall risk to the enterprise and provides a method for determining the relative contribution of each risk source to the overall risk. Examination of the results of these two steps provides valuable insight into the business dynamics of the enterprise.

The last two steps are necessary to determine the optimal total expenditure for risk management and the most efficient allocation of that capital. Optimization also reflects constraints imposed by exogenous factors — the timing of expenditures, level of insurance, level of financial hedging and value-at-risk. In combination, the four-step analytical process lays a firm foundation for management decision making with respect to ERM.

V A Brief Discussion of Exploiting Risk and Keeping Ahead

Risk has two faces. This monograph has focused on risk as a threat. But risk also represents an opportunity. In fact, organizations routinely pursue risk for the chance of increased reward. Companies achieve competitive advantage by correctly identifying which risks the organization can pursue better than its peers.

This advantage can arise in at least two ways (see *Figure 19*). The first relates to the nature of the risk itself. Certain risks, due to their predictability and/or effect on company financials, provide more of a risk to your competition than to your own organization. For example, currency translation risk is less of a concern to the organization whose distribution of cost of goods sold by country is similar to its distribution of revenue by country. The second way risk advantage arises relates to the organization's understanding of the risk and its capabilities to respond. For example, the oil company that, due to its hiring and training practices, has developed industry-leading capabilities in commodity risk analysis, can market these capabilities through a separate profit center.

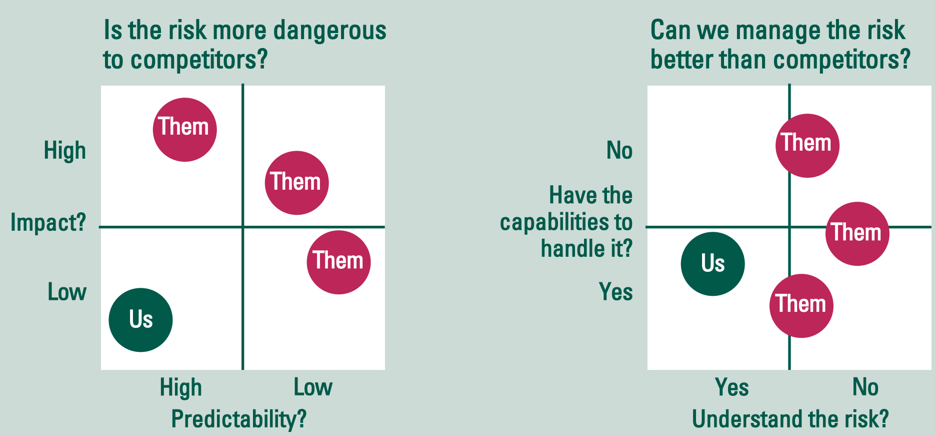
A robust ERM assessment process will be alert to both faces of risk and will form the organization's strategic response accordingly.

In the dynamic risk environment, change is constant. It occurs in the organization's underlying risk factors, in the economic, political/regulatory and competitive landscapes within which the organization operates, and in the organization itself (e.g., its business objectives, the skill sets of its managers and key employees, and even its makeup after such events as downsizing, divestitures, mergers and acquisitions). Continual monitoring of this risk environment is therefore crucial if the organization's ERM program, however successful to date, is to remain relevant. Depending on the nature and degree of these inevitable changes, farseeing management reenters the ERM process at the appropriate step(s). Not surprisingly, several organizations make ERM an integral part of their business and strategic planning processes.

FIGURE 19

If You Understand Risk, It Can Be a Competitive Advantage

Two scenarios



ERM includes identifying those risks that represent areas of competitive advantage.

VI Implementing ERM in Phases

Implementing ERM is clearly a challenge. Most organizations have therefore "started small," undertaking the implementation in discrete, manageable phases.

We can view ERM in three dimensions (see *Figure 20*). The first represents the range of company operations. Some organizations have started small by piloting ERM in one, or a small number, of their business units or locations, for real-time fine-tuning and eventual rollout to the entire enterprise. The second dimension represents the sources of risk (hazard, financial, operational, etc.). Some organizations confine the initial scope of their ERM to a selected subset of these risk sources, for example, property catastrophe risk and currency risk. Eventually, all sources of risk would be layered in, in sequential fashion.

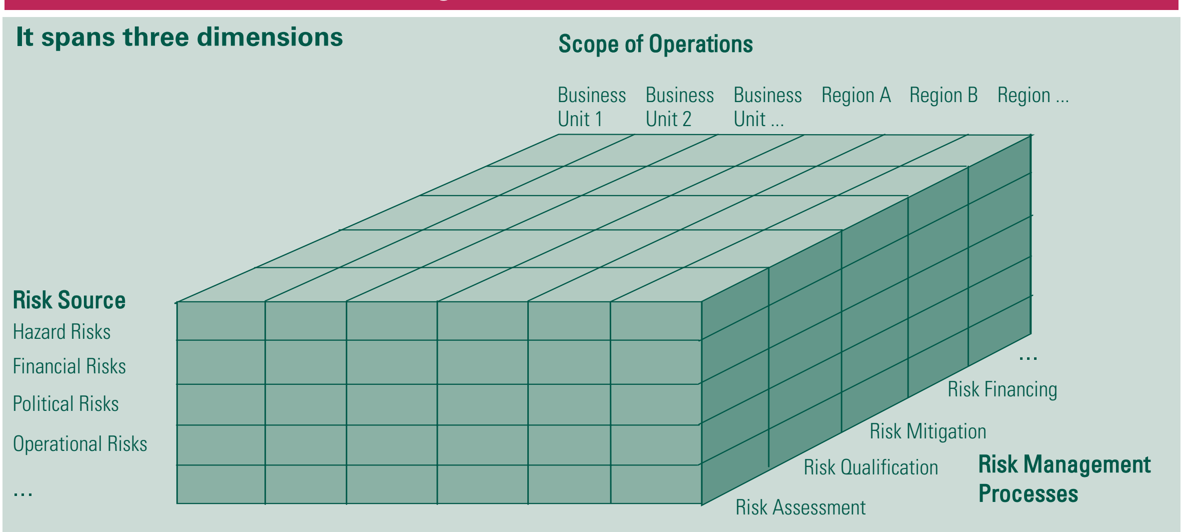
The third dimension represents the types of risk management activities or processes (risk identification, risk measurement, risk financing, etc.). Some organizations confine their initial vision to the identification and prioritization of enterprise-wide risks, with subsequent activities dependent on the results. Others begin by fashioning an integrated risk financing program around a subset of risk sources; these depend on the risk sources for which their financial service providers have integrated products. Still others begin by measuring and modeling virtually all sources of risk, regardless of their priority and the current availability of risk financing products.

While some of these approaches may appear more prudent than others, it is wise to reserve judgment. We believe no single best approach to ERM implementation exists that is appropriate for all organizations. Leading companies successfully employ a number of different phased approaches. The nature and sequence of these phases depend on the culture, strategic imperatives and management style of the organization. However, it is certain that for every organization a phased approach of some sort will be more successful than attempting to do too much, too soon.

Regardless of their starting point, many organizations include in their implementation plans the attempt to ingrain ERM into their cultures through communication, education, training and incentive programs. In some cases, these are coordinated in an extensive formal change management process to help impose the new order of things and achieve sustainable results. Clearly, to be successful, ERM needs to be more than a technique — and needs to be embraced by more than just management. These issues will be explored further in our subsequent publications.

FIGURE 20

The Universe of ERM Is Quite Large...



The scope of ERM is quite large. Organizations have variously "started small" by phasing in their implementation along one or more of ERM's three dimensions.

VII References and Recommended Reading

General risk management

Bernstein, Peter L. (1996). *Against the Gods: The Remarkable Story of Risk*. John Wiley & Sons.

Knight, Rory F. & Pretty, Deborah J. (1998). *The Impact of Catastrophes on Shareholder Value*. The Oxford Executive Research Briefings. A Research Report Sponsored by Sedgwick Group.

Probability assessment and utility theory

Clemen, Robert T. (1996). *Making Hard Decisions*. 2nd edition. Duxbury Press.

Linstone, H. A. & Turoff, M. (1975). *The Delphi Method: Techniques and Applications*. Addison-Wesley Publishing Company Inc.

Oliver, R. M. & Smith, J. Q. (1990). *Influence Diagrams, Belief Nets, and Decision Analysis (Wiley Series in Probability and Mathematical Statistics)*. John Wiley & Sons.

von Winterfeldt, D. & Edwards, W. (1986). *Decision Analysis and Behavioral Research*. Cambridge: Cambridge University Press.

Scenario generation and stochastic differential equations

Merton, Robert C. (1992). *Continuous-Time Finance*. 2nd edition. Blackwell Publishers, Inc.

Mulvey, John M. (1996). "Generating Scenarios for the Towers Perrin Investment System." *Interfaces, An International Journal of INFORMS*. March-April 1996, Vol. 26, Number 2.

Neftci, Salih N. (1996). *An Introduction To The Mathematics of Financial Derivatives*. Academic Press.

Schwartz, Eduardo & Smith, James E. (1999). *Short-Term Variations and Long-Term Dynamics in Commodity Prices*. Duke University, Fuqua School of Business.

Simulation and optimization

Bazaraa, Mokhtar S., Sherali, Hanif D. & Shetty, C. M. (1993). *Nonlinear Programming, Theory and Algorithms*. 2nd edition. John Wiley & Sons, Inc.

Fourer, R., Gay, D. M. & Kernighan, B. W. (1993). *AMPL, A Modeling Language for Mathematical Programming*. Duxbury Press (includes disk with AMPL and optimization solvers).

Hillier, Frederick S. & Lieberman, Gerald J. (1986). I*ntroduction to Operations Research*. 4th edition. Holden-Day, Inc., Oakland, California; Cambridge: Cambridge University Press.

Nelson, Barry L. (1995). *Stochastic Modeling: Analysis & Simulation*. McGraw-Hill, Inc.

Simulation and optimization software

@RISK and @RiskOptimizer, MS Excel add-in for simulation and optimization, developed by Palisade Corp., Newfield, New York.

Fourer, R., Gay, D. M. & Kernighan, B. W. (1993). *AMPL, A Modeling Language for Mathematical Programming*. Duxbury Press (includes disk with AMPL and optimization solvers).

Global CAP:Link, scenario generator for macroeconomic variables worldwide (such as interest rates, exchange rates and major asset classes), developed by Tillinghast – Towers Perrin, New York.

Service Model, discrete-event stochastic simulation software, developed by Promodel Corp, Orem, Utah.

VIII

Acknowledgments

The authors wish to acknowledge the invaluable assistance of their colleagues:

- **Robert Schneier** and **David Watkins,** who shared their expertise and ideas on strategy, organization and corporate culture — and their connection to ERM
- **Andrew Berry** and **Julian Phillips,** who contributed the corporate governance research
- **Peter Watson** and **Brian Merkley,** who analyzed the linkage between performance consistency and share value
- **Kathleen Waslov,** who contributed information on probability assessment methods based on expert testimony
- **Ravin Jesuthasan** and **Emory Todd,** who contributed to the development of the stochastic financial model
- **Anne McKneally,** who helped edit the text.

We also extend our heartfelt appreciation to the following individuals who constituted our Editorial Review Board:

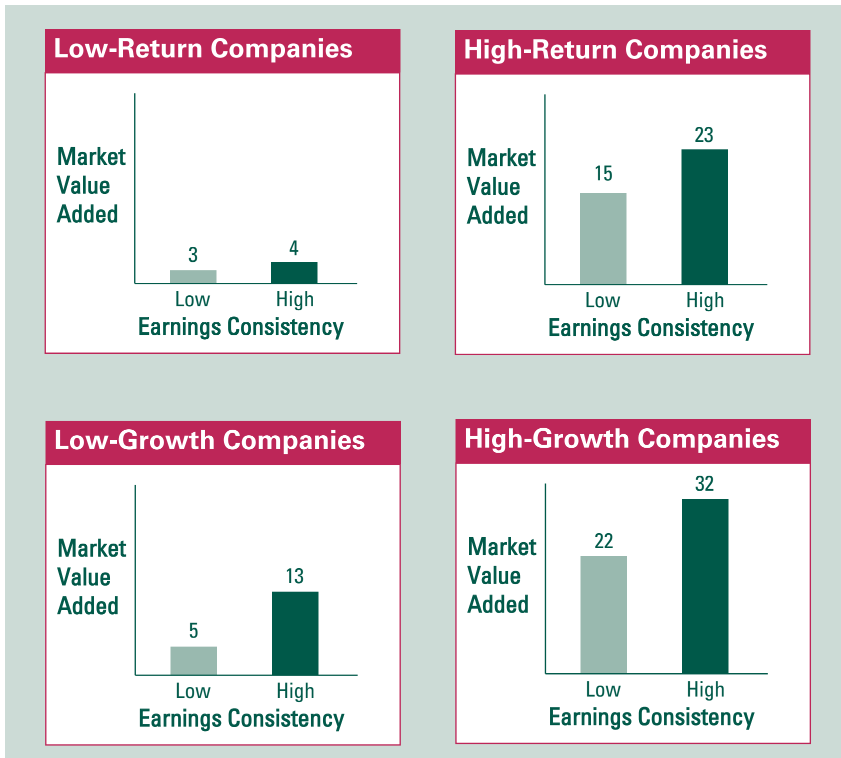
- **Stephen D'Arcy,** Professor of Finance, University of Illinois
- **William Fealey,** Executive Director, Corporate Risk Management, Estée Lauder Companies, Inc.
- **Felix Kloman,** Editor & Publisher, Risk Management Reports
- **André-Richard Marcil,** Director, Control and Integrated Risk Management, Hydro Quebec
- **Robert Markman,** Vice President, Risk Management, United Parcel Service
- **John Mulvey,** Professor of Engineering & Management Systems, Princeton University.

This monograph was immeasurably improved by the generous contributions of these individuals. Any errors that readers may find remain the sole responsibility of the authors.

Appendix A

The Value of Consistency

- Earnings consistency typically explains 25% of annual change in share price
- Primarily affects premium over "warranted" multiple. Example (from the Integrated Petroleum Industry):



The market reacts to perceptions of how well risk is handled.

Source: Towers Perrin consistency analysis of selected industries (see following background information).

Background Information on Towers Perrin Consistency Analysis

Overview

Consistency analysis empirically estimates whether companies with more consistent earnings receive a premium market valuation relative to peers. Since many other factors — in addition to earnings consistency — shape market valuations, we use a series of basic analytic steps to attempt to control for the influence of other factors (e.g., earnings growth and return on capital) and isolate a consistency premium or discount. We use a relatively simple control process since (1) we find that more complicated methods introduce other sources of "noise" into the process and (2) consistency premiums are fairly robust across many industry groups and emerge readily with relatively simple control techniques.

A general description of the control process is provided below. For specific definitions and data sources used in the analysis, please see the Methodology section that follows.

Basic methodology

In performing consistency analysis, Towers Perrin's first step is to identify a relevant industry peer sample for a given company. Using an industry peer group helps filter out the effect of common industry factors (e.g., commodity price movements, regulatory risk) on market valuations. We typically use published industry groupings provided by Valueline or Standard & Poor's.

Next, we create a data set including a market premium measure, earnings growth rate, return on capital and earnings consistency for each peer. We employ historical growth rates and returns as surrogates for the future growth rates and returns that drive valuations. We calculate growth rates, using a least squares (regression) approach to avoid biases caused by point-to-point methodology, and average returns on capital over the measurement window (typically 10 years). To measure the market premium, we employ a standardized market value-added metric since it properly distinguishes between the capital that investors have placed in the business and the market value added to this capital.

Unlike market-to-book ratios, standardized market value added also captures the dollar growth in the value premium over time. Since the measure is standardized (indexed), it can be meaningfully compared across companies. Finally, Valueline's earnings predictability score (0%-100%) is used as the measure of earnings consistency.

We then calculate a median growth rate and return on capital for the peers and break the sample into "high growth" (growth ≥ median) and "low growth" (growth < median) and high-return (return ≥ median) and low-return (return < median) subsets.

The process is repeated one more time by calculating the median earnings predictability score for each of the four subsets and then further breaking each subset into a high earnings consistency (earnings predictability ≥ subset median) and low earnings consistency (earnings predictability < subset median). A total of eight subsets results from both steps.

Finally, an average market premium (standardized market value added) is calculated for each of the eight subsets, and the results are summarized in bar chart form.

Towers Perrin Consistency Analysis Methodology

Data Sources

- **Compustat PC Plus database**
- **Valueline Investment Survey (earnings consistency only)**

Performance Metric Definitions

"Return on Capital"

- **Definition**
  - 10-year (1989-98) average Return on Capital Employed (ROCE)
- **Formula**
  - (Income before Extraordinary Items + Special items) (Beginning Stockholders' Equity + Beginning Total Debt)
  - Perform same calculation for 10 years and take average
- **Comment**
  - Simplified return on invested capital definition (provides some adjustment for restructuring charges and other one-offs but makes simplifying assumption that special items receive no tax deduction)
  - Note: Compustat does not report after-tax special items

"Earnings Growth"

- **Definition**
  - 10-year (1989-98) least-squares EBIT growth rate
- **Formula**
  - Regress log adjusted operating income after depreciation against time to determine growth rate
- **Comment**
  - Growth rate based on regression more accurate than CAGR (which is biased by endpoints)

"Earnings Consistency"

- **Definition**
  - Valueline Earnings Predictability score as reported in Valueline Investment survey
- **Formula**
  - Valueline earnings predictability scoring based on stability of year-to-year comparisons, with recent years being weighted more heavily than earlier ones. The earnings stability is derived from the standard deviation of the percentage changes in quarterly earnings over an eight-year period. Special adjustments are made for comparisons around zero and from plus to minus.

"Market Premium"

- **Definition**
  - 1998 Standardized Market Value Added (MVA) based on 1988 ending invested capital base
- **Formula**
  - Std MVA = MVA % Capital x Indexed Capital = (M/C - 1) x Indexed Capital
  - M/C = (Stock price * Common shares outstanding + Preferred stock + Total debt)/(Shareholders' equity + Total debt)

    — All data reflect year-end 1998
  - Indexed Capital = (1998 Shareholders' equity + 1998 Total debt)/(1988 Shareholders' equity + 1988 Total debt)
- **Comment**
  - MVA captures value of growth (unlike M/B ratio) since it is measured in dollars. Standardizing MVA (by indexing every company's capital to same base year) corrects size bias of measure (so big companies with lots of capital but low M/C don't dominate smaller companies with higher M/C).

Appendix B

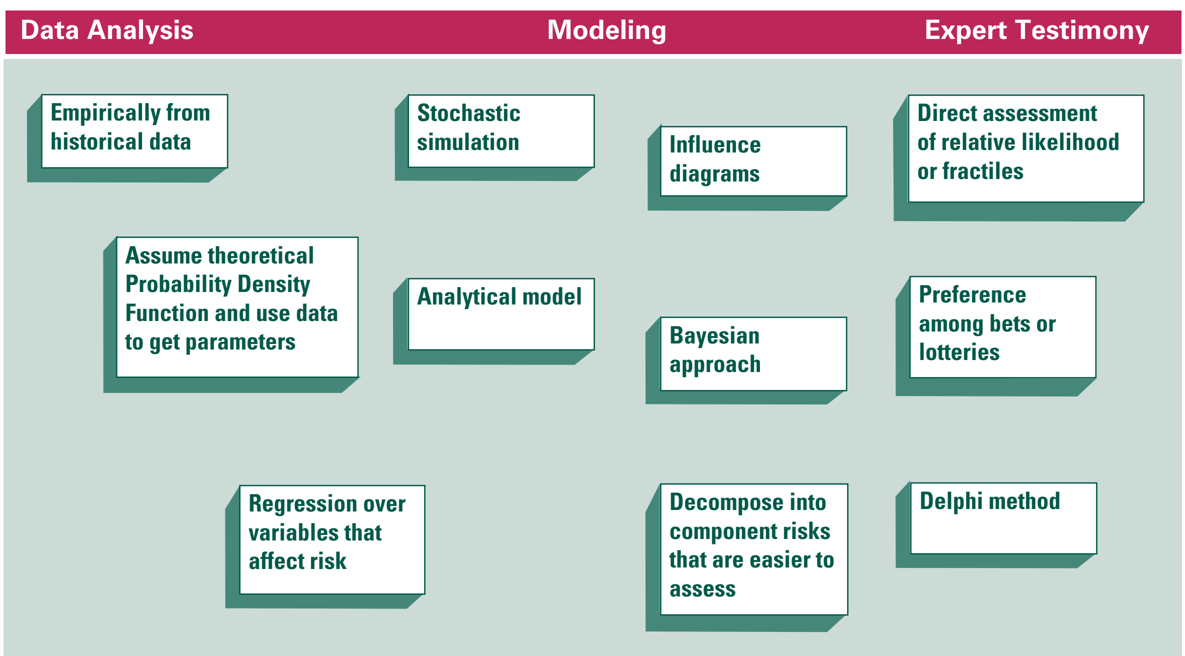
Probability Assessment Methods Based on Expert Testimony

Approaches to modeling risk

To model risk, it is necessary to understand the nature of risk itself. Risk arises from the fact that actual future results could differ from expected or projected results, often materially; one does not know with certainty what will happen in the future. In projecting into the future, one must consider a range of potential outcomes from a given event. Risk assessment aims to evaluate both the impact (financial, reputational, etc.) of each outcome and the likelihood or probability of each outcome occurring. The process develops a probability distribution that captures the impact and likelihood of given risk types or events.

There is a continuum of methods for developing probability distributions. These methods can be grouped into three principal categories: data analysis approaches, expert testimony and modeling (whose methods are often hybrids of methods from the other two categories). The choice of method depends significantly on the amount and type of historical data that are available. The methods also require varying analytical skills and experience. Each method has advantages and disadvantages over the other methods, so it is important to match the method to the facts and circumstances of the particular risk type.

Building a probability distribution of outcomes for each risk type is the first stage in developing an entire risk profile for the organization. In financial terms, each of these distributions needs to be combined with the others — taking into account correlations among risk types — and applied to the organization's financial value tree to develop a unique probability distribution of future financial results for that organization.



Estimating probabilities through expert testimony

Probability distributions for events for which there is sparse data can be estimated through expert testimony. A naive method for assessing probabilities is to ask the expert, e.g., "What is the probability that a new competitor will enter the market?" However, the expert may have difficulty answering direct questions and the answers may not be reliable.

Behavioral scientists have learned from extensive research that the naive method can produce unreliable results due to heuristics and biases. For example, individuals tend to estimate higher probabilities for events that can be easily recalled or imagined. Individuals also tend to anchor their assessments on some obvious or convenient number resulting in distributions that are too narrow. (See Clemen 1996 and von Winterfeldt & Edwards 1986 in the list of references for further examples.) Decision and risk analysts have developed several methods for accounting for these biases. Several of these methods are described below.

Preference among bets

Probabilities are determined by asking the expert to choose which side is preferred on a bet on the underlying events. To avoid issues of risk aversion, the amounts wagered should not be too large. For example, a choice is offered between the following bet and its opposite:

Bet	Opposite Side of Bet
Win \$x if a competitor enters the market	Lose \$x if a competitor enters the market
Lose \$y if no new competition	Win \$y if no new competition

The payoffs for the bet, amounts \$x and \$y, are adjusted until the expert is indifferent to taking a position on either side of the bet. At this point, the expected values for each side of the bet are equal in the expert's opinion. Therefore,

$$\$x\,P(C) - \$y\,(1-P(C)) = -\$x\,P(C) + \$y\,(1-P(C))$$

where P(C) is the probability of a new competitor entering the market. Solving this equality for P(C):

$$P(C) = \$y/(\$x + \$y)$$

For example, if the expert is indifferent to taking a position on either side of the following bet:

Win \$900 if a competitor enters the market

Lose \$100 if no new competition

then the estimated subjective probability of a new competitor entering the market is \$100/(\$100 + \$900) = 0.10.

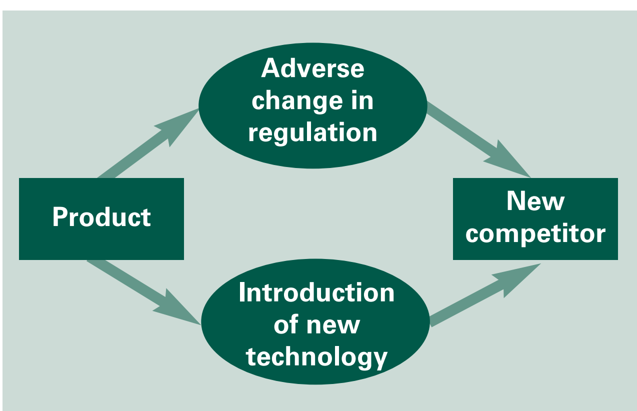
Judgments of relative likelihood

This method involves asking the expert to provide information on the likelihood of an event relative to a reference lottery. The expert is asked to indicate whether the probability of the event occurring is more likely, less likely or equally likely compared to a lottery with known probabilities. Typically, a spinning wheel (a software implementation of the betting wheels in casinos) is used on which a portion of the wheel is colored to represent the event occurring. The relative size of the colored portion is specified. The expert is asked to indicate whether the event is more, less or equally likely to occur than the pointer landing on the colored area if the wheel was spun fairly. The colored area is reduced or increased as necessary depending on the answers until the expert indicates that the two events are equally likely. This method is often used with subjects who are naive about probability assessments.

Decomposition to aid probability assessment

Often, decomposing an event into conditional causal events helps experts assess risk of complex systems. The structure of the conditional causal events can be represented by an influence diagram. Influence diagrams illustrate the interdependencies between known events (inputs), scenarios and uncertainties (intermediate variables) and an event of interest (output). An influence diagram model comprises risk nodes representing the uncertain conditions surrounding an event or outcome. Relationships among nodes are indicated by connecting arrows, referred to as arcs of influence. The graphical display of risks and their relationships to process components and outcomes facilitates visualization of the impacts of external uncertainties.

While this approach increases the number of probability assessments, it also allows input from multiple experts or specialists and helps combine empirical data with subjective data. For example, a new competitor entering the market may be decomposed using an influence diagram such as this one:



The probability of a new competitor, P(C) can be estimated, using a Bayesian approach. The approach uses Bayes' Rule, which is a formal, optimal equation for the revision of probabilities in light of new evidence contained in conditional or causal probabilities.

$$P(C) = \Sigma_i \, P(C_i \mid R_i, T_i) \, P(R_i, T_i)$$

where i is a product index, $P(R_i, T_i)$ is the joint probability of an adverse change in regulation and introduction of new technology, and $P(C_i \mid R_i, T_i)$ is the conditional probability of a new competitor entering a market for product i. This formula is useful when assessing the conditional probabilities $P(C_i \mid R_i, T_i)$ and is easier than a direct calculation of P(C).

Several different experts may be asked to assess the conditional and joint probabilities. For example, one expert (or group of experts) may assess the probability of adverse regulation for a specific product, another expert may assess probability of introduction of new technology, and yet a third may assess the probability of a new competitor given the state of new regulation and technology.

The Delphi technique

Scientists at the Rand Institute developed the "Delphi process" in the 1950s for forecasting future military scenarios. Since then it has been used as a generic strategy for developing consensus and making group decisions, and can be used to assess probabilities from a group of individuals. This process structures group communication and usually involves anonymity of responses, feedback to the group as collective views, and the opportunity for any respondent to modify an earlier judgment. The Delphi process leader poses a series of questions to a group; the answers are tabulated, and the results are used to form the basis for the next round. Through several iterations, the process synthesizes the responses, resulting in a consensus that reflects the participants' combined intuition, experience and expert knowledge.

The Delphi technique can be used to explore or expose underlying assumptions or information leading to differing judgments and to correlate informed judgments on a topic spanning a wide range of disciplines. It is useful for problems that can benefit from subjective judgments on a collective basis.

Pitfalls and biases

Estimating subjective probabilities is never as straightforward as implied in the description of the methods above. There are several pitfalls and biases to be aware of:

- None of the methods works extremely well by itself. Typically, multiple techniques must be used.
- To increase consistency, experts should be asked to assess both the probability of an event and separately the probability of the complement of the event. The two should always add up to 1.0; however, in practice they seldom do without repeated application of the assessment method.
- The events must be defined clearly to eliminate ambiguity. "What is the probability of a new competitor entering the market?" is not unambiguous. "What is the probability that a new competitor will take more than 5% market share of product A in the next two years?" more clearly defines the event.
- When assessing probabilities for rare events, it is generally better to assess odds. Odds of event E is [P(E)/P(complement of E)].

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Risk Management: An Overview

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Swiss Banking School, 17-18 September 2001

Module Overview

- Monday
  - ⋆ Overview and Introduction, McNeil
  - ⋆ Management of Market Risks, Bitz
  - ⋆ Management of Credit Risks, Haller
  - ⋆ Management of Operational Risks, Geiger & Piaz
- Tuesday
  - ⋆ Asset & Liability Management, Enderli & Spillmann
  - ⋆ Risk Management of a Private Bank, Hodler
  - ⋆ Risk Management of a Global Player, Guldimann
  - ⋆ Panel Discussion, Hodler, Guldimann, McNeil
  - ⋆ Closing Remarks, McNeil

How Should You Approach the Module ?

* Search for the common ideas. What concepts and concerns reappear when one discusses the various areas of financial risk - market, credit, operational ?

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* Do all financial institutions face identical challenges? How does RM differ between a global player and a Swiss private bank?

Contents

A. Finance and Risk Management at ETH

ETH Eidgenössische Technische Hochschule Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

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Our main web pages are:

- Members of the group
- Talks in financial and insurance mathematics
- Current courses and seminars
- Education in financial mathematics
- Education in insurance mathematics
- Books for Risk Management
- Probability theory home page
- Seminar on stochastic processes
- Swiss probability seminar
- RiskLab
- The ETH *Riskometer* for online VaR prognoses
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About RiskLab

[General Description] [Vision] [Budget] [Organisational Structure] [Research]

General Description

RiskLab is an inter-university research institute, concentrating on precompetitive, applied research in the general area of (integrated) risk management for finance and insurance. The laboratory, founded in 1994 as a virtual research cooperation, was reorganized in 1999 and is now physically located in ETH's main building. RiskLab is presently co-sponsored by the Swiss Federal Institute of Technology (ETHZ) in Zurich, the Credit Suisse Group, the Swiss Reinsurance Company and UBS AG. Various members of the Department of Mathematics at the ETHZ and the Swiss Banking Institute at the University of Zurich are informally linked to RiskLab. The research carried out at RiskLab combines academic, methodological research with a strong input from and interaction with the industry partners. Besides the research director and two postdocs, several additional researchers and guests are often appointed to RiskLab on the basis of specific projects between industry and academia. RiskLab is open for further institutional partners.

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- Promotion of fundamental and precompetitive applied research in strong connection with practitioners,
- Knowledge exchange between academia and the finance industry,
- Promotion of Zurich (and Switzerland in general) as one of the leading centres of excellence regarding the finance business and the corresponding academic education and research.

Budget

The budget of RiskLab consists of a yearly grant towards the appointment of two post-doctoral research fellows plus infrastructure, IT support and rooms from ETHZ as well as a substantial budget towards the support of project oriented, applied research from the finance industry partners (Credit Suisse Group, Swiss Reinsurance Company and UBS AG).

Organisational Structure

- The **Supervisory Board** (Patronat) currently consists of the Chief Risk Officers of the industry partners and the Vice President for Research of ETHZ.
- The **Executive Board** currently consists of delegated experts from the industry partners, three professors from ETHZ and the Research Director. One of the professors acts as Director/President of the Executive Board.
- The **Research Director**, appointed by the Executive Board, runs RiskLab and supervises the

My Own Work

A book provisionally entitled Quantitative Methods in Risk Management is currently in preparation. Publication 2002-2003 ?
Authors: Paul Embrechts, Rüdiger Frey, Alexander McNeil

Aims:

- To provide practitioners of RM with a reference work on the quantitative (mathematical and statistical) tools their work often requires.

- To supply a course text for masters level courses on quantitative risk management, e.g. in a financial engineering programme. A joint University of Zurich and ETH programme starts Autumn 2002.

B. A Brief History of Risk Management

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- The late 20th century saw a "revolution" on financial markets. Derivatives and other financial innovations.
- Large derivatives losses and other financial incidents followed.
- Banks became subject to regulatory capital requirements, internationally coordinated by the Basle Committee of the Bank of International Settlements.

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- 1973. CBOE, Chicago Board Options Exchange starts operating.

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- 1999. Financial Services Act repealing many key provisions of Glass-Steagall. Bank holding companies will continue to expand the range of their financial services; further convergence of finance and insurance likely.

Consequences

Enormous growth in both volume and complexity of products traded on the financial markets.

Example 1
Average daily trading volume at New York stock exchange:
1970: 3.5 million shares 1990: 40 million shares

Example 2: Global market in OTC derivatives (nominal value).

	1995	1998
FOREX contracts	\$13 trillion	\$18 trillion
Interest rate contracts	\$26 trillion	\$50 trillion
All types	\$47 trillion	\$80 trillion

Source BIS; see [Crouhy et al., 2001]. $\$1$ trillion $= \$1 \times 10^{12}$.

First Problems Occur

The period 1993-1996 saw some spectacular derivatives-based losses:

⋆ Orange County (1.7 billion US$)

⋆ Metallgesellschaft (1.3 billion US$)

⋆ Barings (1 billion US$)

Although, to be fair, "classical banking" produced its own large losses.e.g. 50 billion CHF of bad loans written off by the Big Three in early nineties.

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- **Operational Risk** - risk associated with possibility of human error, IT failure, dishonesty, natural disaster, terrorism etc.
- **Liquidity risk** - risk that positions cannot be unwound quickly enough at critical times due to lack of market liquidity.

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  First methodological progress on market risk front.

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  - ⋆ 1996. BIS Amendment prescribing standardized model but allowing internal market VaR models for larger banks.
  - ⋆ 2001. Consultative process for new BIS Accord. Move toward internal credit models. Consideration of operational risk.

Why is the Regulator Concerned?

"Banks collect deposits and play a key role in the payment system. National governments have a very direct interest in ensuring that banks remain capable of meeting their obligations; in effect they act as a guarantor, sometimes also as lender of last resort. They therefore wish to limit the cost of the safety net in the case of bank failure. By acting as a buffer against unanticipated losses, regulatory capital helps to privatize a burden that would otherwise be borne by national governments."
[Crouhy et al., 2001]

C. The VaR Concept

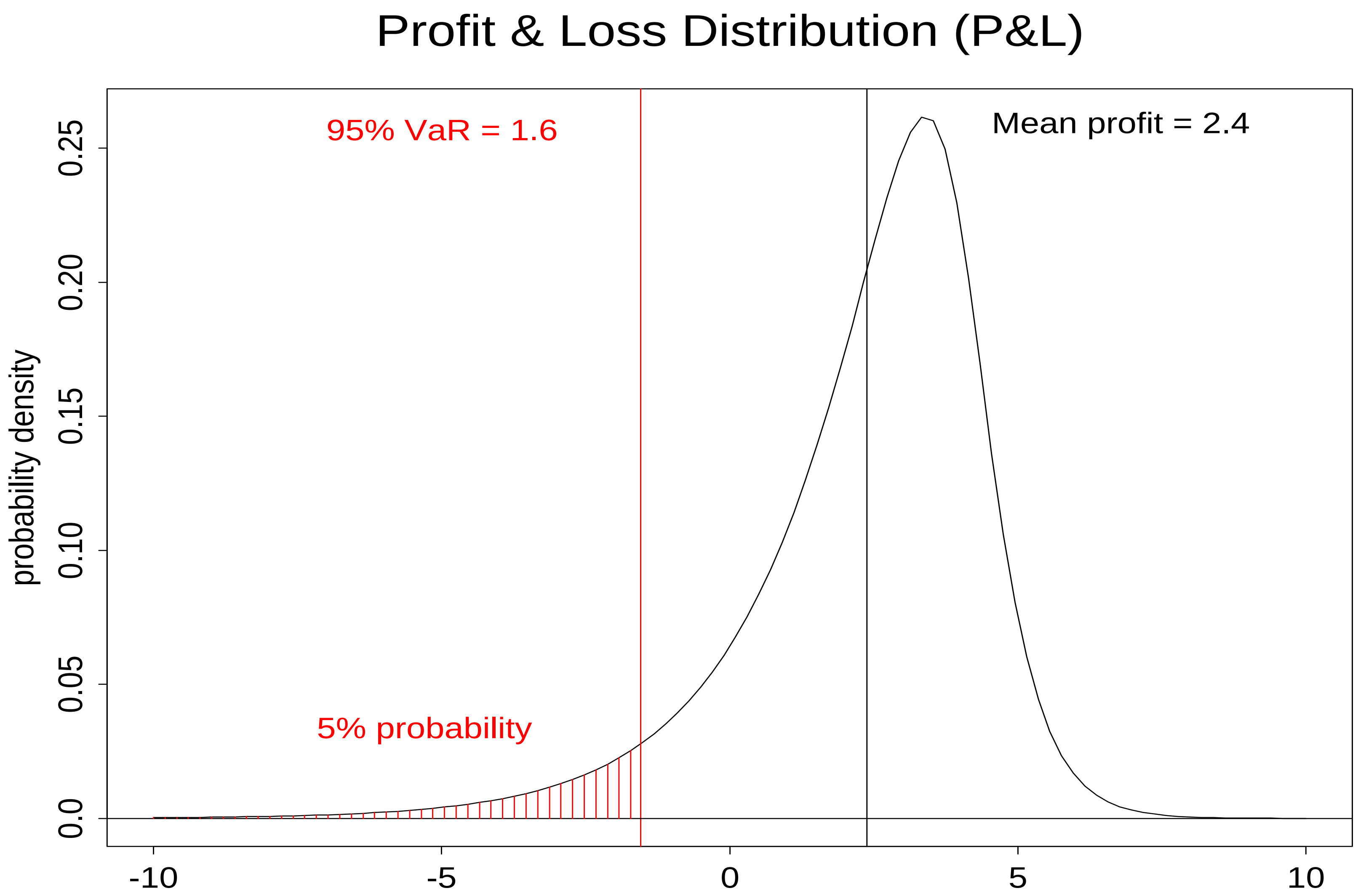
Consider a portfolio/position and potential profits and losses over a fixed time horizon - e.g. 1 day or 10 days.

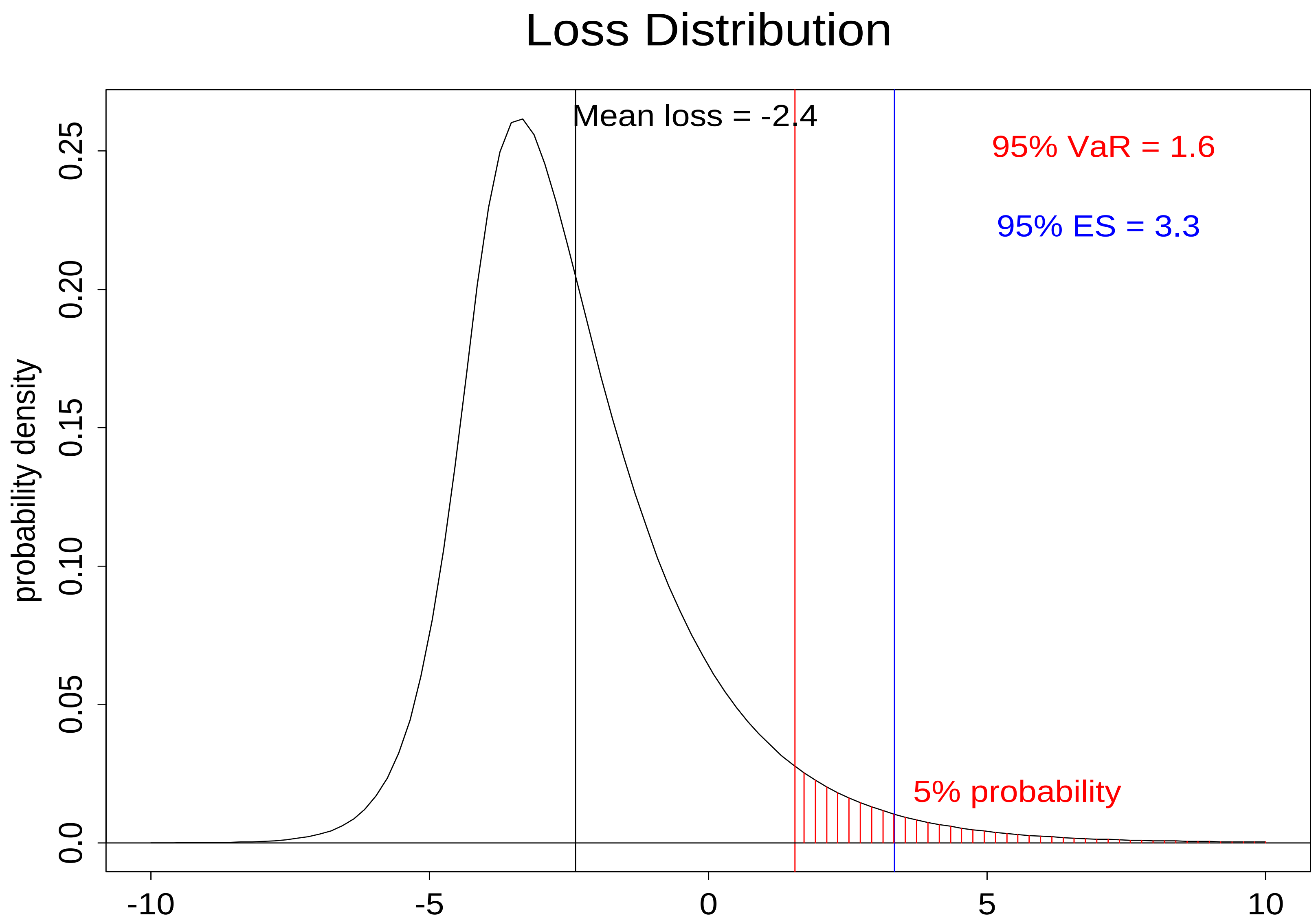
VaR is a percentile (or quantile) of the profit and loss (P&L) distribution with the property that, with a small given probability, we stand to incur that loss or more over the fixed time horizon.

Example. 10-day 99% VaR of 1M$

Interpretation.
If we hold our current portfolio position fixed for 10 days then
Probability (we lose 1M$ or more) $=$ 1%
Probability (we lose up to 1M$) $=$ 99%.

VaR in Visual Terms





Loss Distribution
Mean loss = -2.4
95% VaR = 1.6
95% ES = 3.3
5% probability
probability density
0.0
0.05
0.10
0.15
0.20
0.25
-10
-5
0
5
10

Var - badly defined!

The VaR bible is Philippe Jorion's book.[Jorion, 2001].

The following "definition" is very common:

"VaR is the $maximum$ expected loss of a portfolio over a given time horizon with a certain confidence level."

It is however mathematically meaningless and potentially misleading.
In no sense is VaR a maximum loss!
We can lose more, sometimes much more, depending on the heaviness of the tail of the loss distribution.

The VaR Discipline in Market Risk

Aside from problems of definition/interpretation, the VaR concept has been instrumental in introducing a culture of quantitative (statistical) risk analysis into banks.

1. Estimation of the distribution of future profits and losses for fixed holding period and portfolio
   - single position
   - trading book for a particular market
   - entire position of the bank

2. Estimation of risk measures (VaR) based on estimated P&L.

3. Use of these risk measures to manage enterprise.

A Simple Example: Portfolio of Equities

Today is day t. We are interested in a horizon h (say 10 days). We have an equity portfolio of 3 equities with value given by

$$V_t = \alpha_1 S_{1,t} + \alpha_2 S_{2,t} + \alpha_3 S_{3,t},$$

α_i is number of units of equity i, $S_{i,t}$ is price of equity i.

Our unknown profit/loss is given by $V_{t+h} - V_t$.

To estimate P&L distribution we use historical information concerning changes in the 3 underlying equity values. The underlying equities are known as the risk factors affecting the P&L.

The form of relationship between the risk factors and the value is known as the mapping.

VaR Estimation Methodology

A number of techniques are in widespread use:

- Analytic variance-covariance approach.

  Assumptions:

  - Changes in risk factor values are assumed to have a (multivariate) normal distribution.
  - Changes in value of portfolio are approximated by linear function of changes in risk factors.

  Problems: both normality and linearity.
  Why should risk factor changes be normal? Thin tails may underestimate risk.

VaR Estimation Methodology

- The historical simulation approach

  Observations from the P&L are simulated by examining what would happen if historical observed risk factor changes recurred.

  Problems: relies on availability and relevance of historical risk factor data.

- The Monte Carlo approach

  Assume more complex models for the risk factors and their dynamics. Simulate observations from resulting P&L using computer programs.

  Problems. computer intensive; what model to choose?

VaR: Deeper Problems

Aside from the statistical issue of how to estimate VaR, more fundamental issues have been raised. Many have asked

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Alternative risk measures: expected shortfall (a.k.a. conditional VaR) - the expected size of a loss exceeding VaR.

Example of a VaR Paradox

Consider 100 corporate bonds, $X_1, \ldots, X_{100}$ with 1-year maturity. Each has face value 100, pays interest at 2% and has default rate 1%, per annum.

The P&L of a single bond is

$$X_i = \begin{cases} 2 & \text{with probability } 99\%, \\ -100 & \text{with probability } 1\%. \end{cases}$$

Now consider two portfolios:
A. 100 of bond X_1
B. One each of $X_1, \ldots, X_{100}$.

Which is riskier?

VaR Paradox II

The 95% 1-year VaR of portfolio A is -200.
(Informally: we are 95% certain of making a gain of 200 dollars.)

The 95 % 1-year VaR of portfolio B is > 0.

Paradoxically, the diversified portfolio B is riskier than A in VaR terms. This is clearly nonsensical.

This phenomenon relates to the non-subadditivity of VaR, which makes it poor for decentralized risk management.

Note, the trader who buys position A is 'gaming the VaR'.

D. LTCM. Back to the Drawing Board?

The core strategy of LTCM was relative-value trades. The nature of the bet is to take long and short positions in closely related titles whose yields are expected to soon converge, e.g. German government bonds and Italian government bonds prior to EMU. Since the return is small leverage was used to create attractive returns. Before the crisis LTCM had leverage ratio of 25:1. Of the $125 billion on its balance sheet only $5 billion was equity; the rest was borrowed.

Unfortunately the Russian ruble crisis led to a flight to quality and the divergence of values that were expected to converge. The net result was huge losses - 4.4$ billion - of which 1.9$ billion was incurred by the partners and 2.5 by other investors (700M$ by UBS).

Was VaR to Blame?

LTCM actually used VaR methodology. According to LTCM the fund was structured so that the risk should have been no great than investing in the S&P 500.

"The non-fault bankruptcy". Myron Scholes in *Economist* 25.09.99.

"VaR, the product of portfolio theory, is used for short-run day-to-day profit and loss-risk exposures. Now is the time to encourage the BIS and other regulatory bodies to support studies on stress test and concentration methodologies. Planning for crises is much more important than VaR analysis. And such new methodologies are the correct response to recent crises in the financial industry." [Scholes, 2000].

VaR Wasn't to Blame

"The story of LTCM should not be taken as an indictment of VaR systems, which after all, performed reasonably well for the banking sector in 1998. Instead it provides a number of useful risk management lessons. First it illustrates the danger of optimization biases, or traders 'gaming the system'. LTCM's strategy can be interpreted as a constrained optimization, i.e. maximizing expected returns subject to a constraint on VaR. This strategy led to its demise, as it created huge leverage and extreme sensitivity to instability in the correlations." [Jorion, 2000]

Too Little Rocket Science?

"In a sense, maybe the problem wasn't too much rocket science, but too little. Extreme, synchronized rises and falls in financial markets occur infrequently - but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which many things go wrong at the same time the "perfect storm" scenario."

Business Week, September 21 1998.

First Lesson of all RM Disasters

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$$\mathrm{RM} \approx \mathrm{MR}$$

E. Towards Better Quantitative Methods

Risk Management poses difficult quantitative problems. Much of conventional statistics is to do with "the average", "the normal", or "the expected". Risk management has more to do with the extreme, the abnormal and the unexpected.

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Three technical issues are:

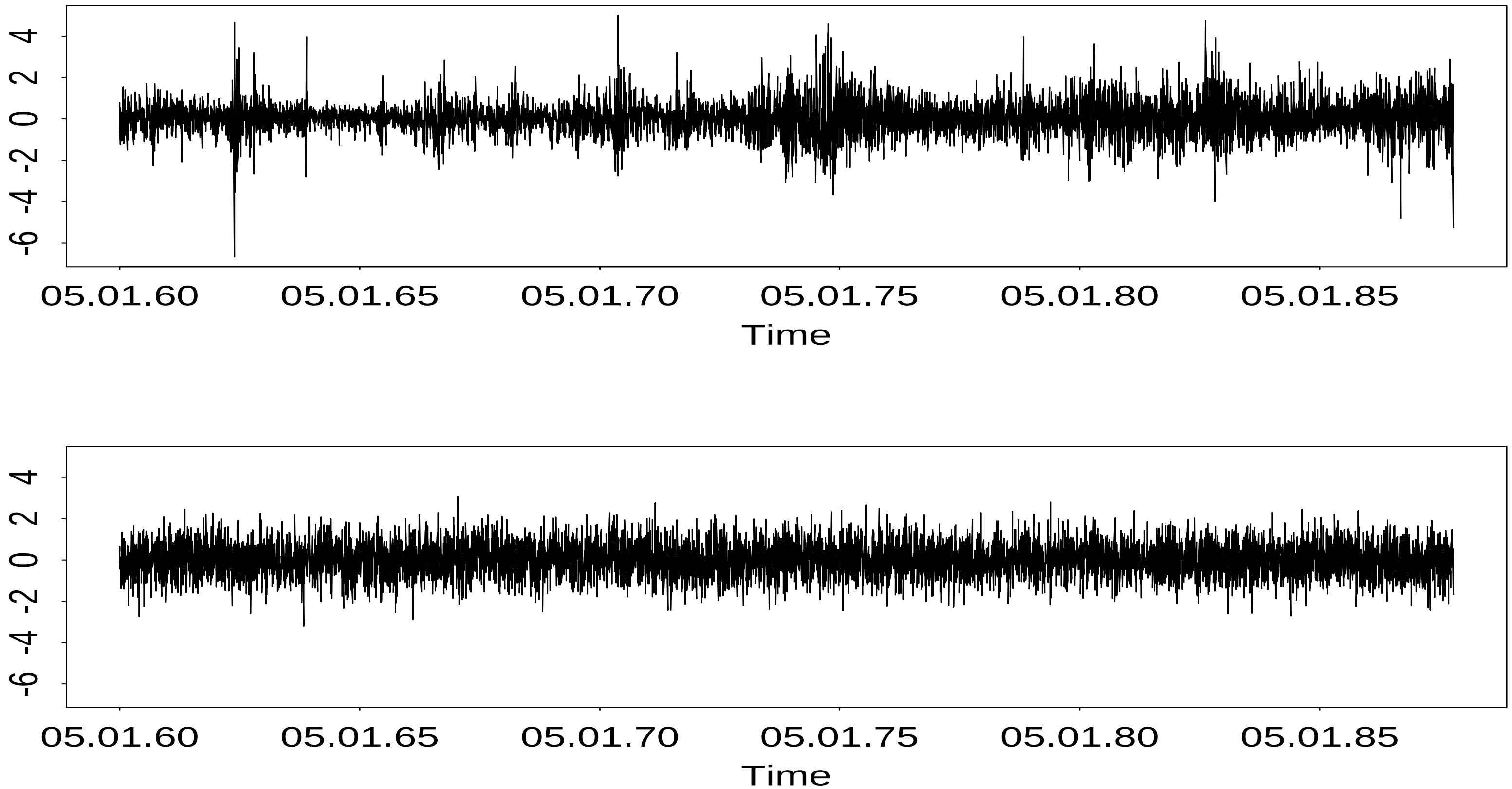
* How to model volatility?
* How to model extremes and stress events?
* How to model correlation and concentration risk?

Volatility

Any financial asset with an element of market risk shows volatility. The scale of this volatility generally contradicts the standard model of finance - geometric Brownian motion - which is the basis of pricing theory.

The implication is that the models with which we measure risk, should probably be different to the models with which we price risky assets.

Stock-market data versus simulated normal



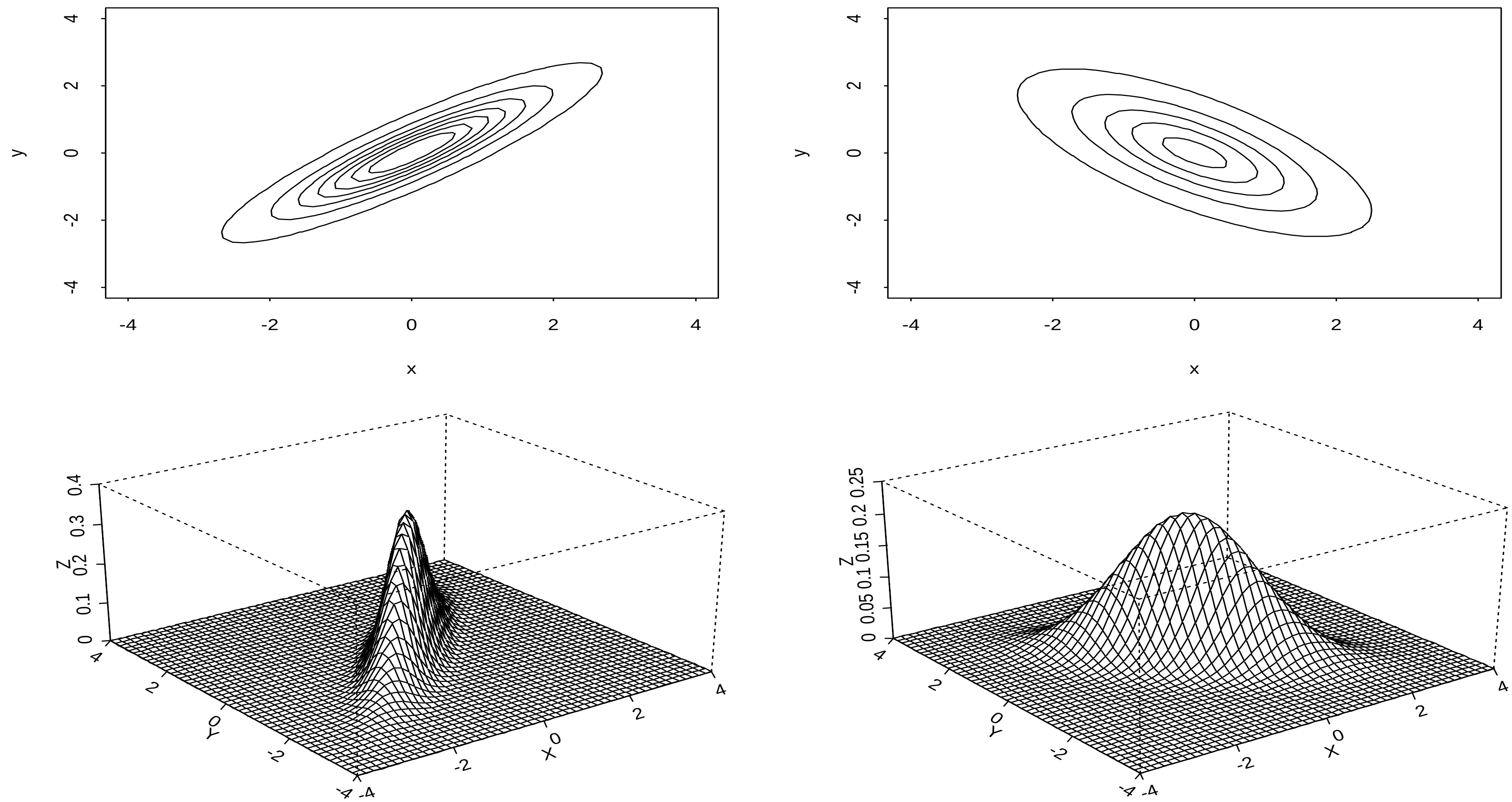
The Stylized Facts of Empirical Finance

Consider daily returns on a stock price, exchange rate, commodity price or other financial instrument, or portfolio of instruments.

We consistently observe the following stylized facts:

- Returns not iid but correlation low
- Absolute returns highly correlated
- Volatility changes randomly with time
- Returns are heavier–tailed than normal distribution
- Extremes appear in clusters

How Normal is the Normal Distribution ?



Standard (bivariate) normal distributions ($\rho = 0.9, -0.7$).

Extreme Values

Above and beyond this persistent background of volatility there is the phenomenon of extreme returns.

Econometric forecasting technology (such as GARCH models and stochastic-volatility models) can go some way to predicting at least short-term volatility development.

But the standard versions of these models (which assume normality of return shocks) fail to explain the frequency and severity of the most extreme movements.

By working with more realistic statistical distributions (heavy-tailed distributions) we can often get a truer risk appraisal. This is the essential idea of extreme value theory. The consideration of stress scenarios is also vital.

Eidgenössische Technische Hochschule Zürich — Department of Mathematics — D-MATH

The ETH *Riskometer*

Market Risk Summary for Major Indices on 18/04/00

Dynamic Risk Measures

Index	VaR (95%)	ESfall (95%)	VaR (99%)	ESfall (99%)	Volatility
S&P 500	3.98	5.99	7.16	9.46	40.1
Dow Jones	3.66	5.43	6.47	8.47	37.4
DAX	3.08	4.21	4.89	6.12	29.3

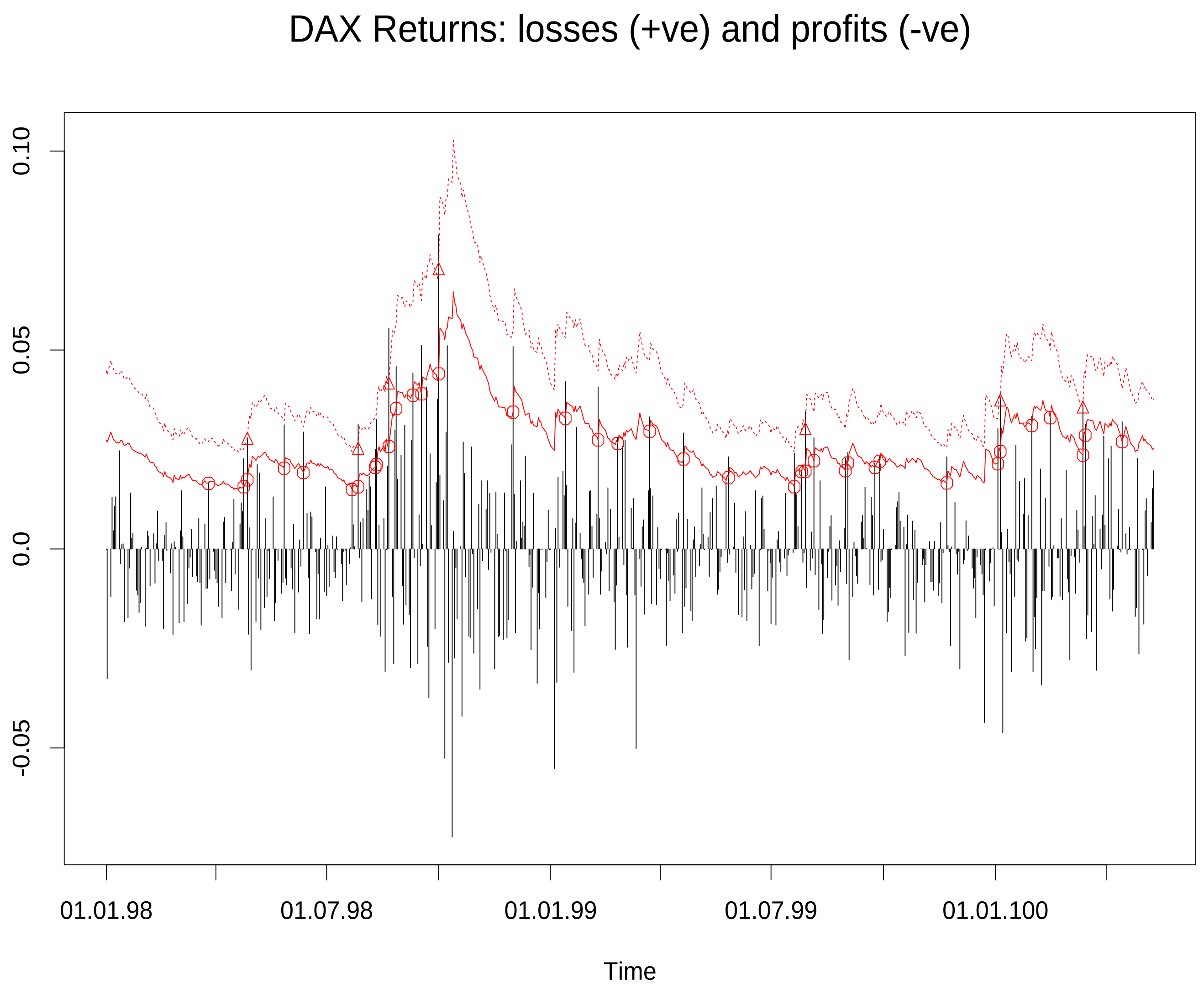
- **VaR and ESfall** prognoses are estimates of potential daily losses expressed as percentages.
- **Volatility** is an annualized estimate expressed as a percentage; click on column heading for recent history.
- **Data** are kindly provided by Olsen & Associates.
- **Developers are** Alexander McNeil and Rüdiger Frey in the group for financial and insurance mathematics in the mathematics department of ETH Zürich.
- **Our methods**, which combine econometric modelling and extreme value theory, are described in our research paper; there are postscript and pdf versions.

VaR Backtests & Violation Summary

- DAX backtest table or picture
- Dow Jones backtest table or picture
- S&P backtest table or picture

In all backtest pictures the 95% VaR is marked by a solid red line and the 99% VaR by a dotted red line. Circles and triangles indicate violation respectively.

Alexander McNeil (mcneil@math.ethz.ch)


DAX Returns: losses (+ve) and profits (-ve)
0.10
0.05
0.0
-0.05
01.01.98
01.07.98
01.01.99
01.07.99
01.01.100
Time

Correlation Confusion

"Among nine big economies, stock market correlations have averaged around 0.5 since the 1960s. In other words, for every 1 per cent rise (or fall) in, say, American share prices, share prices in the other markets will typically rise (fall) by 0.5 per cent."

The Economist, 8th November 1997

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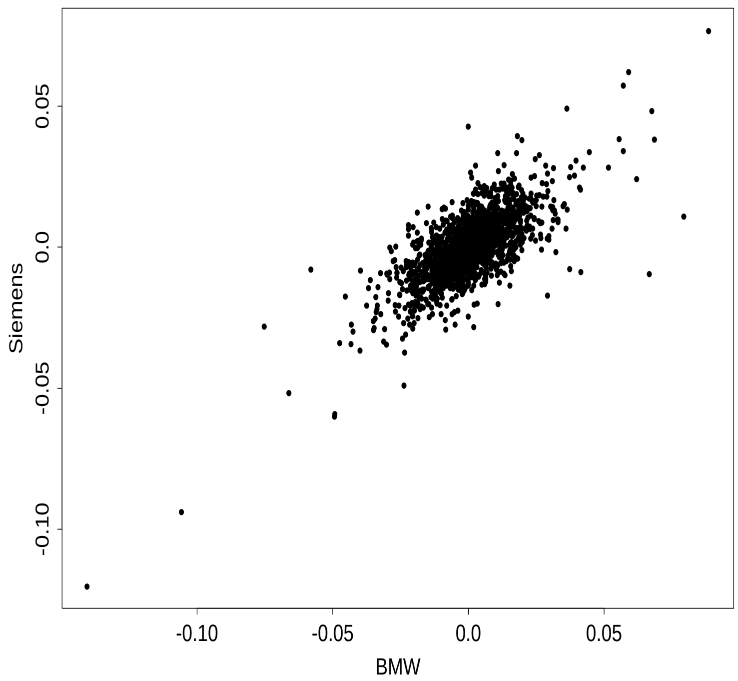
"A correlation of 0.5 does not indicate that a return from stock-market A will be 50% of stockmarket B's return, or vice-versa...A correlation of 0.5 shows that 50% of the time the return of stockmarket A will be positively correlated with the return of stock-market B, and 50% of the time it will not."

The Economist (letter), 22nd November 1997

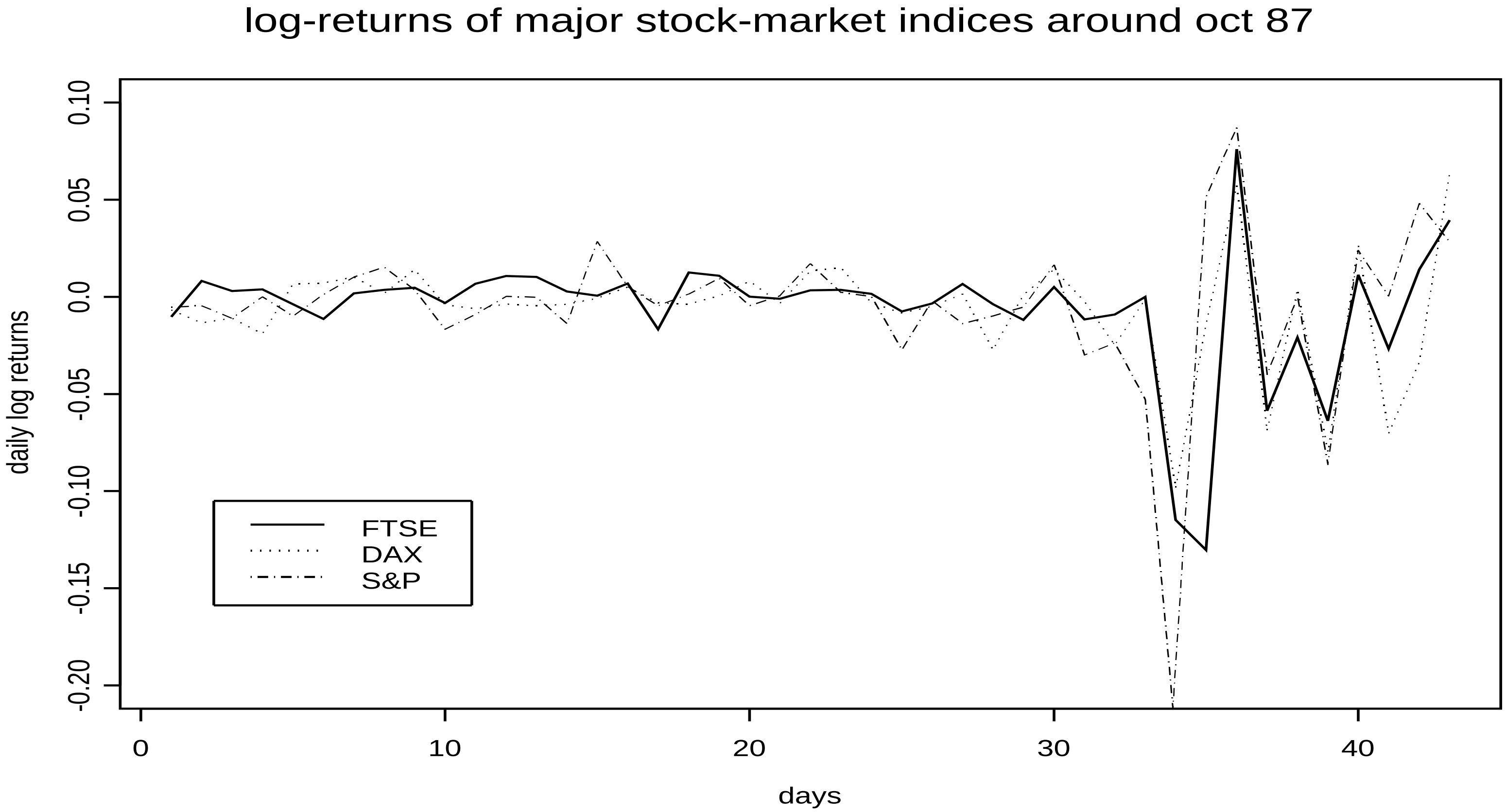
Concentration Risk: Extremes Occur Together

"Correlations are higher in stress periods than in normal periods."

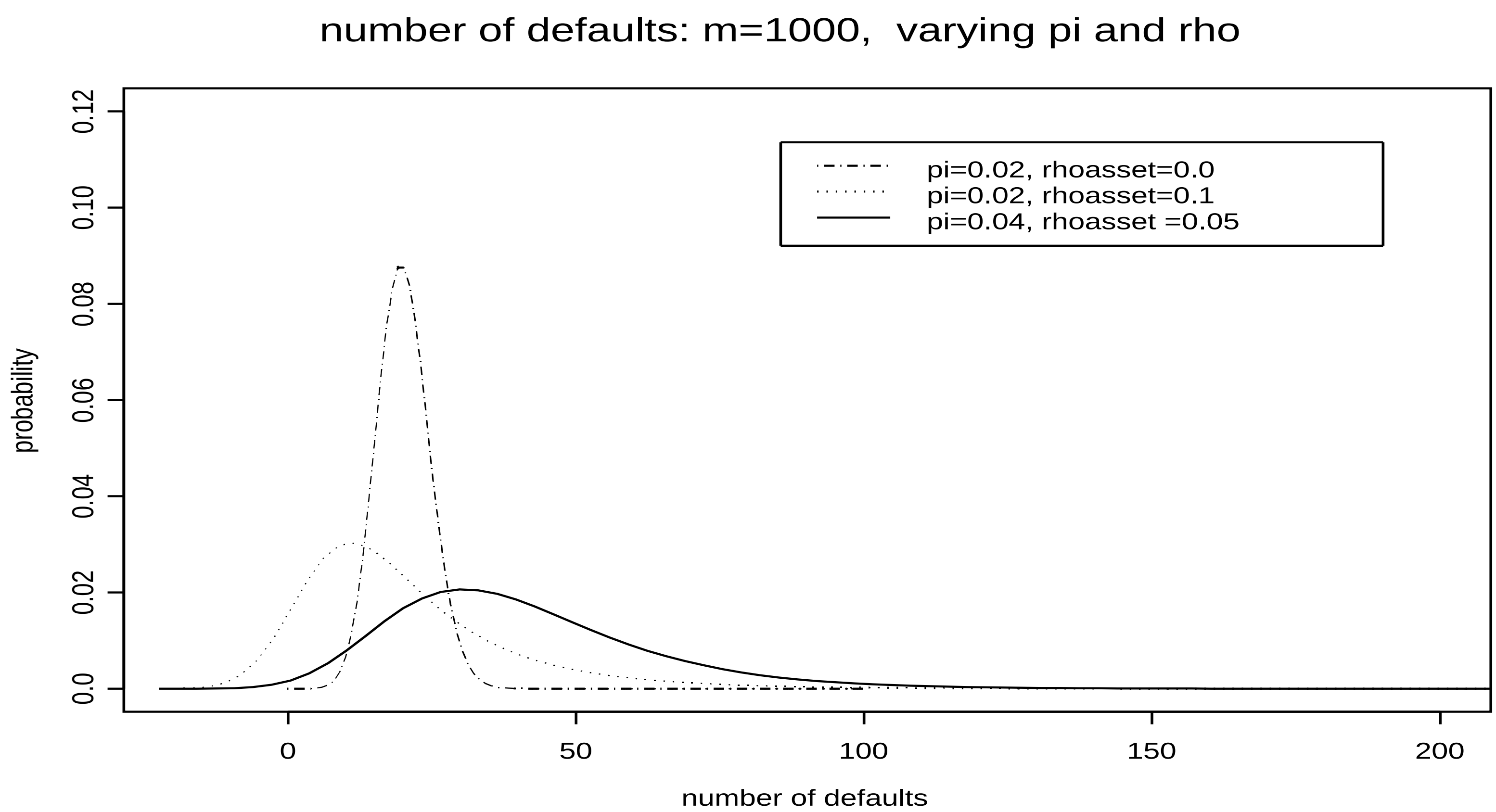


This multivariate stylized fact may express the observation that extreme moves of many financial assets are synchronous.

Extremes Occur Together II



Dependent defaults and credit losses



Distribution of number of defaults in portfolio of 1000 firms.
Dependence between defaults has a large influence on distribution.

Further Technical Reading

- On Extreme Values [Embrechts et al., 1997]
- On Volatility and Extremes [McNeil and Frey, 2000]
- On Dependence and Correlation [Embrechts et al., 2001]
- On Correlation and Credit [Frey and McNeil, 2001]

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- Operational Risk. On the agenda, but less amenable to quantitative approaches.
- Liquidity Risk. Very topical since LTCM, but extremely challenging.
- Risk Integration. Market-credit integration has been addressed, but hardly mastered.

References

[Artzner et al., 1999] Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9:203–228.

[Black and Scholes, 1973] Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.

[Crouhy et al., 2001] Crouhy, M., Galai, D., and Mark, R. (2001). *Risk Management*. McGraw-Hill, New York.

[Embrechts et al., 1997] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.

[Embrechts et al., 2001] Embrechts, P., McNeil, A., and Straumann, D. (2001). Correlation and dependency in risk management: properties and pitfalls. In Dempster, M. and Moffatt, H., editors, *Risk Management: Value at Risk and Beyond*. Cambridge University Press.

[Frey and McNeil, 2001] Frey, R. and McNeil, A. (2001). Modelling dependent defaults. Preprint, ETH Zürich. available from `http://www.math.ethz.ch/~mcneil`.

[Jorion, 2000] Jorion, P. (2000). Risk management lessons from long-term capital management. *European Financial Management*, 6:277–300.

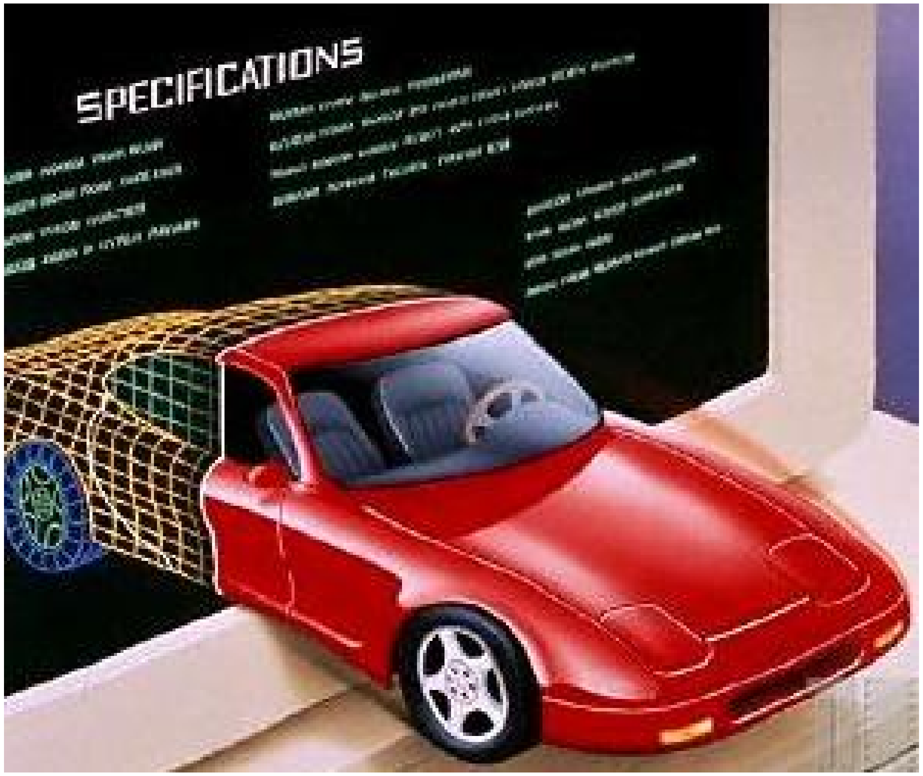
[Jorion, 2001] Jorion, P. (2001). *Value at Risk: the New Benchmark for Measuring Financial Risk*. McGraw-Hill, New York.

[McNeil and Frey, 2000] McNeil, A. and Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of Empirical Finance*, 7:271–300.

[Scholes, 2000] Scholes, M. (2000). Crisis and risk management. *American Economic Review*, pages 17–22.

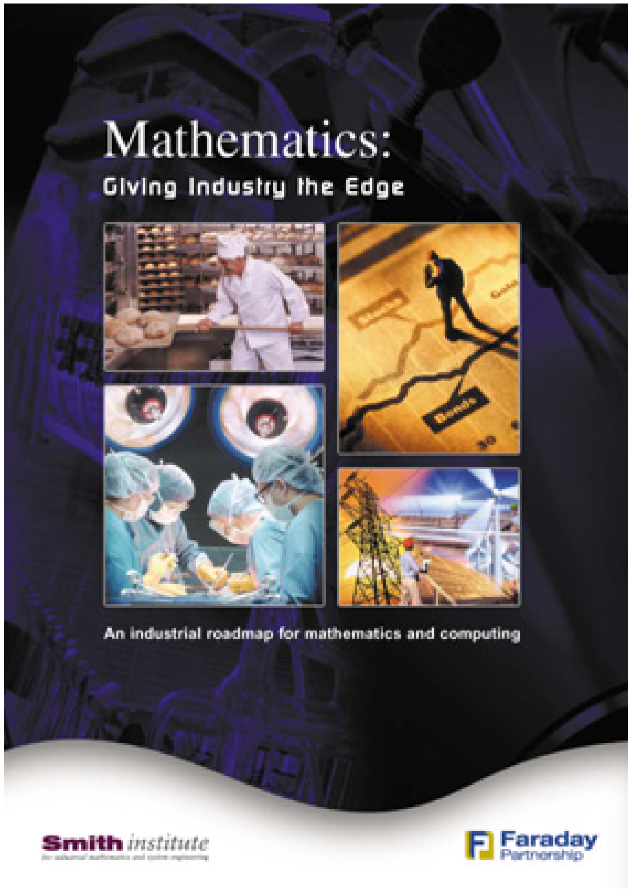
[Steinherr, 1998] Steinherr, A. (1998). *Derivatives. The Wild Beast of Finance*. Wiley, New York.

Mathematics in Industry and Government



Douglas N. Arnold

Institute for Mathematics and its Applications



Mathematics is the most versatile of all the sciences. It is uniquely well placed to respond to the demands of a rapidly changing economic landscape. Just as in the past, the systematic application of mathematics and computing to the most challenging industrial problems will be a vital contributor to business performance. The difference now is that the academic community must broaden its view of mathematics in industry and its expertise must be managed in more imaginative ways.

Mathematics now has the opportunity more than ever before to underpin quantitative understanding of industrial strategy and processes across all sectors of business. Companies that take best advantage of this opportunity will gain a significant competitive advantage: mathematics truly gives industry the edge.

Academic mathematics is insufficiently connected to mathematics outside the university. One of the greatest—and most difficult—opportunities for academic mathematics is to build closer connections to industry.

Academic mathematical science must strike a better balance between theory and application. At one extreme, a narrowly inward-looking community will miss both the opportunities that arise outside the mathematical sciences and the opportunities that are part of scientific and technological developments. At the other extreme, an exclusive concern with applications and collaborative research would severely limit the mathematical sciences and deprive the scientific community of the full benefits of mathematical inquiry. At present, the balance is tilted too far towards inwardness.

A narrow vision of mathematics in academic departments translates into a narrow education for graduate students, most of whom are orinted toward careers only in academic mathematics.

Observations and opinions

- The potential impact of contemporary mathematics on science, on technology, and on industry is vast.
- Unfortunately, the actual impact—though great—is no where near as large as it should be.
- In significant part, this results from the decision of many mathematicians to address themselves to internally generated challenges rather than to the challenges that arise from the complexities of the modern world.
- Industrial mathematicians almost always face problems coming from outside mathematics.
- Industrial managers are convinced of the power of mathematics. . . they hire 25% of mathematics doctorates.

A problem from outside mathematics

Planning for and responding to the deliberate release of infectious agents is a clear example of a problem that mathematics cannot solve, but to which it can contribute immensely.

For a smallpox attack for example, many critical decisions have to be made. Examples:

- who to vaccinate (direct contacts of infected, neighborhoods of infected, essential personnel, the city, the country,. . . , healthy, at-risk, young, old, . . .)
- prophylactic vaccination?
- quarantine policy
- value of early detection
- value of diagnostic testing
- dealing with uncertainty

Math can help!

SIR model of mathematical epidemiology

Daniel Bernoulli published a mathematical study of smallpox spread in 1760. In the 1920's Kermack and McKendrick formulated the SIR model:

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I, \quad \frac{dR}{dt} = \gamma I,$$

where $S + I + R = 1$ give the division of the population into susceptible, infective, and recovered segments, $\beta > 0$ the *infection rate*, $\gamma > 0$ the *removal rate*.

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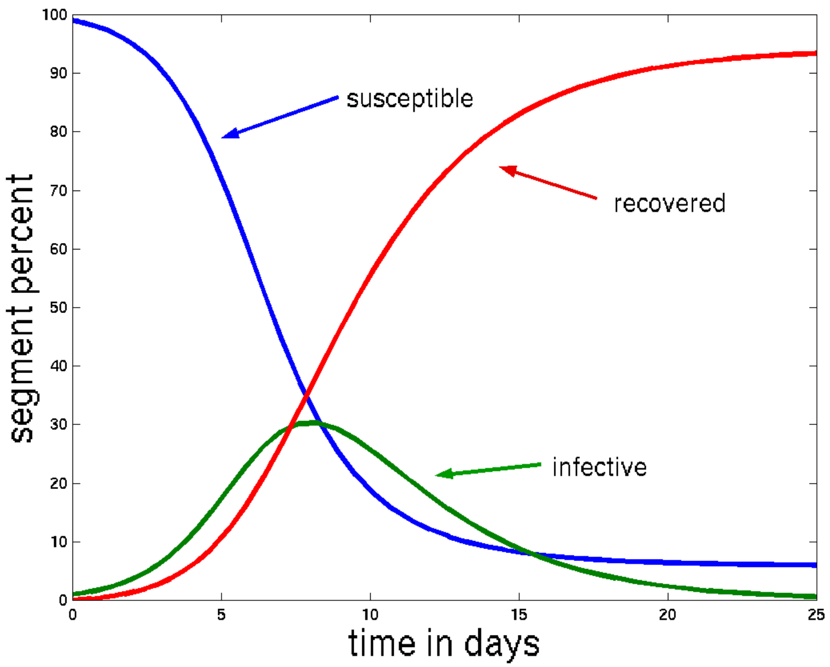
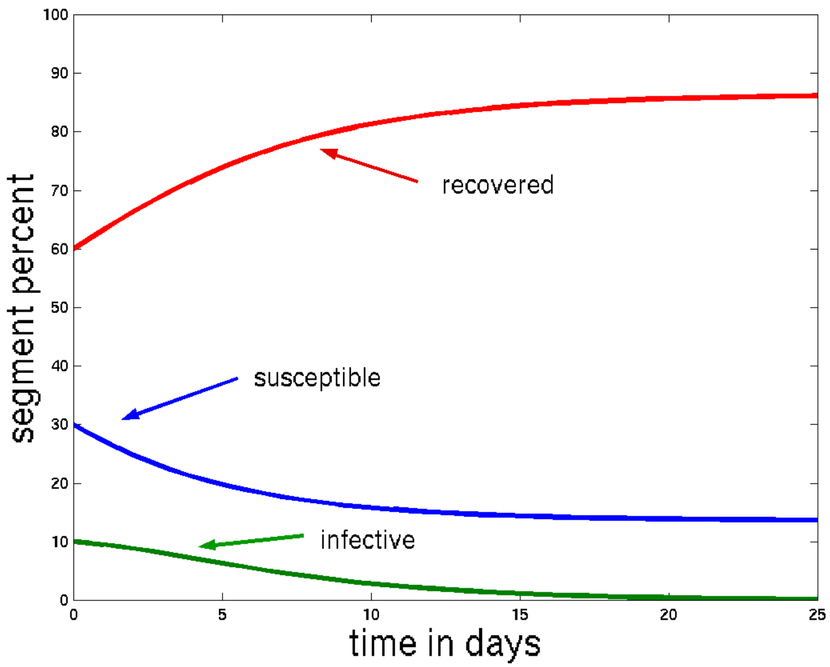
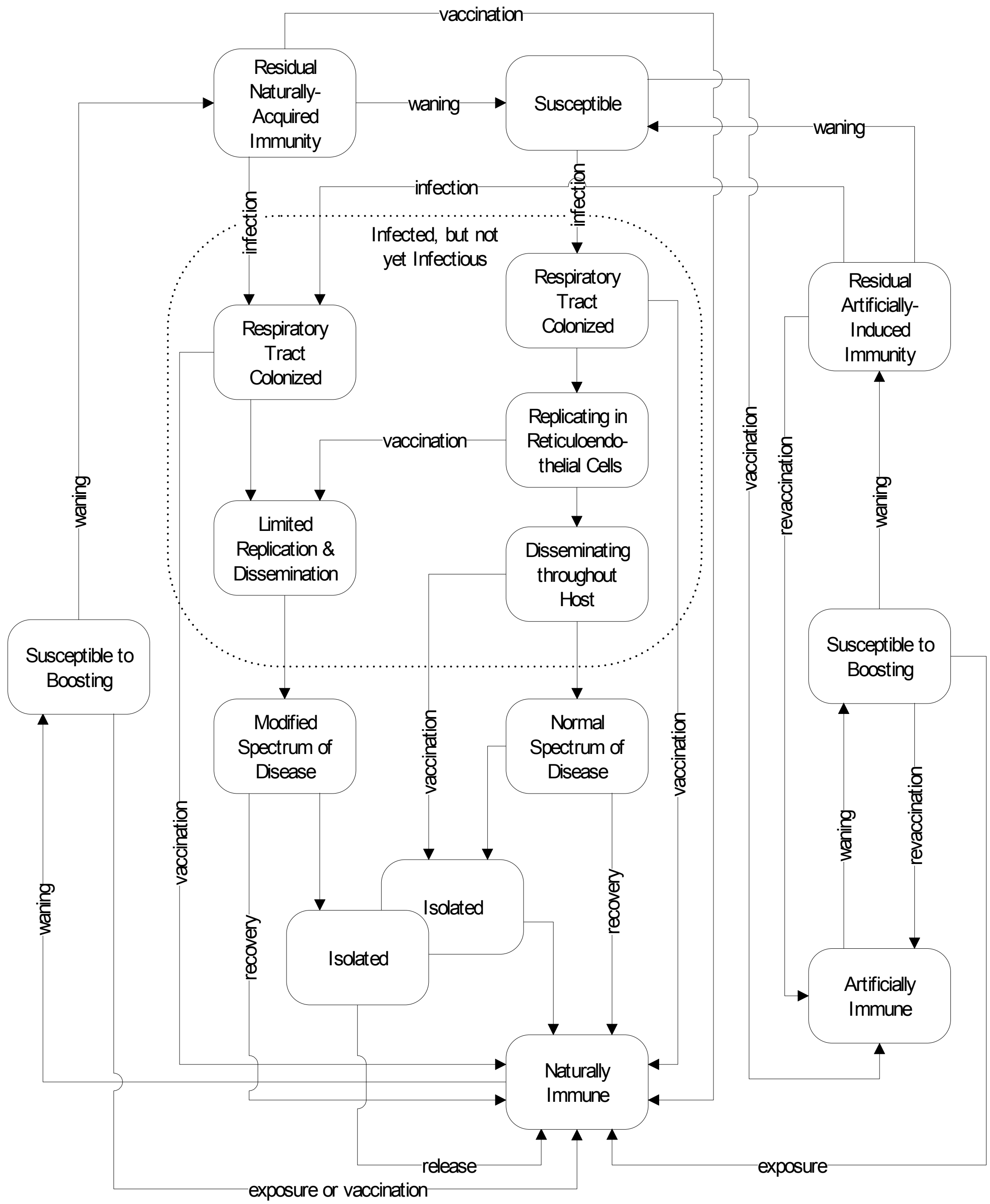
Theorem. *Let* $S(0), I(0) > 0$, $R(0) = 1 - S(0) - I(0) \geq 0$ *be given. For the solution of the SIR model with* $S(0) > \gamma/\beta$, $I(t)$ *increases initially until it reaches its maximum value and then decreases to zero at* $t \to \infty$. *Otherwise* $I(t)$ *decreases monotonically to zero as* $t \to 0$.





herd immunity

Smallpox modeling at the Center for Disease Control



Multi-patch generalization of Keeling and Gilligan, 2000

- *Treating patch-patch heterogeneity by Lloyd and May's (1996) approach*
- *Incorporating spatial spread (city-city to transnational) by modeling transportation networks, rates via Rvachev et al. (1977) approach*
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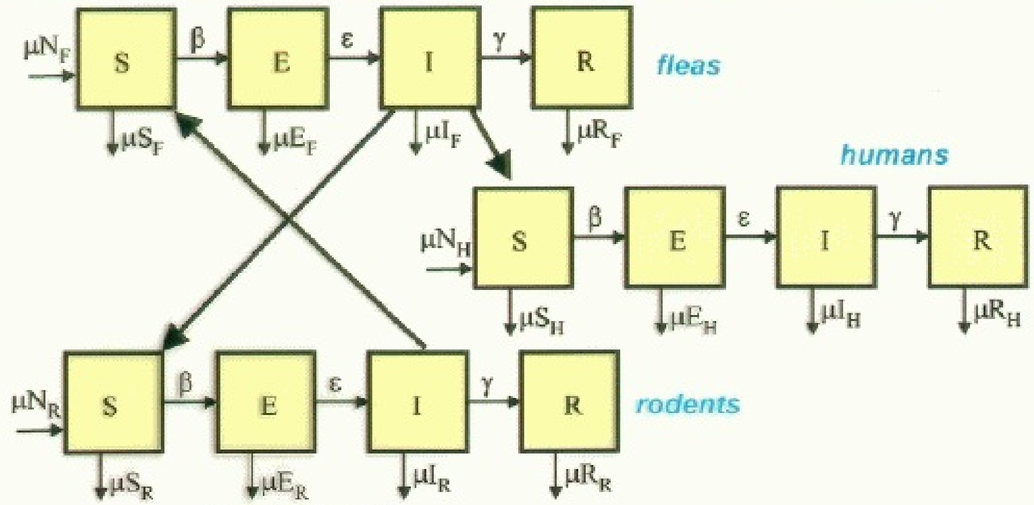

Includes essential dimensions of plague epidemiology

- *Human, rodent and flea interactions*
- *Patch-patch ecological variation*
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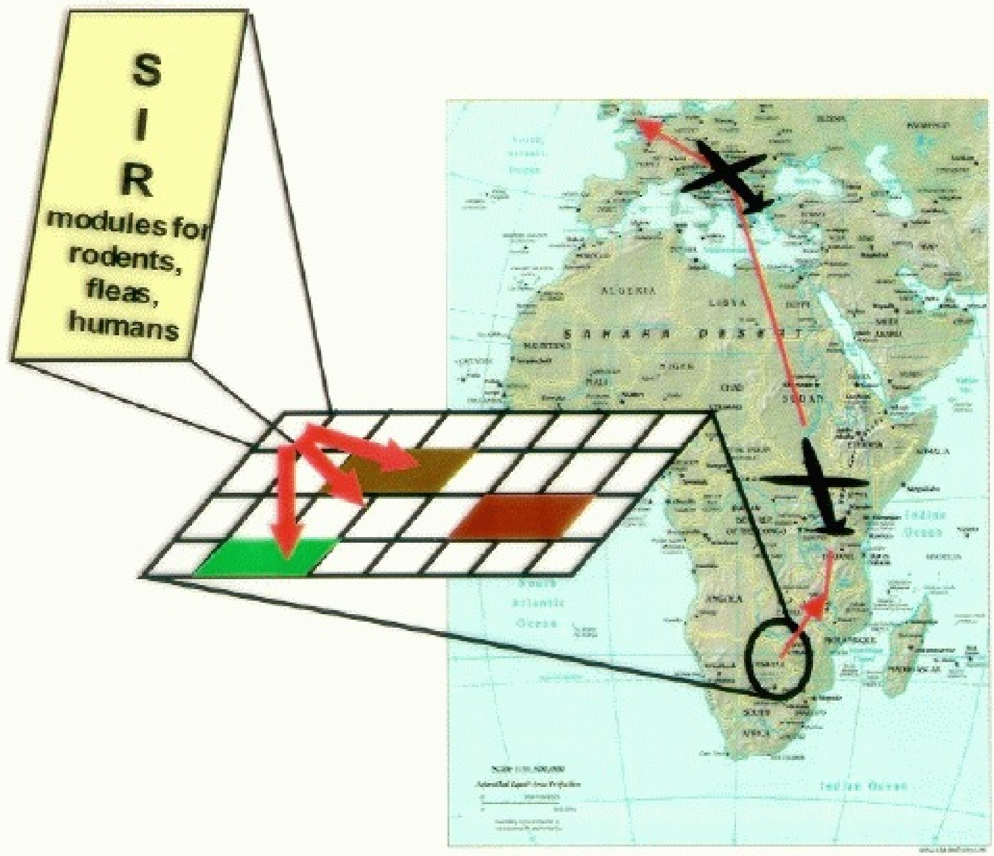

Evaluation to include

- *Single-patch incidence, prevalence, R_0*
- *Patch-patch disease propagation and spatial spread*

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www.nytimes.com
The New York Times ON THE WEB

July 7, 2002

U.S. to Vaccinate 500,000 Workers Against Smallpox

By WILLIAM J. BROAD

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Until last month, officials had said they would soon vaccinate a few thousand health workers and would respond to any smallpox attack with limited vaccinations of the public. Since 1983, only 11,000 Americans who work with the virus and its related diseases have received a vaccination, according to the Centers for Disease Control and Prevention.

The plan to increase the number of "first responders" who receive the vaccination to roughly 500,000 from 15,000 and to prepare for a mass undertaking of vaccinations in effect acknowledges that the government's existing program is insufficient to fight a large outbreak.

Mathematical techniques relevant to bioterrorism

- mathematical epidemiology
- ODE, dynamical systems
- PDE
- numerical analysis, scientific computation
- probability, statistics
- graph theory, network analysis
- game theory
- control theory
- optimization
- . . .

Industries using mathematics

Aerospace	Financial services
Automation and control	Geosciences
Automotive	Healthcare
Computing	Information Technology
Defense	Manufacturing
Energy	Telecommunication
Transportation	Shipping

scores of others and increasing

Areas and Applications (MII '98)

Mathematical Area	Application
Algebra and number theory	Cryptography
Computational fluid dynamics	Aircraft and automobile design
Differential equations	Aerodynamics, porous media, finance
Discrete mathematics	Communication and information security
Formal systems and logic	Computer security, verification
Geometry	Computer-aided engineering and design
Nonlinear control	Operation of mechanical and electrical systems
Numerical analysis	Essentially all applications
Optimization	Asset allocation, shape and system design
Parallel algorithms	Weath modeling and prediction, crash simulation
Statistic	Design of experiments, analysis of large data sets
Stochastic processes	Signal analysis

Are all mathematical fields of interest to industry?

Just about, but some more so than others.

> *What kind of mathematics is useful? Every kind, but at Kodak partial differential equations are useful more often than topology. – Peter Castro*

Industry hired 50% of the 2001 PhDs in statistics, 43% in numerical analysis, and 10% of those in geometry/topology.

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Field of specialization is a secondary condition in industry. An academic mathematician very well may spend his career working around the area of their thesis, but an industrial mathematician almost never does.

> *We never know what kind of mathematics is the right kinds, so an "algebraist for life" is not the right kind of mathematician.*

An industrial mathematician must be a generalist, learning whatever kind of mathematics the problem calls for. She should be interested in all kinds of mathematics, and also in things other than mathematics.

Depth in one area is certainly a plus, especially if the area seems relevant to the industry, but breadth is more important.

What do mathematicians bring to industry?

- logical thinking
- the ability to abstract and recognize underlying structure
- knowing the right questions, recognizing the wrong ones
- familiarity with a wide variety of problem-solving tools

Problems never come in formulated as mathematical problems. A mathematician's biggest contribution to a team is often an ability to state the right question.

What can't mathematics do for industry?

Solve its problems.

There are countless problems in industry that require deep mathematics, but almost none that can be solved by mathematics alone.

> *The strength of the mathematical sciences is that they are pervasive in many applications. The challenge is that they are only a part of each application. – Shmuel Winograd*

∴ a mathematician in industry must be part of a team.

∴ communication skills and social skills matter (while, according to popular opinion, these are positively harmful for an academic mathematician).

Traits of successful industrial mathematicians

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- balance between breadth and depth
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- computational skills
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- IMA Industrial Postdocs
- Hot topics workshops
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Closing remarks

- Industry provides a rich source of problems involving a wide range of advanced mathematics.
- A math job in industry can provide intellectual challenge, a good salary, and a chance for real impact.
- The distinction between industrial mathematics and academic mathematics is more one of attitude than content.
- Future potential is tremendous potential. Mathematics can, and should, have much greater impact in the future.
- Traditional graduate math training helps develop several skills useful in industry, but downplays others.
- Many grad programs are adapting. Many programs for students are available (workshops, internships, conferences).

Action item

Encourage your students (and faculty) to think deeply about how they want to spend their lives, to collect information about the alternatives, to look outward as well as inward, to avail themselves of non-traditional and interdisciplinary programs, and to keep an open mind.

Two useful references

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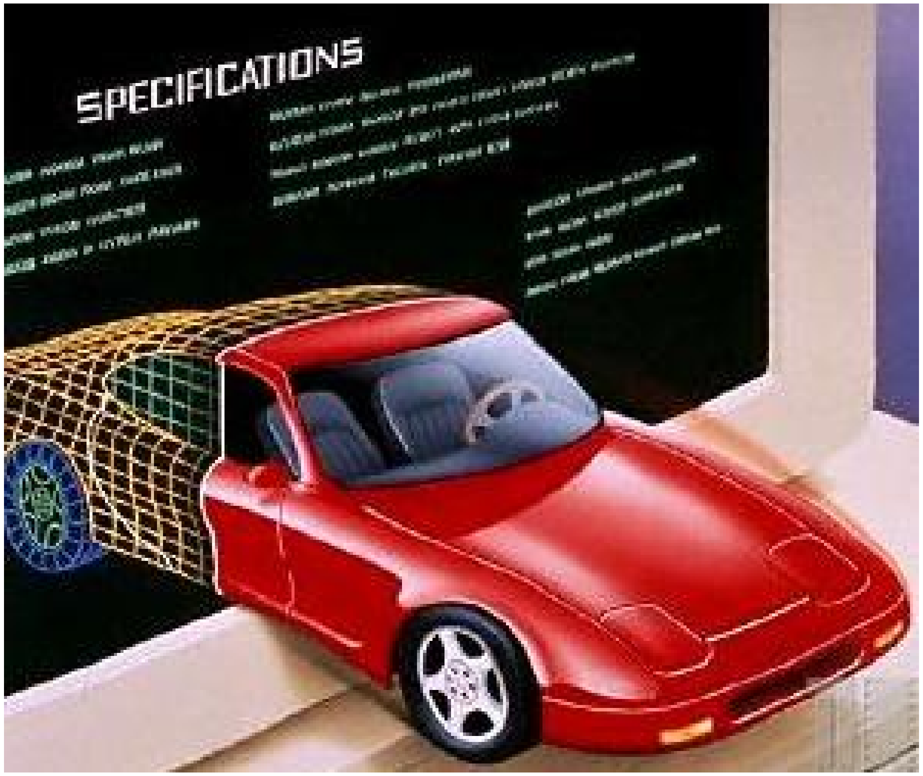

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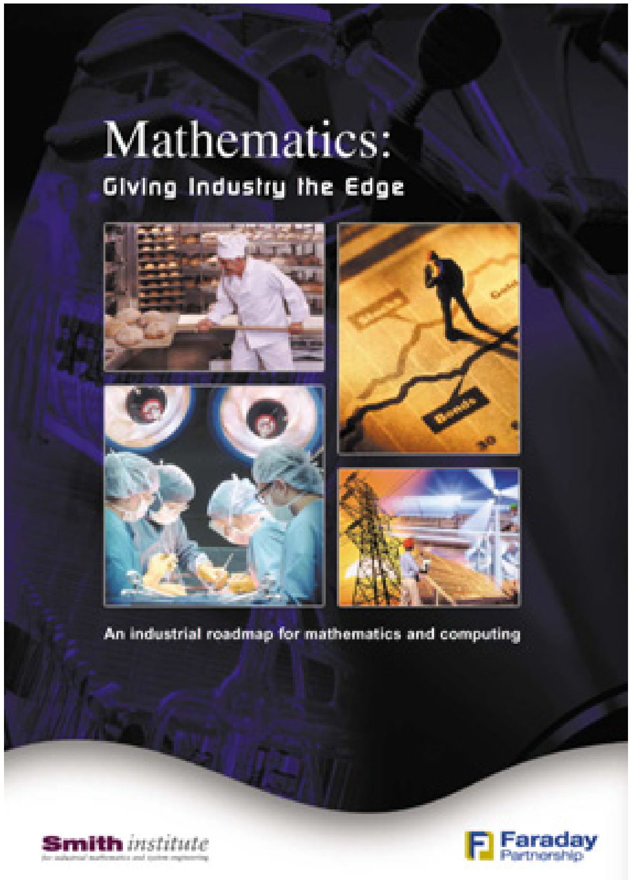

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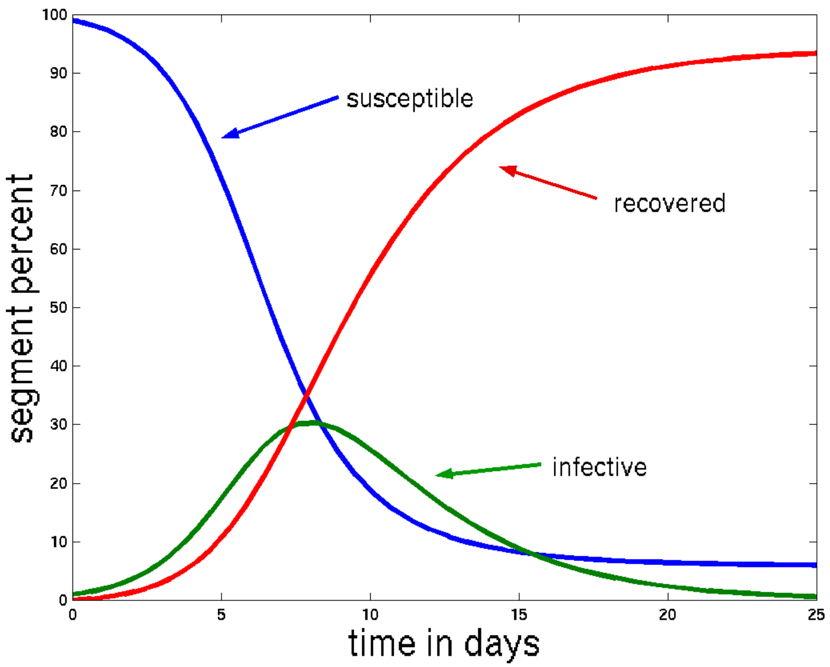

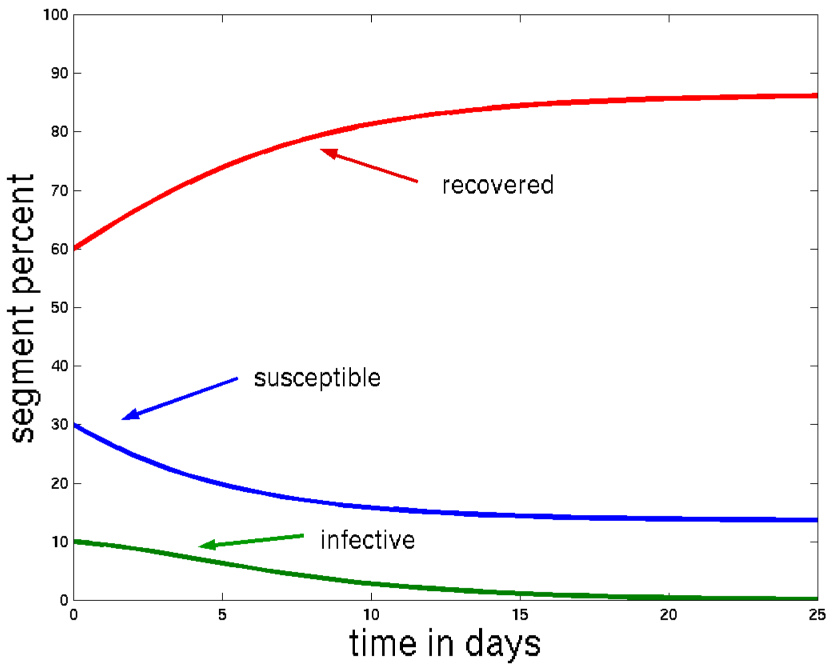

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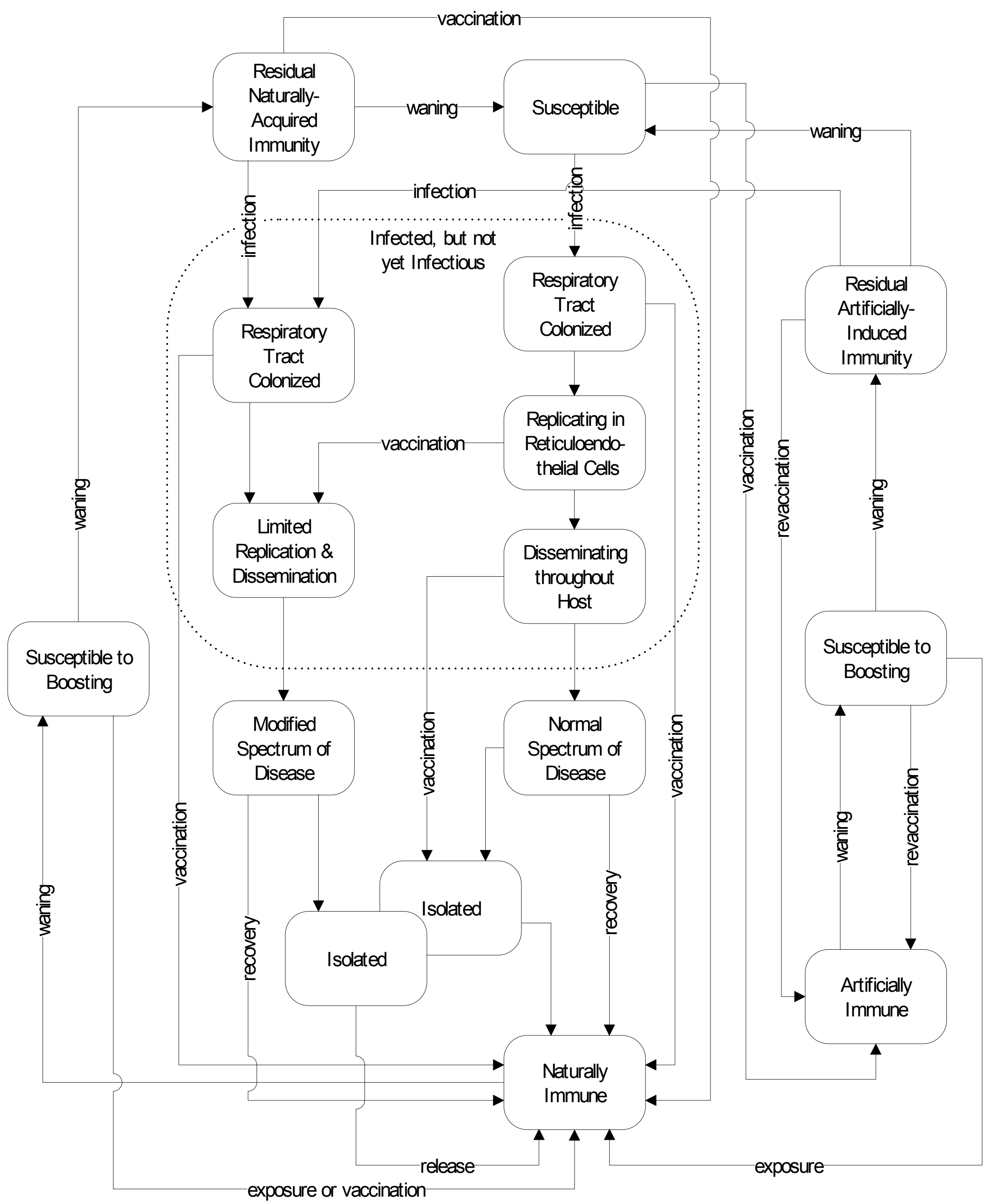

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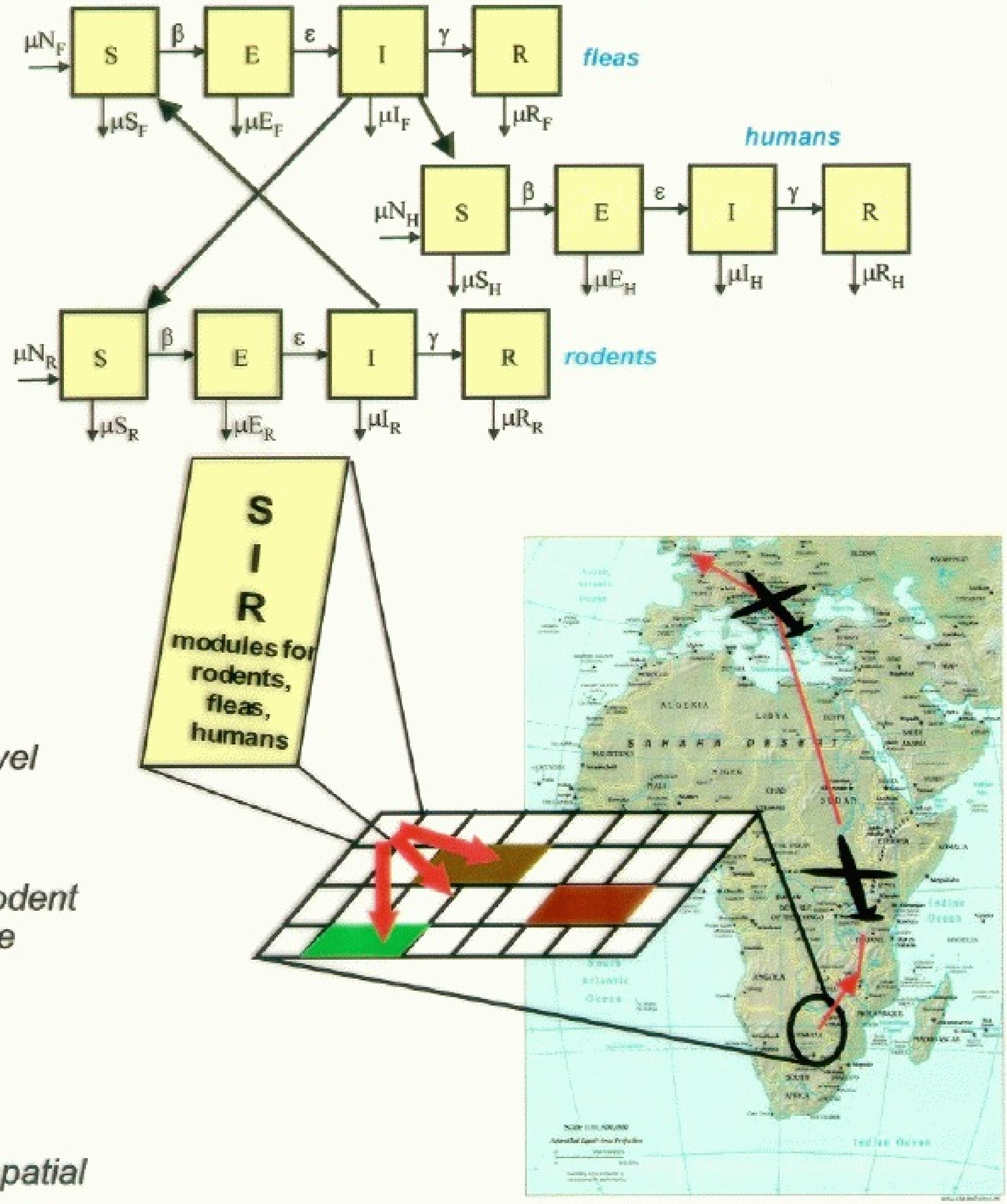

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La proyección social de las Matemáticas

Mesa redonda

Coordinador: *Ramón Á. Orive Rodríguez*

Profesor Titular de Matemática Aplicada del Departamento de Análisis Matemático y Decano de la Facultad de Matemáticas de la Universidad de La Laguna

Ponentes:

Javier Ariz Tellería
Licenciado en Ciencias Biológicas e Investigador del Área de Pesca del Centro Oceanográfico de Canarias
Luis Balbuena Castellano
Catedrático de Matemáticas del IES "Viera y Clavijo" de La Laguna y miembro fundador de la Sociedad Canaria "Isaac Newton" de Profesores de Matemáticas
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Resumen

Esta mesa redonda supone la culminación del Módulo I, sobre *Matemáticas y Sociedad*, del Curso Interdisciplinar "Sociedad, Ciencia, Tecnología y Matemáticas". Con la participación de profesionales cuyo trabajo está total o muy directamente relacionado con las Matemáticas, que actuarán como ponentes, y con las preguntas o comentarios que vayan surgiendo desde la audiencia, pretendemos que en el transcurso de esta mesa-coloquio se den cita algunas de las tendencias y controversias actuales acerca de la proyección social de las Matemáticas, tratando de abarcar la mayor variedad posible de aspectos de esta realidad poliédrica, algunos de las cuales pueden ser:

- Aplicabilidad de las Matemáticas – y los matemáticos – a otras ciencias experimentales y sociales en la sociedad actual.
- Las Matemáticas como herramienta imprescindible en la técnica y en la industria.
- Necesidad de una mayor conexión entre Universidad y Empresa.

- Situación de la investigación en Matemáticas en España y transferencia de resultados.
- Conexión entre Matemáticas y Sociedad. Conveniencia y necesidad de una mayor divulgación social de la belleza y la aplicabilidad de las Matemáticas, así como de un mayor esfuerzo pedagógico en la transmisión de los conocimientos matemáticos.
- Necesidades de adecuación de los actuales planes de estudio de Matemáticas en las universidades españolas, máxime ante el reto inminente de la convergencia europea en materia universitaria.
- Necesidad de definir nuevas salidas profesionales para los matemáticos, y de favorecer posibles mecanismos de integración de éstos en equipos multiprofesionales.

El análisis de la situación académica y socio-económica actual, que propicia la necesidad de este coloquio, así como de este Curso Interdisciplinar y de otras actividades similares que se están llevando a cabo en otras muchas universidades, arroja un balance muy claro: pese a que nuestra sociedad está cada vez más matematizada e informatizada -no sólo en lo referente a las tradicionalmente denominadas ciencias "puras" y "duras" sino también en lo tocante a las ciencias humanas y sociales e, incluso, en casi todo aquello que afecta a la vida cotidiana de las personas-, las Matemáticas siguen siendo unas grandes desconocidas. Esta fue, a grandes rasgos, la principal conclusión que llevó a declarar el año 2000 como Año Mundial de las Matemáticas y a organizar, seguidamente, toda una serie de actividades destinadas a intentar acercar a la sociedad la realidad de esta disciplina. En esta misma línea, cabe destacar las sucesivas reuniones de Decanos y Directores de Matemáticas que se han venido celebrando anualmente en España desde 1999.

Es indudable que buena parte de ese desconocimiento social de las Matemáticas viene determinado por la falta de capacidad y/o interés de los propios matemáticos por conectar con el resto de los agentes sociales, profesionales y empresariales cuya práctica diaria demanda el uso constante de Matemáticas. De ahí que, sin menoscabo de los debates "internos" de la comunidad matemática sobre la situación actual (planes de estudio, salidas profesionales, ...), sea imprescindible recabar la opinión y la experiencia de los "otros", que también las utilizan o necesitan. En este sentido, creemos que la selección de los participantes en esta mesa redonda, matemáticos y no matemáticos, todos ellos profesionales de prestigio muy interesados en nuestra disciplina, servirá para extraer valiosas conclusiones acerca de la realidad actual de las Matemáticas, su aplicabilidad y su conexión con la sociedad.

Bibliografía

C. Andradas, E. Zuazua (coordinadores): *Informe sobre la investigación matemática en España en el periodo 1990-1999.* Comité Español para el Año Mundial de las Matemáticas CEAMM2000.
[Disponible en http://www.rsme.es/inicio/informem.pdf].

J. Bruna: Una reflexión sobre los estudios de matemáticas y sus perspectivas. *La Gaceta de la RSME* **3**, no. 1 (2000), 43-64.

P.A. Griffiths: Mathematics at the turn of the millennium. *Amer. Math. Monthly* **107**, no. 1 (2000), 1-14. [Versión en castellano: Las Matemáticas ante el cambio de milenio. *La Gaceta de la RSME* **3**, no. 1 (2000), 23-41. Disponible en http://www.mat.ucm.es/deptos/am/guzman/griffithsgaceta/griffithsgac.html].

J.L. Vázquez: The importance of Mathematics in the development of Science and Technology. *Boletín SEMA* **19** (2001), 69-112. [Versión (reelaborada) en castellano: Matemáticas, Ciencia y Tecnología: Una relación profunda y duradera. Ambas versiones disponibles en http://www.uam.es/personal_pdi/ciencias/jvazquez/jlvescr.html].

Documento de trabajo del proyecto piloto auspiciado por la CRUE sobre la integración de los estudios españoles de Matemáticas en el espacio europeo de enseñanza superior. [Disponible en http://www.matematicas.us.es/Bolonia/documento.htm].

El papel de las matemáticas en la empresa. Recopilación de intervenciones en los "Encuentros Universidad-Sociedad". Universidad del País Vasco, Bilbao, febrero de 2000. [Disponible en http://www.ehu.es/gizartekontseilua/pdf/encuentros/matematicas.pdf].

En Internet

http://www.rsme.es/comis/educ/debate.htm
Debate: "La enseñanza de las Matemáticas en España"
Comisión de Educación de la RSME.

http://www.ams.org/ams/mathmoments.html
Mathematical Moments
Programa de la American Mathematical Society para promover el conocimiento y la apreciación del papel que las Matemáticas juegan en la naturaleza, la ciencia, la tecnología y la cultura.

http://plus.maths.org
Plus Magazine
Revista electrónica mensual patrocinada por la Universidad de Cambridge que introduce a los lectores en la belleza y la aplicabilidad de las Matemáticas.

http://ochoa.mat.ucm.es/~guzman/00edumatuniv/index.html
Problemas de la Educación Matemática en la Universidad
Material puesto a disposición de los asistentes en la mesa redonda celebrada en abril de 2001 en la Facultad de Matemáticas de la Universidad Complutense de Madrid.

http://www.ams.org/new-in-math/happening.htm
What's Happening in the Mathematical Sciences
Serie de publicaciones divulgativas sobre la actualidad de la investigación matemática, escrita por B. Cipra y editada por la American Mathematical Society.

Documento de trabajo sobre la integración de los estudios españoles de matemáticas en el espacio europeo de enseñanza superior

Octubre-2002

Documento de trabajo sobre la integración de los estudios españoles de matemáticas en el espacio europeo de enseñanza superior

Presentación

Este documento es fruto del consenso del grupo de matemáticas del proyecto piloto, auspiciado por la CRUE, para impulsar el proceso de Bolonia en las universidades españolas. Consideramos necesario someterlo a discusión entre la comunidad matemática española. Sumándonos a la declaración del grupo europeo "estamos convencidos de que cualquier clase de acción en las direcciones que aquí señalamos solamente será posible y fructífera cuando se haya alcanzado un amplio acuerdo. Por supuesto, todos los matemáticos pertenecientes al grupo recibirán gustosos cualquier comentario sobre este documento".

Antecendentes

En mayo de 2001 la Comisión Europea puso en marcha un programa piloto *Tuning educational structures in Europe* para facilitar e impulsar la construcción del espacio europeo de enseñanza superior previsto en los acuerdos de Bolonia y Praga.

En dicho programa se seleccionaron cinco titulaciones y para cada una se constituyó una red con universidades de los distintos países de la UE. El proyecto finalizó el 31 de mayo de 2002. Entre sus objetivos estaban:

- El diseño de los contenidos básicos de cada titulación (*core* curriculum) y su perfil profesional.
- El incremento de la transparencia, mediante las herramientas del proceso de Bolonia y la presentación de ejemplos de "buena práctica"
- El análisis y la asignación de créditos europeos (ECTS) a las distintas materias.
- El desarrollo de métodos para el análisis de los elementos comunes y de los diferenciadores en los curricula de las titulaciones del proyecto.

Las titulaciones seleccionadas y las universidades españolas que se incorporaron a cada una de ellas fueron:
Administración de Empresas (Universidad de Salamanca)
Geología (Universidad de Barcelona)
Historia (Universidad de Valencia)
Matemáticas (Universidad Autónoma de Madrid, Universidad de Cantabria)
Ciencias de la Educación (Universidad de Deusto)

En el anexo 1 se encuentran las conclusiones de la red de matemáticas

La Conferencia de Rectores de Universidades Españolas (CRUE) a través de su sectorial académica (CASUE) decidió, en su reunión de Barcelona de octubre de 2001, utilizar esta experiencia para desarrollar proyectos piloto para la implantación del Suplemento Europeo al Título, una de las herramientas fundamentales del proceso de Bolonia. Se decidió partir de las cinco titulaciones del programa *Tuning educational*

structures in Europe más las tres siguientes por haber desarrollado experiencias piloto previas:
Química (Universidad Complutense, área de sinergia del *Tuning*)
Turismo (Universidad de Barcelona)
Lingüística (Universidad de Cádiz)

En estas ocho titulaciones se decidió incorporar a todas las universidades españolas que quisiesen participar en el proyecto, formando 8 redes coordinadas por las universidades ya mencionadas.

El Consejo de Universidades (actual Consejo de Coordinación Universitaria), representado en la reunión por Dña Teresa Díez Iturrioz, mostró su disposición a colaborar en el desarrollo del proyecto en aquellos aspectos de la competencia del Consejo.

El trabajo se desarrolló mediante reuniones internas de cada grupo por disciplina y reuniones interdisciplinares de coordinación.

Universidades participantes en el grupo de matemáticas
Universidad Autónoma de Barcelona
Universidad de Santiago de Compostela
Universidad de Sevilla
Universidad de Cantabria (co-cordinadora)
Universidad Autónoma de Madrid (co-coordinadora)

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Objetivos del grupo de matemáticas

- Analizar y completar los campos del Suplemento Europeo al Título actual de licenciado en matemáticas en las 5 universidades en versiones en español e inglés.
  (Propuesta final en el anexo 2)
- Realizar un estudio sobre la valoración y métodos de asignación de créditos europeos a las distintas materias del curriculum de matemáticas actual.
  (Informe en el anexo 3)
- Diseñar una propuesta para debate sobre la estructura grado/postgrado/doctorado y sus objetivos en el caso de los estudios de matemáticas.
  (Esquema en el anexo 4)

- A la luz de las conclusiones del proyecto europeo *Tuning educational structures in Europe*, profundizar en los contenidos básicos del grado de licenciado en matemáticas describiendo, por materias, los objetivos de aprendizaje, los contenidos mínimos y las habilidades o destrezas a exigir.
  (Propuesta en el anexo 5)

Anexos

- Documento final del grupo de matemáticas del proyecto europeo *Tuning educational structures in Europe.*
- Documento completo de la propuesta para el Suplemento Europeo al Título actual de matemáticas con ejemplos de las 5 universidades.
- Informe y datos obtenidos respecto a la valoración de créditos europeos (ECTS).
- Esquema general de estructura de los estudios.
- Propuesta de contenidos básicos por materias y destrezas a adquirir para la obtención del grado de licenciado en matemáticas: formación generalista + tres perfiles profesionales.
- Ejemplos de posibles postgrados (másters).

Anexo 1

Documento final del grupo de matemáticas del proyecto europeo *Tuning educational structures in Europe*

(versiones en inglés y castellano)

Towards a common framework for Mathematics degrees in Europe

THE MATHEMATICS TUNING GROUP[1]

In the wake of the *Bologna Declaration* [B], signed in 1999 by Ministers responsible for Higher Education from 29 European countries, and its follow up, the *Prague Communiqué* [P], a group of universities established the project "Tuning educational structures in Europe" [T1, T2]. It was co-ordinated by the Universities of Deusto and Groningen and benefited from the financial support of the European Union. As its name suggests, the main objective of the project was to study how to "tune" (*not* to make uniform) educational structures in Europe, and thereby aid the development of the European Higher Education Area. This in turn should help mobility and improve the employability of European graduates.

Mathematics was one of the areas included in Tuning, and this paper reflects the unanimous consensus of the mathematics group of the project. But since the group does not pretend to have any representative role, we think it is necessary to make this document available for comment to the wider community of European mathematicians. We believe that any kind of action along the lines we sketch will only be possible and fruitful when a broad agreement has been reached. Indeed any mathematician member of the group welcomes comments on the document. E-mail addresses appear at the end.

The Mathematics Tuning Group is happy to express its thanks to the co-ordinators of the Tuning Project, Julia González (Universidad de Deusto) and Robert Wagenaar (Rijksuniversiteit Groningen), as well as to the European Commission, for creating the conditions for fruitful and pleasant interactions between its members.

Summary

- This paper refers only to universities (including technical universities), and none of our proposals apply to other types of institutions.
- The aim of a "common framework for mathematics degrees in Europe" is to facilitate an automatic recognition of degrees in order to help mobility.
- The idea of a common framework must be combined with an accreditation system.
- The two components of a common framework are similar (although not necessarily identical) structures and a basic common core curriculum (allowing for some degree of local flexibility) for the first two or three years.
- Beyond the basic common core curriculum, and certainly in the second cycle, programmes could diverge significantly. Since there are many areas in mathematics, and many of them are linked to other fields of knowledge, flexibility is of the utmost importance.
- Common ground for all programmes will include calculus in one and several real variables and linear algebra.

[1] Group members are listed at the end of the paper.

- We propose a broad list of further areas that graduates should be acquainted with in order to be easily recognised as mathematicians. It is not proposed that all programmes include individual modules covering each of these areas.
- We do not present a prescriptive list of topics to be covered, but we do mention the three skills we consider may be expected of any mathematics graduate:
- the ability to conceive a proof,
- the ability to model a situation mathematically,
- the ability to solve problems using mathematical tools.
- The first cycle should normally allow time to learn some computing and to meet at least one major area of application of mathematics.
- We should aim for a wide variety of flavours in second cycle programmes in mathematics. Their unifying characteristic feature should be the requirement that all students carry out a significant amount of individual work. To do this, a minimum of 90 ECTS credits[2] seems necessary for the award of a Master's qualification.
- It might be acceptable that various non-identical systems coexist, but large deviations from the standard (in terms of core curriculum or cycle structure) need to be grounded in appropriate entry level requirements, or other program specific factors, which can be judged by external accreditation. Otherwise, such degrees risk not benefiting from the automatic European recognition provided by a common framework, even though they may constitute worthy higher education programmes.

1. A common framework: what it should and shouldn't be or do

1.1 The only possible aim in agreeing a "common European framework" should be to facilitate the automatic recognition of mathematics degrees in Europe in order to help mobility. By this we mean that when somebody with a degree in mathematics from country A goes to country B:

- He/she will be legally recognised as holding such a degree, and the Government of country B will not require further proof of competence.

- A potential employer in country B will be able to assume that he/she has the general knowledge expected from somebody with a mathematics degree.

Of course, neither of these guarantees employment: the mathematics graduate will still have to go through whatever procedures (competitive exams, interviews, analysis of his/her curriculum, value of the degree awarding institution in the eyes of the employer,...) are used in country B to obtain either private or public employment.

1.2 One important component of a common framework for mathematics degrees in Europe is that all programmes have similar, although not necessarily identical, structures. Another component is agreeing on a basic common core curriculum while allowing for some degree of local flexibility.

[2] ECTS stands for "European Credit Transfer System". ECTS credits measure the learning outcomes attained by students. The basic general assumption is that the learning outcomes that an average full time student is expected to attain in one academic year are worth 60 ECTS credits. Therefore, the workload required to get 60 ECTS credits should correspond to what an average full time student is expected to do in one academic year.

1.3 We should emphasise that by no means do we think that agreeing on any kind of common framework can be used as a tool for automatic transfer between Universities. These will always require consideration by case, since different programmes can bring students to adequate levels in different but coherent ways, but an inappropriate mixing of programmes may not.

1.4 In many European countries there exist higher education institutions that differ from universities both in the level they demand from students and in their general approach to teaching and learning. In fact, in order not to exclude a substantial number of students from higher education, it is essential that these differences be maintained. We want to make explicit that **this paper refers only to universities (including technical universities)**, and that any proposal of a common framework designed for universities would not necessarily apply to other types of institutions.

2. Towards a common core mathematics curriculum

2.1 General remarks

At first sight, mathematics seems to be well suited for the definition of a core curriculum, especially so in the first two or three years. Because of the very nature of mathematics, and its logical structure, there will be a common part in all mathematics programmes, consisting of the fundamental notions. On the other hand, there are many areas in mathematics, and many of them are linked to other fields of knowledge (computer science, physics, engineering, economics, etc.). Flexibility is of the utmost importance to keep this variety and the interrelations that enrich our science.

There could possibly be an agreement on a list of subjects that must absolutely be included (linear algebra, calculus/analysis) or that should be included (probability/statistics, some familiarity with the mathematical use of a computer) in any mathematics degree. In the case of some specialised courses, such as mathematical physics, there will certainly be variations between countries and even between universities within one country, without implying any difference of quality of the programmes.

Moreover, a large variety of mathematics programmes exist currently in Europe. Their entry requirements vary, as do their length and the demands on the student. It is extremely important that this variety be maintained, both for the efficiency of the education system and socially, to accommodate the possibilities of more potential students. To fix a single definition of contents, skills and level for the whole of European higher education would exclude many students from the system, and would, in general, be counterproductive.

In fact, the group is in complete agreement that programmes could diverge significantly beyond the basic common core curriculum (e.g. in the direction of "pure" mathematics, or probability - statistics applied to economy or finance, or mathematical physics, or the teaching of mathematics in secondary schools). The presentation and level of rigour, as well as accepting there is and must continue to be variation in emphasis and, to some extent, content, even within the first two or three years, will make all those programmes recognisable as valid mathematics programmes.

As for the second cycle, not only do we think that programmes could differ, but we are convinced that, to reflect the diversity of mathematics and its relations with other fields, all kinds of different second cycles in mathematics should be developed, using in particular the specific strengths of each institution.

2.2 The need for accreditation

The idea of a basic core curriculum must be combined with an accreditation system. If the aim is to recognise that a given program fulfils the requirement of the core curriculum, then one has to check on three aspects:

- a list of contents
- a list of skills
- the level of mastery of concepts

These cannot be reduced to a simple scale.

To give accreditation to a mathematics programme, an examination by a group of peer reviewers, mostly mathematicians, is considered essential. The key aspects to be evaluated should be:

- the programme as a whole
- the units in the programme (both the contents and the level)
- the entry requirements
- the learning outcomes (skills and level attained)
- a qualitative assessment by both graduates and employers

The group does not believe that a (heavy) system of European accreditation is needed, but that universities in their quest for recognition will act at the national level. For this recognition to acquire international standing, the presence on the review panel of mathematicians from other countries seems necessary.

3. Some principles for a common core curriculum for the first degree (Bachelor) in mathematics

We do not feel that fixing a detailed list of topics to be covered is necessary, or even convenient. However, we do think that it is possible to give some guidelines for the common content of a "European first degree in mathematics", and more important, for the skills that all graduates should develop.

3.1 Contents

3.1.1 All mathematics graduates will have knowledge and understanding of, and the ability to use, mathematical methods and techniques appropriate to their programme. Common ground for all programmes will include

- calculus in one and several real variables
- linear algebra.

3.1.2 Mathematics graduates must have knowledge of the basic areas of mathematics, not only those that have historically driven mathematical activity, but also others of more modern origin. Therefore graduates should normally be acquainted with most, and preferably all, of the following:

- basic differential equations
- basic complex functions
- some probability
- some statistics
- some numerical methods
- basic geometry of curves and surfaces
- some algebraic structures
- some discrete mathematics

These need not be learned in individual modules covering each subject in depth from an abstract point of view. For example, one could learn about groups in a course on (abstract) group theory or in the framework of a course on cryptography. Geometric ideas, given their central role, could appear in a variety of courses.

3.1.3 Other methods and techniques will be developed according to the requirements and character of the programme, which will also largely determine the levels to which the developments are taken. In any case, all programmes should include a substantial number of courses with mathematical content.

3.1.4 In fact, broadly two kinds of mathematics curricula currently coexist in Europe, and both are useful. Let us call them, following [QAA][3], "theory based" and "practice based" programmes. The weight of each of the two kinds of programmes varies widely depending on the country, and it might be interesting to find out whether most European university programmes of mathematics are "theory based" or not.

Graduates from theory-based programmes will have knowledge and understanding of results from a range of major areas of mathematics. Examples of possible areas are algebra, analysis, geometry, number theory, differential equations, mechanics, probability theory and statistics, but there are many others. This knowledge and understanding will support the knowledge and understanding of mathematical methods and techniques, by providing a firmly developed mathematical context.

Graduates from practise-based programmes will also have knowledge of results from a range of areas of mathematics, but the knowledge will commonly be designed to support the understanding of models and how and when they can be applied. Besides those mentioned above, these areas include numerical analysis, control theory, operations research, discrete mathematics, game theory and many more. (These areas may of course also be studied in theory-based programmes.)

3.1.5 It is necessary that all graduates will have met at least one major area of application of mathematics in which it is used in a serious manner and this is considered essential for a proper appreciation of the subject. The nature of the application area and the manner in which

[3] This document was considered extremely useful and met with unanimous agreement from the group. In fact we have quoted it almost verbatim at some points.

it is studied might vary depending on whether the programme is theory-based or practice-based. Possible areas of application include physics, astronomy, chemistry, biology, engineering, computer science, information and communication technology, economics, accountancy, actuarial science, finance and many others.

3.2 Skills

3.2.1 For a standard notion like integration in one variable, the same "content" could imply:

- computing simple integrals
- understanding the definition of the Riemann integral
- proving the existence and properties of the Riemann integral for classes of functions
- using integrals to model and solve problems of various sciences.

So, on one hand the contents must be clearly spelled out, and on the other various skills are developed by the study of the subject.

3.2.2 Students who graduate from programmes in mathematics have an extremely wide choice of career available to them. Employers greatly value the intellectual ability and rigour and the skills in reasoning that these students will have acquired, their firmly established numeracy, and the analytic approach to problem-solving that is their hallmark.

Therefore, the three key skills that we consider may be expected of any mathematics graduate are:

1. the ability to conceive a proof,
2. the ability to model a situation mathematically,
3. the ability to solve problems using mathematical tools.

It is clear that, nowadays, solving problems should include their numerical and computational resolution. This requires a sound knowledge of algorithms and programming and the use of available software.

3.2.3 Note also that skills and level are developed progressively through the practice of many subjects. We do not start a mathematics programme with one course called "how to make a proof" and one called "how to model a situation", with the idea that those skills will be acquired immediately. Instead, it is through practice in all courses that these develop.

3.3 Level

All graduates will have knowledge and understanding developed to higher levels in particular areas. The higher-level content of programmes will reflect the title of the programme. For example, graduates from programmes with titles involving statistics will have substantial knowledge and understanding of the essential theory of statistical inference and of many applications of statistics. Programmes with titles such as mathematics might range quite widely over several branches of the subject, but nevertheless graduates from such programmes will have treated some topics in depth.

4. The second degree (Master) in mathematics

We have already made explicit our belief that establishing any kind of common curriculum for second cycle studies would be a mistake. Because of the diversity of mathematics, the different programmes should be directed to a broad range of students, including in many cases those whose first degree is not in mathematics, but in more or less related fields (computer science, physics, engineering, economics, etc.). We should therefore aim for a wide variety of flavours in second cycle programmes.

Rather than the contents, we think that the common denominator of all second cycles should be the level of achievement expected from students. A unifying characteristic feature could be the requirement that all second cycle students carry out a significant amount of individual work. This could be reflected in the presentation of a substantial individual project.

We believe that, to be able to do real individual work in mathematics, the time required to obtain a Master's qualification should be the equivalent of at least 90 ECTS credits. Therefore, depending on the national structure of first and second cycles, a Master would typically vary between 90 and 120 ECTS credits.

5. A common framework and the Bologna agreement

5.1 How various countries implement the Bologna agreement will make a difference on core curricula. In particular, 3+2 may not be equivalent to 5, because, in a 3+2 years structure, the 3 years could lead to a professional diploma, meaning that less time is spent on fundamental notions, or to a supplementary 2 years, and in that case the whole spirit of the 3 years programme should be different.

5.2 Whether it will be better for mathematics studies to consist of a 180 ECTS Bachelor, followed by a 120 ECTS Master (a 3+2 structure in terms of academic years), or whether a 240+90 (4+1+project) structure is preferable, may depend on a number of circumstances. For example, a 3+2 break up will surely facilitate crossing between fields, where students pursue Masters in an area different from that in which they obtained their Bachelor degree.

One aspect that can not be ignored, at least in mathematics, is the training of secondary school teachers. If the pedagogical qualification must be obtained during the first cycle studies, these should probably last for 4 years. On the other hand, if secondary school teaching requires a Master (or some other kind of postgraduate qualification), a 3 years Bachelor may be adequate, with teacher training being one of the possible postgraduate options (at the Master's level or otherwise).

5.3 The group did not attempt to solve contradictions that could appear in the case of different implementations of the Bologna agreement (i.e. if three years and five years university programmes coexist; or different cycle structures are established: 3+1, 3+2, 4+1, 4+1+project, 4+2 have all been proposed). As we said before, it might be acceptable that various systems coexist, but we believe that large deviations from the standard (such as a 3+1 structure, or not following the principles stated in section 3) need to be grounded in appropriate entry level requirements, or other program specific factors, which can be judged by external accreditation. Otherwise, such degrees risk not benefiting from the automatic European recognition provided by a common framework, even though they may constitute worthy higher education programmes.

6. References

[B] http://www.bologna-berlin2003.de/pdf/bologna_declaration.pdf

[P] http://www.bologna-berlin2003.de/pdf/Prague_communiquTheta.pdf

[QAA] The Benchmark document on Mathematics, Statistics and Operational Research, from the UK Quality Assurance Agency for Higher Education, http://www.qaa.ac.uk/crntwork/benchmark/phase2/mathematics.pdf.

[T1] The official sites of the project *Tuning educational structures in Europe*: http://www.relint.deusto.es/TuningProject/index.htm, http://www.let.rug.nl/TuningProject/index.htm

[T2] Information on the project *Tuning educational structures in Europe* in the European Commission site: http://europa.eu.int/comm/education/tuning.html

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Hacia un marco común para los títulos de Matemáticas en Europa

THE MATHEMATICS TUNING GROUP[1]

Tras la firma de la *Declaración de Bolonia* [B] en 1999 por Ministros responsables de la Educación Superior de 29 países europeos, y de su continuación, el *Comunicado de Praga* [P], un grupo de universidades puso en marcha el proyecto "*Tuning educational structures in Europe*" [T1, T2]. Lo han coordinado las universidades de Deusto y Groningen y ha obtenido financiación de la Unión Europea. Como su nombre indica, el objetivo principal del proyecto fue estudiar la forma de "afinar" (como los distintos instrumentos de una orquesta, *no* uniformizar) las estructuras educativas europeas, y colaborar así en la construcción del Espacio Europeo de la Educación Superior. Esto debería a su vez contribuir a la movilidad y mejorar las posibilidades laborales de los titulados europeos.

Uno de los campos incluidos en el proyecto *Tuning* fue el de las matemáticas, y este documento refleja el consenso unánime del grupo de matemáticas del proyecto. Pero dado que el grupo no pretende tener ningún papel representativo, consideramos necesario someterlo a discusión entre la comunidad de matemáticos europeos. Estamos convencidos de que cualquier clase de acción en las direcciones que aquí señalamos solamente será posible y fructífera cuando se haya alcanzado un amplio acuerdo. Por supuesto, todos los matemáticos pertenecientes al grupo recibirán gustosos cualquier comentario sobre este documento. Sus direcciones electrónicas aparecen al final.

El Grupo *Tuning* de Matemáticas quiere mostrar su agradecimiento a los coordinadores del proyecto *Tuning*, Julia González (Universidad de Deusto) y Robert Wagenaar (Rijksuniversiteit Groningen), y a la Comisión Europea por crear las condiciones que permitieron una agradable y provechosa comunicación entre sus miembros.

Resumen

- Este documento se refiere únicamente a las universidades (incluyendo las politécnicas), y ninguna de nuestras propuestas se aplica a otros tipos de instituciones de educación superior.
- La finalidad de disponer de un "marco común para los títulos de matemáticas en Europa" es la de facilitar un reconocimiento automático, que contribuya a la movilidad.
- La idea de un marco común debe ir ligada a la de un sistema de acreditación.
- Las dos componentes de un marco común son unas estructuras similares (aunque no necesariamente idénticas) y una parte troncal, básica y común, en los contenidos de los dos o tres primeros años del plan de estudios (permitiendo cierto grado de flexibilidad local).
- Más allá de la parte básica y troncal del plan de estudios, y sin duda en todo el segundo ciclo, los planes podrían diverger de modo significativo. Puesto que hay muchas áreas en matemáticas, y están enlazadas con otros campos del conocimiento, la flexibilidad es de la máxima importancia.

[1] Ver relación de miembros del grupo al final del documento.

- La base común de todos los planes de estudios incluirá el cálculo en una y varias variables reales y el álgebra lineal.
- Proponemos una amplia lista de otras materias que nuestros graduados deberían conocer para ser inmediatamente reconocidos como matemáticos. No se propone que todos los planes incluyan asignaturas específicas que se dediquen a cada uno de estos temas.
- No presentamos una lista obligatoria de temas que haya que estudiar, pero sí que mencionamos tres destrezas que cualquier graduado en matemáticas debería poseer:
  (a) la capacidad de idear demostraciones,
  (b) la capacidad de modelizar matemáticamente una situación,
  (c) la capacidad de resolver problemas con técnicas matemáticas.
- El primer ciclo normalmente debería incluir el aprendizaje de algo de computación y la adquisición del conocimiento de al menos uno de los más importantes campos de aplicación de las matemáticas.
- Se debería procurar que los segundos ciclos de matemáticas fueran de muy diversa índole. Su característica común debería ser que todos los estudiantes lleven a cabo una apreciable cantidad de trabajo individual. Para conseguir esto, parece necesario un mínimo de 90 créditos ECTS[2] para obtener un título de *Master*.
- Puede ser aceptable que coexistan titulaciones con diversos diseños, pero en el caso de que se den desviaciones significativas del estándar (en lo relativo a los contenidos mínimos o a la estructura cíclica), éstas han de estar fundamentadas en unos requisitos de ingreso adecuados o en otros factores específicos del plan que puedan ser juzgados en la acreditación externa. De otro modo, tales títulos corren el riesgo de no beneficiarse del reconocimiento automático europeo que dará el marco común, aunque puedan constituir títulos válidos de educación superior.

1. Un marco común: lo que significa y lo que no significa.

- El único objetivo posible de acordar un “marco común europeo” debería ser el de facilitar un reconocimiento automático de los títulos de matemáticas en Europa para contribuir a la movilidad. Esto significaría que cuando una persona con un título en matemáticas obtenido en un país A se traslada a un país B:

1. Se le reconocerá oficialmente el título, y para ello las autoridades del país B no le exigirán ninguna otra prueba de su capacidad.

2. Quienquiera que vaya a contratarle en el país B podrá suponer que el poseedor del título tiene los conocimientos generales que se esperan de alguien con un título en matemáticas.

Naturalmente, ninguna de estas facilidades garantizará la obtención de un empleo: el titulado en matemáticas tendrá que pasar por cualesquiera procedimientos (oposiciones, entrevistas, análisis de su currículum vitae, valoración por parte del empresario de la universidad en la que obtuvo el título,...) que se utilicen en el país B para obtener un empleo, ya sea público o privado.

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[2] ECTS son las siglas de *European Credit Transfer System*. Los créditos ECTS se utilizan para medir el aprendizaje de los alumnos. Por definición, los resultados del aprendizaje que se espera que un alumno medio a tiempo completo pueda obtener en un año académico, valen 60 créditos ECTS. En consecuencia, la carga de trabajo que se precisa para obtener 60 créditos ECTS debería corresponder a lo que se espera que un estudiante medio a tiempo completo realice durante un año académico.

1.2 Una componente importante del marco común de los títulos europeos de matemáticas es que todos los planes tengan estructuras similares, aunque no necesariamente idénticas. Otra componente es un acuerdo sobre una parte troncal, básica y común del contenido del plan que permita cierto grado de flexibilidad local.

1.3 Queremos insistir en que de ningún modo pensamos que un acuerdo sobre un marco común pueda usarse como un instrumento para los traslados automáticos entre universidades. Los traslados deberán considerarse caso a caso, puesto que diferentes planes de estudios pueden llevar a los estudiantes hasta los mismos niveles de formas diferentes pero todas ellas coherentes, mientras que una mezcla inadecuada de varios planes puede no servir para el mismo fin.

1.4 En muchos países europeos existen instituciones de educación superior que difieren de las universidades tanto en el nivel que exigen a sus estudiantes como en su enfoque general de la enseñanza y el aprendizaje. Para no excluir de la enseñanza superior a un número importante de estudiantes, en la práctica es esencial mantener estas diferencias. Queremos declarar expresamente que **este documento se refiere solamente a las universidades (incluyendo las politécnicas)**, y que cualquier propuesta de un marco común diseñado para las universidades no sería automáticamente aplicable a instituciones de otro tipo.

2. Hacia una troncalidad común

2.1. Consideraciones generales

A primera vista, las matemáticas parecen idóneas para la definición de unos contenidos comunes, por ejemplo, para los dos o tres primeros años. Por la naturaleza misma de las matemáticas, y por su estructura lógica, habrá una parte común a todos los planes de estudios de matemáticas, que constará de las nociones fundamentales. Pero por otra parte, existen muchas áreas de las matemáticas, y muchas de ellas están relacionadas con otros campos del conocimiento (informática, física, ingeniería, economía, etc.). La flexibilidad es de la máxima importancia para preservar esta variedad y las interrelaciones que enriquecen nuestra ciencia.

Podría alcanzarse un acuerdo sobre una lista de materias que con toda seguridad deben estar incluidas (álgebra lineal, cálculo/análisis) o que debieran estar incluidas (probabilidad/estadística, cierta familiaridad con la utilización matemática de un ordenador) en cualquier título de matemáticas. En el caso de algunos temas especializados, como física matemática, sin duda habrá variaciones entre países e incluso entre universidades del mismo país, sin que deba deducirse ninguna diferencia de calidad entre los distintos planes de estudios.

Por otra parte, actualmente existen en Europa planes de estudios de matemáticas muy variados, con diferentes requisitos de acceso y con distintas duraciones de las enseñanzas y distintos niveles de exigencia sobre los estudiantes. Es enormemente importante que se mantenga esta variedad, tanto para la eficiencia del sistema educativo como desde el punto de vista social, con objeto de conseguir atender a las demandas del mayor número posible de alumnos potenciales. La fijación de una única definición de los contenidos, las destrezas y los niveles para la totalidad de la educación superior europea excluiría del sistema a muchos estudiantes y, en conjunto, resultaría contraproducente.

De hecho en el grupo hay un acuerdo total acerca de que los planes puedan diverger de modo significativo en lo que sea adicional a la parte troncal básica (por ejemplo en la dirección de la matemática “pura”; o de la probabilidad-estadística aplicada a la economía o a las finazas; o de la física matemática; o de la enseñanza de las matemáticas en la educación secundaria). Lo que hará que esos planes sean reconocibles como planes válidos de matemáticas será su forma de presentación y su nivel de rigor, admitiendo que hay y debe seguir habiendo variantes en la importancia que se dé a cada tema y, hasta cierto punto, en el contenido, incluso dentro de los dos o tres primeros años.

En cuanto al segundo ciclo, no sólo pensamos que los distintos planes pueden diferir, sino que estamos convencidos de que, para reflejar la diversidad de las matemáticas y de sus relaciones con otros campos, se deberían desarrollar en las diferentes instituciones todo tipo de segundos ciclos diferentes en matemáticas, aprovechando en particular los aspectos en los que más destaque cada institución.

2.2 La necesidad de la acreditación

La idea de una troncalidad básica debe combinarse con un sistema de acreditación. Con el objetivo de reconocer que un programa dado cumple con los requisitos de la troncalidad, hay que comprobar tres aspectos:

- una lista de contenidos
- una lista de destrezas o competencias
- el nivel del dominio de los conceptos

No es posible reducir estos aspectos a una simple escala.
Para conceder la acreditación a un plan de matemáticas es imprescindible un análisis por parte de un grupo de evaluadores académicos, de los cuales la mayor parte serán matemáticos. Los aspectos clave a ser evaluados deberían ser:

(a) el plan de estudios en su conjunto
(b) las unidades o asignaturas (tanto en contenido como en nivel)
(c) los requisitos de acceso al plan
(d) los objetivos del aprendizaje (las destrezas y el nivel alcanzado)
(e) una evaluación cualitativa tanto por los graduados como por quienes les contratan.

El grupo no cree que se necesite un (elaborado) sistema de acreditación europeo, sino que las universidades, buscando el reconocimiento, actuarán a nivel nacional. Para que este reconocimiento tenga valor internacional, parece necesario que entre los evaluadores se incluyan matemáticos de otros países.

3. Algunos principios para la troncalidad común del primer título (Bachelor) en matemáticas

No creemos que sea necesario, ni siquiera oportuno, fijar una lista detallada de los temas a cubrir. Sin embargo, creemos que es posible dar algunas directrices sobre el contenido común de un “primer título europeo en matemáticas”, y, lo que es más importante, sobre las destrezas que todos los titulados deberían poseer.

3.1 Contenido

3.1.1 Todos los titulados en matemáticas conocerán y entenderán, y serán capaces de usar, los métodos y las técnicas apropiados a su plan de estudios. La parte común de todos los planes incluirá

- cálculo en una y varias variables reales
- álgebra lineal.

3.1.2 Los titulados en matemáticas han de conocer las áreas básicas de las matemáticas, no solo las que históricamente han guiado la actividad matemática, sino también otras de origen más moderno. En consecuencia los titulados normalmente habrán de conocer la mayoría de las siguientes materias, y preferiblemente todas:

- ecuaciones diferenciales a nivel básico
- funciones de variable compleja a nivel básico
- algo de probabilidad
- algo de estadística
- algo de métodos numéricos
- geometría de curvas y superficies a nivel básico
- algunas estructuras algebraicas
- algo de matemáticas discretas.

No es necesario que estos temas se aprendan en asignaturas o módulos individuales que cubran en profundidad y desde un punto de vista abstracto cada materia. Por ejemplo, un estudiante podría aprender sobre los grupos en un curso de teoría de grupos (abstracta) o en el marco de un curso sobre criptografía. Las ideas geométricas podrían aparecer en varias asignaturas, dado su papel central.

3.1.3 De acuerdo con el carácter y las exigencias del plan de estudios, se desarrollarán otros métodos y otras técnicas, cuyos niveles serán definidos por el propio plan. En cualquier caso, todos los planes incluirán un número importante de asignaturas con contenido matemático.

3.1.4 En la práctica y hablando en términos algo imprecisos, hay dos tipos de estudios de matemáticas que coexisten actualmente en Europa, y ambos tipos de estudios son útiles. Podemos llamarlos, siguiendo [QAA][3], “basados en la teoría” y “basados en la práctica”. La incidencia de cada uno de estos dos tipos de enseñanzas varía ampliamente según el país, y podría ser interesante averiguar si la mayor parte de los estudios universitarios europeos de matemáticas son “basados en la teoría” o no.
Los graduados en planes de estudios basados en la teoría tendrán conocimiento y comprensión de los resultados de varios de los campos más importantes de las matemáticas. Ejemplos de tales campos son el álgebra, el análisis, la geometría, la teoría de números, las ecuaciones diferenciales, la mecánica, la teoría de la probabilidad y la estadística, pero hay otros muchos. Sobre este conocimiento y esta comprensión se apoyarán el conocimiento y la comprensión de los métodos y técnicas matemáticos, otorgándoles un contexto matemático bien fundamentado.

Los graduados en planes de estudios basados en la práctica también tendrán conocimiento de los resultados de varios campos matemáticos, pero este conocimiento normalmente estará

[3] El grupo consideró enormemente útil este documento, y mostró un acuerdo unánime con su contenido. Incluso se han utilizado al pie de la letra algunas de sus frases.

diseñado para apoyar la comprensión de modelos y de cómo pueden aplicarse. Además de los mencionados más arriba, estos campos incluyen el análisis numérico, la teoría de control, la investigación operativa, las matemáticas discretas, la teoría de juegos y muchos otros. (Naturalmente estos campos también pueden estudiarse en las enseñanzas más teóricas.)

3.1.5 Es necesario que todos los titulados conozcan al menos una de las más importantes áreas de aplicación de las matemáticas, en la que el uso de las matemáticas sea esencial para entender verdaderamente la materia. La naturaleza y la forma en que se estudia esta área de aplicación puede variar en función de si el plan de estudios está basado en la teoría o en la práctica. Algunas de las posibles áreas de aplicación pueden ser la física, la astronomía, la química, la biología, la ingeniería, la computación, la tecnología de la información y las comunicaciones, la economía, la contabilidad, las ciencias actuariales, las finanzas y muchas otras.

3.2 Destrezas

3.2.1 Para un concepto como la integración en una variable, el mismo "contenido" podría significar:

- calcular integrales sencillas
- comprender la definición de la integral de Riemann
- conocer las demostraciones de la existencia y de las propiedades de la integral de Riemann para ciertas clases de funciones
- usar las integrales para modelizar y resolver problemas en diversas ciencias.

Concluimos que por una parte el contenido ha de ser detallado claramente, y que por otra mediante el estudio de una misma materia se desarrollan varias destrezas.

3.2.2 Los estudiantes que se gradúan en matemáticas disponen de una amplia variedad de posibilidades de empleo. Los empresarios valoran en alto grado la capacidad y el rigor intelectual, y las habilidades de razonamiento que estos estudiantes han adquirido, así como sus demostradas capacidades numéricas y el enfoque analítico a la solución de problemas que constituyen sus cualidades más distintivas.
Por tanto, las tres destrezas clave que consideramos que cualquier titulado en matemáticas debería adquirir son:

(a) la capacidad para idear demostraciones
(b) la capacidad para modelizar matemáticamente una situación
(c) la capacidad para resolver problemas con técnicas matemáticas.

Hoy en día está claro que resolver un problema debe incluir su resolución numérica y computacional. Para esto se requiere un firme conocimiento de algoritmos y de programación, así como del uso del software actualmente existente.

3.2.3 Conviene resaltar también que estas destrezas y el nivel de las mismas se desarrollan de forma progresiva a través de la práctica de varias materias. No se empiezan los estudios de matemáticas con una asignatura llamada "cómo hacer una demostración" y con otra llamada "cómo modelizar una situación" con la idea de que estas destrezas se adquieran inmediatamente, sino que se desarrollan practicándolas en todas las asignaturas.

3.3 Nivel

Todos los graduados habrán desarrollado el conocimiento y la comprensión a un alto nivel en algún área en particular. El nombre de los estudios o del título reflejará su contenido de materias a alto nivel. Por ejemplo, los poseedores de títulos que incluyan “estadística” tendrán un conocimiento y una comprensión sustanciales de la teoría central de la inferencia estadística y de muchas aplicaciones de la estadística. Quienes posean un título en “matemáticas” pueden tener conocimientos de muy distintas partes de las matemáticas, pero en todo caso habrán tratado en profundidad algunos temas.

4. El segundo título (Master) en matemáticas

Ya hemos dejado claro nuestro convencimiento de que sería un error establecer cualquier clase de currículum troncal para los estudios de segundo ciclo. Dada la diversidad de las matemáticas, los diferentes planes deberían dirigirse a una amplia gama de estudiantes, incluyendo muchos cuyo primer título no sea en matemáticas sino en otros campos más o menos relacionados (informática, física, ingeniería, economía, etc.). En consecuencia se debería procurar que los segundos ciclos de matemáticas fueran de muy diversa índole.

Pensamos que el denominador común de todos los segundos ciclos debería residir, más que en el contenido, en el nivel que se espera que los alumnos alcancen. Una característica unificadora podría ser el requisito de que todos los estudiantes de segundo ciclo lleven a cabo una apreciable cantidad de trabajo individual, lo que se podría plasmar en la presentación de un proyecto individual de cierta consideración.

Creemos que, en orden a lograr el nivel necesario para realizar un verdadero trabajo individual en matemáticas, el tiempo requerido para obtener un título de *Master* debería ser al menos el equivalente de 90 créditos ECTS. Por tanto el número de créditos ECTS de un *Master* estará comprendido normalmente entre 90 y 120, dependiendo de cuál sea la duración de cada uno de los dos ciclos en los distintos países.

5. Un marco europeo y el acuerdo de Bolonia

5.1 La forma en que los diferentes países implementen el acuerdo de Bolonia tendrá trascendencia sobre la troncalidad común. En particular, 3+2 puede no ser equivalente a 5, porque en una estructura de 3+2 años los 3 primeros años podrían conducir a un título profesional, lo que significaría que se invierte menos tiempo en las nociones fundamentales, o podrían conducir a los 2 años siguientes, en cuyo caso el espíritu del plan de estudios de los 3 años sería diferente.

5.2 Si es mejor que los estudios de matemáticas estén formados por un *Bachelor* de 180 créditos ECTS seguidos por un *Master* de 120 créditos ECTS (es decir, una estructura 3+2, en términos de años académicos), o si por el contrario es preferible una estructura 240+90 (es decir, 4+1+proyecto), dependerá de varias circunstancias. Por ejemplo, una estructura 3+2 seguramente facilitará la movilidad entre materias para estudiantes que decidan seguir un *Master* en un área distinta de aquella en la obtuvieron su *Bachelor*.

Un aspecto que no se puede ignorar, al menos en matemáticas, es la formación de los profesores de enseñanza secundaria. En caso de que la cualificación pedagógica haya de obtenerse durante los estudios de primer ciclo, éstos probablemente deberían durar 4 años. Pero si el ser profesor de enseñanza secundaria exige un *Master* (o algún otro tipo de cualificación postgraduada), entonces un *Bachelor* de 3 años puede ser adecuado, y en este caso la formación pedagógica sería una de las posibles opciones de postgrado (a nivel de *Master* o a otro nivel).

5.3 El grupo no ha intentado resolver las contradicciones que podrían aparecer en el caso de que haya diferentes implementaciones del acuerdo de Bolonia (es decir, si coexisten planes universitarios de tres años con otros de cinco años; o si se establecen diferentes estructuras cíclicas, ya que se han propuesto todos estos esquemas: 3+1, 3+2, 4+1, 4+1+proyecto, 4+2). Como se ha dicho más arriba, podría ser aceptable que coexistan diversos sistemas, pero creemos que si hay grandes alejamientos del estándar (como la estructura 3+1, o el incumplimiento de los principios enunciados en la sección 3) éstos tienen que estar fundamentados en unos requisitos adecuados sobre los niveles de acceso o en otros factores particulares del plan de estudios, que puedan ser juzgados en la acreditación externa. De otro modo, tales títulos corren el riesgo de no beneficiarse del reconocimiento automático europeo que dará el marco común, aunque puedan constituir títulos válidos de educación superior.

6. Referencias

[B] http://www.bologna-berlin2003.de/pdf/bologna_declaration.pdf

[P] http://www.bologna-berlin2003.de/pdf/Prague_communiquTheta.pdf

[QAA] Documento para la evaluación comparada de los títulos de Matemáticas, Estadística e Investigación Operativa, de la *Quality Assurance Agency for Higher Education* del Reino Unido.

http://www.qaa.ac.uk/crntwork/benchmark/phase2/mathematics.pdf.

[T1] Los sitios oficiales del proyecto *Tuning educational structures in Europe*: http://www.relint.deusto.es/TuningProject/index.htm, http://www.let.rug.nl/TuningProject/index.htm

[T2] Información sobre el proyecto *Tuning educational structures in Europe* en el sitio de la Comisión Europea: http://europa.eu.int/comm/education/tuning.html

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Anexo 2

Documento completo de la propuesta para el Suplemento Europeo al Título actual de matemáticas:

En castellano con ejemplos de las cinco universidades.
En inglés ejemplo de la UAM.

SUPLEMENTO EUROPEO AL TÍTULO

EJEMPLO DE LICENCIADO EN MATEMÁTICAS

1. INFORMACIÓN SOBRE LA IDENTIDAD DEL POSEEDOR DE LA TITULACIÓN

1.1 **Apellido(s):** García Pérez [en la versión informática: un solo campo con espacio en blanco entre los dos apellidos]

1.2 **Nombre(s):** Carmen

1.3 **Fecha de nacimiento *(día/mes/año)*:** 01/01/1979

1.4 **Número de identificación de estudiante o código (*si se conoce*):** [temporalmente se utilizará código Erasmus + DNI, aunque se apoyará el proyecto ScanNet]

E MADRID04 + DNI
E BARCELO02 + DNI
E SANTAND01 + DNI
E SANTIAG01 + DNI
E SEVILLA01 + DNI

2. INFORMACIÓN SOBRE LA TITULACIÓN

2.1 **Denominación de los estudios y (*si procede*) título conferido (*en la lengua original*):**

Estudios: Licenciatura en Matemáticas.
Título conferido: Licenciada en Matemáticas (oficial y válido en todo el territorio nacional).

2.2 **Principal(es) campo(s) de estudio de la titulación:** Matemáticas (Álgebra, Análisis Matemático, Geometría, Estadística, Análisis Numérico).

2.3 **Nombre (*en la lengua original*) y naturaleza de la institución que la concede:**

Universidad Autónoma de Madrid, universidad pública.
Universidad Autónoma de Barcelona, universidad pública.
Universidad de Cantabria, universidad pública.
Universidad de Santiago de Compostela, universidad pública.
Universidad de Sevilla, universidad pública.

2.4 **Nombre (*en la lengua original*) y naturaleza de la institución (*si es diferente de la anterior*) en la que se cursaron los estudios:**

[US] 44,5% de los estudios cursados en la Universidad de Cádiz
55,5% de los estudios cursados en la Universidad de Sevilla

2.5 **Lengua(s) de enseñanza/examen:**

[UAM] español (castellano).
[UAB] catalán y/o castellano.
[UC] español (castellano)
[USC] español (castellano) y/o gallego.
[US] español (castellano)

3. INFORMACIÓN SOBRE EL NIVEL DE LA TITULACIÓN

3.1 **Nivel de la titulación:** Licenciado (ver 8).

3.2 **Duración oficial del programa:**

[UAM] 4 años, pero existe la posibilidad de completarlo en 5 años. En ambos casos el tiempo total presencial con los profesores en clases teóricas, de problemas y prácticas es de 2500 horas en el conjunto de la titulación. El tiempo total estimado de trabajo para el estudiante, incluyendo clases, trabajo individual, exámenes y su preparación, es anualmente de 1500 horas aproximadamente.

[UAB] 5 años, pero existe la posibilidad de completarlo en 4 años. En ambos casos el tiempo total presencial con los profesores en clases teóricas, de problemas y prácticas es de 2850 horas en el conjunto de la titulación. El tiempo total estimado de trabajo para el estudiante, incluyendo exámenes y su preparación, es anualmente de 1400 horas aproximadamente.

[UC] 5 años. El tiempo total de trabajo en clases teóricas, de problemas y prácticas es de 2500 horas (=300 créditos). El tiempo total estimado de trabajo para el estudiante, incluyendo exámenes y su preparación, es de 1500 horas al año aproximadamente

[USC] No se prescribe una duración oficial, pero se recomienda completarlo en 5 años. El tiempo total de trabajo en clases teóricas, de problemas y prácticas es de 3000 horas. El tiempo total estimado de trabajo para el estudiante, incluyendo exámenes y su preparación, es de 1600 horas al año aproximadamente.

[US] 5 años. El tiempo total presencial con los profesores en clases teóricas, de problemas y prácticas es de 3000 horas en el conjunto de la titulación. El tiempo total estimado de trabajo para el estudiante, incluyendo clases, trabajo individual, exámenes y su preparación, es de 1500 horas al año aproximadamente.

3.3 **Requisito(s) de acceso:**
Bachillerato + Prueba de Acceso a la Universidad. Los estudiantes que poseen el título de Diplomado en Estadística pueden acceder directamente al segundo ciclo, cursando ciertos Complementos de Formación.

4. INFORMACIÓN SOBRE EL CONTENIDO Y LOS RESULTADOS OBTENIDOS

4.1 **Forma de estudio:** Programa presencial. Los estudiantes pueden organizar temporalmente su plan de estudios como deseen. [Sería conveniente que se estableciesen criterios para distinguir entre estudiantes a tiempo completo y a tiempo parcial]

4.2 **Requisitos del programa:**
[UAM] El estudiante tiene que completar 2500 horas presenciales con los profesores distribuidas de la siguiente forma:

- 18 asignaturas troncales (1410 horas presenciales),
- 4 asignaturas obligatorias (290 horas presenciales),
- 7 asignaturas optativas (400 horas presenciales),
- asignaturas de libre elección (400 horas presenciales) que el estudiante puede escoger entre todos los cursos ofrecidos por la universidad en cualquier disciplina u otras actividades curriculares.

Cada asignatura debe ser aprobada de forma independiente.

[UAB] El estudiante tiene que completar 2850 horas presenciales con los profesores distribuidas de la siguiente forma:

- 16 asignaturas troncales (1290 horas presenciales),
- 10 asignaturas obligatorias (930 horas presenciales),
- 6/7 asignaturas optativas (330 horas presenciales aproximadamente),
- asignaturas de libre elección (30 0 horas presenciales) que el estudiante puede escoger entre todos los cursos ofrecidos por la universidad en cualquier disciplina u otras actividades curriculares.

Cada asignatura debe ser aprobada de forma independiente.

[UC] El estudiante tiene que superar 21 asignaturas, equivalentes a 5 años de estudio a tiempo completo. Todas las asignaturas de los cuatro primeros años son comunes para todos los alumnos, y en el último año cada estudiante puede optar por 2, a elegir entre una oferta de a lo sumo 5 asignaturas. Cada asignatura debe ser superada individualmente, y para ello se evalúa en una escala de 0 a 10; para superarla hay que obtener al menos un 5.

[USC] El estudiante tiene que completar 3000 horas presenciales con los profesores distribuidas de la siguiente forma:

- 9 asignaturas obligatorias (645 horas),
- 20 asignaturas troncales (1550 horas)
- 505 horas a completar entre 51 asignaturas optativas Estas asignaturas se dividen en 21 asignaturas optativas vinculadas a alguna orientación y 30 asignaturas optativas no vinculadas a orientaciones, 300 horas de asignaturas de libre elección. Previa solicitud del estudiante, podrán imputarse como materias de libre elección curricular las materias optativas de la propia titulación, las materias troncales, obligatorias y optativas de otras titulaciones, las materias diseñadas específicamente para la libre elección y cursos,

seminarios y otras actividades a los que la USC reconozca previamente la posibilidad de ser imputados como de libre elección.

Cada asignatura y cada curso debe ser aprobado individualmente (no hay ningún sistema de compensación anual).

[US] El estudiante tiene que completar 3000 horas presenciales con los profesores distribuidas de la siguiente forma:

- 18 asignaturas troncales (1260 horas presenciales),
- 8 asignaturas obligatorias (570 horas presenciales)
- 13 asignaturas optativas (870 horas presenciales), a elegir entre la oferta de la Facultad,
- Asignaturas de libre elección (300 horas presenciales), que el estudiante puede escoger entre los cursos y actividades ofrecidos por la universidad en cualquier disciplina u otras actividades extracurriculares

Cada asignatura debe ser aprobada de forma independiente.

4.3 **Datos del programa (por ejemplo, módulos o unidades cursados) y especificación de las calificaciones/créditos obtenidos:**
(si esta información figura en una certificación oficial, utilícese en este apartado)

4.3.1 **Asignaturas troncales y obligatorias:**

Primer curso

Asignatura	**Horas presenciales**	**Calificación**	**Año**	**Observaciones**
Cálculo I	120	Convalidada	97-98	Convalidada
Álgebra Lineal	120	Convalidada	97-98	Convalidada
Informática	100	Aprobado	97-98	
Cálculo II	120	Matrícula de Honor	97-98	
Geometría I	120	Aprobado	98-99	
Conjuntos y Números	120	Sobresaliente	97/98	

Segundo curso

Asignatura	**Horas presenciales**	**Calificación**	**Año**	**Observaciones**
Cálculo III	80	Notable	98-99	
Ec. Diferen. Ordinarias	80	Aprobado	99-00	
Probabilidad I	100	Notable	98-99	
Cálculo Numérico I	100	Sobresaliente	98-99	
Geometría II	80	Aprobado	99-00	
Topología	80	Aprobado	98-99	
Modelización I	80	Aprobado	99-00	
Física para Matemáticos	80	Aprobado	99-00	

Tercer curso

Asignatura	Horas presenciales	**Calificación**	**Año**	**Observaciones**
Álgebra I	80	Sobresaliente	00-01	EQ Erasmus
Teo. Integral y Medida	80	Aprobado	00-01	EQ Erasmus
Variable Compleja I	80	Notable	00-01	EQ Erasmus
Álgebra II	80	Aprobado	00-01	EQ Erasmus
Ec. Dif, y Anál. Funcional	80	Aprobado	00-01	EQ Erasmus
Probabilidad II	80	Matrícula de Honor	00-01	EQ Erasmus

Cuarto curso

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Geometría III	90	Sobresaliente	01-02	
Cálculo Numérico II	90	Aprobado	01-02	

4.3.2 Asignaturas optativas:

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Historia de las Matemáticas	80	Notable	99-00	
Matemática Discreta	80	Sobresaliente	00-01	EQ Erasmus
Teoría de Números	80	Notable	00-01	EQ Erasmus
Estadística I	80	Notable	01-02	
Estadística II	80	Notable	01-02	
Modelización II	80	Aprobado	01-02	
Lógica	80	Notable	01-02	

4.3.3 Asignaturas de libre elección:

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Inglés (nivel medio)	50	Sobresaliente	97-98	
Inglés (nivel superior)	70	Notable	98-99	
Bases de Datos	50	Notable	00-01	EQ Erasmus
Derecho de la Empresa	60	Notable	00-01	EQ Erasmus
Prácticas en Empresas	120	Apto	01-02	
Actividades extracurriculares	50	Apto		

4.3.4 Asignaturas cursadas en equivalencia:

En la Universität Würzburg, Alemania (EQ Erasmus)

Funktionentheorie
Algebra
Elementare Zahlentheorie
Funktionalanalysis
Einführung zu JAVA
Deutsch als Fremdsprache (Mittelstufe)
Deutsch als Fremdsprache (Oberstufe)
Deutsch als Fremdsprache (Landeskunde)

4.3.1 Asignaturas troncales y obligatorias

Primer curso

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Álgebra Lineal	75	Notable	99-00	Adaptada
Elementos de Análisis Matemático	75	Aprobado	99-00	
Física General	60	Aprobado	99-00	Adaptada
Informática	90	Notable	99-00	Adaptada
Análisis Matemático I	90	Aprobado	99-00	Adaptada
Cálculo Numérico I	60	Aprobado	99-00	Adaptada
Elementos de Geom. Diferencial y Top.	75	Notable	99-00	Adaptada
Geometría	75	Notable	99-00	Adaptada

Segundo curso

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Ampliación de Geometría	60	Aprobado	99-00	
Análisis Matemático II	60	Aprobado	99-00	Adaptada
Cálculo de Probabilidades	60	Notable	99-00	Adaptada
Cálculo Numérico II	60	Aprobado	99-00	Adaptada
Física Teórica	75	Aprobado	99-00	
Ampl. Teoría de Func.Varias Variabl.	75	Notable	99-00	Adaptada
Ecuaciones Diferenciales Ordinarias	75	Aprobado	99-00	Adaptada

Tercer curso

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Estadística Matemática	60	Notable	99-00	Adaptada
Geometría local de Curvas y Superf.	90	Aprobado	99-00	
Álgebra	75	Aprobado	99-00	Adaptada
Variable Compleja y Análisis Fourier	60	Notable	99-00	Adaptada

Cuarto curso

Asignatura	Horas presenciales	Calificación	Año	Observaciones
Análisis Funcional	45	Aprobado	00-01	
Cálculo Numérico III	90	Aprobado	00-01	
Estructuras Algebraicas	90	Notable	00-01	
Variable Compleja	60	Notable	00-01	
Variedades Diferenciables	45	Aprobado	00-01	
E. D. P. y Análisis Funcional	90	Aprobado	00-01	
Elementos de Homología Clásica	60	Aprobado	00-01	

4.4 Sistema de calificación y, si se conoce, la distribución de las calificaciones:

Cada asignatura se evalúa en una escala de 0 a 10 puntos. Esta calificación numérica tiene asociada una calificación cualitativa, de la siguiente forma:

Calificación Cualitativa	**Calificación Numérica**
Suspenso	Entre 0 y 4,9 puntos
Aprobado	Entre 5 y 6,9 puntos
Notable	Entre 7 y 8,9 puntos
Sobresaliente	Entre 9 y 10 puntos
Matrícula de Honor	Sobresaliente + Mención especial

Para superar una asignatura es preciso obtener, al menos, 5 puntos. Por tato una asignatura se supera con las calificaciones de Matrícula de Honor, Sobresaliente, Notable o Aprobado, y no se supera si se obtiene la calificación de Suspenso.

Matrícula de Honor significa haber obtenido un Sobresaliente y una mención especial y se puede conceder, como máximo, una Matrícula de Honor por cada 20 estudiantes matriculados en la asignatura.

Algunas actividades se califican sólo como Apto/No Apto. No tienen calificación numérica y no se tienen en cuenta en el cálculo de la puntuación media..

No Presentado significa que el alumno ha abandonado voluntariamente la asignatura.

La distribución de calificaciones de los estudiantes que han superado las asignaturas de Matemáticas en los últimos cuatro años ha sido

UAM

Aprobado	65,74 %
Notable	23,50 %
Sobresaliente	7,55 %
Matrícula de Honor	3,21 %

UAB

Aprobado	62,52 %
Notable	26,17 %
Sobresaliente	9,35 %
Matrícula de Honor	1,98 %

UC

Aprobado	67,57 %
Notable	22,34 %
Sobresaliente	6,27 %
Matrícula de Honor	3,82 %

USC

Aprobado	62,12 %
Notable	31,82 %
Sobresaliente	4,54 %
Matrícula de Honor	1,52 %

US

Aprobado	68,29 %
Notable	24,39 %
Sobresaliente	4,88 %
Matrícula de Honor	2,44 %

La observación "EQ" significa que el alumno ha cursado una asignatura equivalente.
La observación "Convalidada" significa que el alumno ha cursado una asignatura similar y no se le asigna calificación.
La observación “Adaptada” significa que el alumno ha cursado una asignatura similar, pero en un Plan de Estudios de Licenciado en Matemáticas diferente del actual.

4.5 **Clasificación general de la titulación (en la lengua original):**
Puntuación media 1.9 [si el estudiante obtiene Premio Extraordinario, señalarlo. En este caso añadir también:
[UAM] Sólo puede ser premiado con la distinción de "Premio Extraordinario" un estudiante de cada 125 por promoción.
[UAB] Sólo pueden ser premiados con la distinción de "Premio Extraordinario" dos estudiantes por cada promoción entre los que hayan obtenido una calificación global igual o superior a 2,3.
[UC] La distinción de "Premio Extraordinario" es asignada a lo sumo a un estudiante por promoción por una Comisión de la Facultad de Ciencias en atención a un expediente de excepcional mérito.
[USC] Sólo puede ser premiado con la distinción de "Premio Extraordinario" un estudiante de cada cinco que hayan obtenido la calificación de Sobresaliente en la obtención del Grado de Licenciado.

[UC] [Si el estudiante presenta una Tesina o supera el Examen de Grado de Licenciatura, señalarlo. En este caso añadir también:

- Como complemento a su formación el estudiante ha superado el Examen de Grado con la calificación de ¿¿??
- Como complemento a su formación e iniciación a la investigación el estudiante ha elaborado y defendido la Tesina de Licenciatura titulada ¿¿?? que recibió la calificación de ¿¿??]

La puntuación media de los Licenciados en Matemáticas en la Universidad Autónoma de Madrid durante los últimos cuatro años es 1.3.

La puntuación media se calcula mediante el criterio numérico siguiente:

Aprobado o Convalidada	1 punto
Notable	2 puntos
Sobresaliente	3 puntos
Matrícula de Honor	4 puntos

5. INFORMACIÓN SOBRE LA FUNCIÓN DE LA TITULACIÓN

5.1 **Acceso a ulteriores estudios:** Después de la Licenciatura se puede acceder a:

- Diploma de Estudios Avanzados (no sólo en Matemáticas). Si es seguido de una Tesis de Investigación, se obtiene el título de Doctor.
- Másters y títulos de especialización en diferentes campos.
- Certificado de Aptitud Pedagógica, necesario para ser profesor permanente en el sistema público de educación secundaria.

5.2 **Rango profesional (si procede):** Matemático no es una profesión regulada oficialmente. Un Licenciado en Matemáticas, como todo Licenciado, puede optar a las categorías más altas de la función pública. El título de Licenciado en Matemáticas cualifica para la formulación matemática, análisis, resolución y, en su caso, tratamiento informático de problemas en diversos campos interdisciplinares de las ciencias básicas, ciencias sociales y de la vida, ingeniería, finanzas, consultoría, etc..., con vistas a las aplicaciones, la investigación y/o la docencia.

6. INFORMACIÓN ADICIONAL

6.1 **Información adicional:**

[UAM] El título de Licenciado en Matemáticas en la Universidad Autónoma de Madrid no está dirigido hacia la formación especializada en ninguna rama de las Matemáticas. Ofrece una amplia selección de asignaturas optativas, de manera que el estudiante puede diseñar un curriculum adaptado a sus futuros intereses profesionales. Es posible, pero no obligatorio, conseguir créditos por realizar prácticas en una empresa o industria.

[UAB] El título de Licenciado en Matemáticas en la Universidad Autónoma de Barcelona no está dirigido hacia la formación especializada en ninguna rama de las Matemáticas. Ofrece una selección de asignaturas optativas, de orientación aplicada y teórica, de manera que el estudiante puede diseñar un curriculum adaptado a sus expectativas profesionales. Escogiendo bien los créditos optativos y de libre elección, el estudiante puede obtener una mayor especialización laboral cursando 30 créditos adicionales. Se pueden conseguir créditos optativos por realizar prácticas en una empresa o un centro de enseñanza de secundaria y también por la realización de un Trabajo Dirigido.

[USC] El título de Licenciado en Matemáticas en la Universidad de Santiago no está dirigido hacia la formación especializada en ninguna rama de las Matemáticas. Ofrece una amplia selección de asignaturas optativas, que se encuadran en las orientaciones de Matemática Aplicada, Matemática Pura y de Estadística e Investigación Operativa. Estas orientaciones hacen posible que el estudiante pueda diseñar un curriculum adaptado a sus expectativas profesionales. Es posible, pero no obligatorio, conseguir créditos por realizar prácticas en una empresa o industria y por trabajos académicamente dirigidos.

[US] El título de Licenciado en Matemáticas por la Universidad de Sevilla pretende dar una formación lo más amplia posible de muy diversas ramas de las Matemáticas, tanto puras como aplicadas. Los estudiantes adquieren conocimientos y destrezas que les capacitan para adaptarse con facilidad tanto al ejercicio profesional como a la docencia y la investigación.

6.2 Otras fuentes de información:

- Sobre el sistema educativo español: www.mecd.es,
- Sobre la Universidad Autónoma de Madrid: www.uam.es,
- Sobre la Titulación de Matemáticas en la UAM: www.uam.es/matem
- Sobre las universidades catalanas: http://dursi.gencat.es
- Sobre la Universidad Autónoma de Barcelona: www.uab.es,
- Sobre la Titulación de Matemáticas en la UAB: http://mat.uab.es/seccio
- Sobre la Universidad de Cantabria: www.unican.es,
- Sobre la Facultad de Ciencias: www.fciencias.unican.es
- Sobre la titulación de Licenciado en Ciencias Matemáticas de la Universidad de Cantabria: http://campusvirtual.unican.es/planes/CMATEMAA.htm
- Sobre la Universidad de Santiago: www.usc.es,
- Sobre la Titulación de Matemáticas en la USC: www.usc.es/intro/facescg.htm
- Sobre la Universidad de Sevilla: www.us.es
- Sobre la Titulación de Matemáticas en la Universidad de Sevilla: www.matematicas.us.es

7. CERTIFICACIÓN DEL SUPLEMENTO

7.1 **Fecha:** 10 de octubre de 2001

7.2 **Firma:** [Secretario General IMPRESA] [Decano/Administrador del Centro/... ORIGINAL]

7.3 **Cargo:** Secretario General de la Universidad ... Decano/Administrador de la Facultad ...

7.4 **Sello o tampón oficial:** SELLO SECO

8. INFORMACIÓN SOBRE EL SISTEMA NACIONAL DE ENSEÑANZA SUPERIOR

(N.B. Las instituciones que tienen previsto expedir el Diploma Supplement deben consultar las notas explicativas sobre su cumplimentación.)

THE DIPLOMA SUPPLEMENT

EJEMPLO DE LICENCIADO EN MATEMÁTICAS EN LA UAM

1 INFORMATION IDENTIFYING THE HOLDER OF THE QUALIFICATION

1.1 **Family name(s):** García Pérez [en la versión informática: un solo campo con espacio en blanco entre los dos apellidos]
1.2 **Given name(s):** Carmen
1.3 **Date of birth *(day/month/year)*:** 01/01/1979
1.4 **Student identification number or code *(if available)*:** EMADRID04 + DNI [temporalmente se utilizará código Erasmus + DNI, aunque se apoyará el proyecto ScanNet]

2 INFORMATION IDENTIFYING THE QUALIFICATION

2.1 **Name of studies and *(if applicable)* title conferred *(in original language)*:**
Studies: Licenciatura en Matemáticas
Title conferred: Licenciada en Matemáticas (state recognised)
2.2 **Main field(s) of study for the qualification:** Mathematics (Algebra, Mathematical Analysis, Geometry, Statistics, Numerical Analysis)
2.3 **Name and status of awarding institution *(in original language)*:** Universidad Autónoma de Madrid, universidad pública
2.4 **Name and status of institution *(if different from 2.3)* administering studies *(in original language)*:**
2.5 **Language of instruction/examination:** Spanish

3 INFORMATION ON THE LEVEL OF THE QUALIFICATION

3.1 **Level of qualification:** Licenciado (see 8).
3.2 **Official length of programme:** 4 years, but there is the choice of doing it in 5 years. In both cases the total time of contact with the lecturers in theoretical classes, problems sessions and laboratory work is 3000 hours during the degree. The total estimated working time for the student, including classes, individual study, exams and preparation for them, is annually about 1500 hours.
3.3 **Access requirements(s):** Bachillerato + University Entrance Examination. Students holding the degree of Diplomado en Estadística are admitted directly into the second cycle taking some complementary courses.

4 INFORMATION ON THE CONTENTS AND RESULTS GAINED

4.1 **Mode of study:** Presential programme. Students are free to organise temporally their study plan in any way they want. [Sería conveniente que se estableciese con claridad la distinción entre estudiantes a tiempo completo y a tiempo parcial].
4.2 **Programme requirements:** The student has to complete 2500 contact hours with the lecturers corresponding to:

- 18 core subjects (1410 contact hours)
- 4 compulsory subjects (290 contact hours)
- 7 elective subjects (400 contact hours)
- free choice subjects (400 contact hours) that the student may take among all subjects offered by the university in all disciplines or other extracurricular activities.

Each subject must be passed independently.

4.3 **Programme details: (e.g. modules or units studied), and the individual grades/marks/credits obtained:**
(if this information is available on an official transcript this should be used here)

4.3.1 Core and compulsory subjects:

First year

Subject	Contact hours	Grade	Year	Observations
Conjuntos y Números (Sets and Numbers)	120	Sobresaliente	97-98	
Cálculo I (Calculus I)	120	Recognised	97-98	Recognised
Álgebra Lineal (Linear Algebra)	120	Recognised	97-98	Recognised
Informática (Computer Science)	100	Aprobado	97-98	
Cálculo II (Calculus II)	120	Matrícula de Honor	97-98	
Geometría I (Geometry I)	120	Aprobado	98-99	

Second year

Subject	Contact hours	Grade	Year	Observations
Cálculo III (Calculus III)	80	Notable	98-99	
Ec. Diferen. Ordinarias (Ordinary Differential Equations)	80	Aprobado	99-00	
Probabilidad I (Probability I)	100	Notable	98-99	
Cálculo Numérico I (Numerical Analysis I)	100	Sobresaliente	98-99	
Geometría II (Geometry II)	80	Aprobado	99-00	
Topología (Topology)	80	Aprobado	98-99	
Modelización I (Mathematical Models I)	80	Aprobado	99-00	
Física para Matemáticos (Physics)	80	Aprobado	99-00	

Third year

Subject	Contact hours	Grade	Year	Observations
Álgebra I (Algebra I)	80	Sobresaliente	00-01	EQ Erasmus
Teo. Integral y Medida (Measure Theory)	80	Aprobado	00-01	EQ Erasmus
Variable Compleja I (Complex Variables I)	80	Notable	00-01	EQ Erasmus
Álgebra II (Algebra II)	80	Aprobado	00-01	EQ Erasmus
Ec. Difer, y Anál. Funcional (Differential Equations and Functional Analysis)	80	Aprobado	00-01	EQ Erasmus
Probabilidad II (Probability II)	80	Matrícula de Honor	00-01	EQ Erasmus

Fourth year

Subject	Contact hours	Grade	Year	Observations
Geometría III (Geometry III)	90	Sobresaliente	01-02	
Cálculo Numérico II (Numerical Analysis III)	90	Aprobado	00-01	

4.3.2 Elective subjects

Subject	Contact hours	Grade	Year	Observations
Historia de las Matemáticas (History of Math)	80	Notable	99-00	
Matemática Discreta (Discrete Mathematics)	80	Sobresaliente	00-01	EQ Erasmus
Teoría de Números (Number Theory)	80	Notable	00-01	EQ Erasmus
Estadística I (Statistics I)	80	Notable	01-02	
Estadística II (Statistics II)	80	Notable	01-02	
Modelización II (Mathematical Models II)	80	Aprobado	01-02	
Lógica (Mathematical Logic)	80	Notable	01-02	

4.3.3 Free choice subjects

Subject	Contact hours	Grade	Year	Observations
Inglés, nivel medio (English, intermediate level)	50	Sobresaliente	97-98	
Inglés, nivel superior (English, advanced level)	70	Notable	98-99	
Bases de Datos (Databases)	50	Notable	00-01	EQ Erasmus
Derecho de la Empresa (Enterprise Law)	60	Notable	00-01	EQ Erasmus
Prácticas en Empresas (Practices in companies)	120	Apto	01-02	
Extracurricular activities	50	Apto		

4.3.4 Subjects taken in equivalence

At the Universität Würzburg, Germany (EQ Erasmus)

Funktionentheorie (Function Theory)
Algebra I (Algebra I)
Elementare Zahlentheorie (Elementary Number Theory)
Funktionalanalysis (Functional Analysis)
Einführung zu JAVA (Introduction to JAVA)
Deutsch als Fremdsprache, Mittelstufe (German, intermediate level)
Deutsch als Fremdsprache, Oberstufe (German, advanced level)
Deutsch als Fremdsprache, Landeskunde (German regional studies)

4.4 Grading scheme and, if available, grade distribution guidance:

Each subject is graded in a scale from 0 a 10 points. Each numeric grade correspond to a qualitative grade as follows:

Qualitative Grade	Numeric Grade
Suspenso	Entre 0 y 4,9 puntos
Aprobado	Entre 5 y 6,9 puntos
Notable	Entre 7 y 8,9 puntos
Sobresaliente	Entre 9 y 10 puntos
Matrícula de Honor	Sobresaliente + Mención especial

To pass a subject it is necessary to get at least 5 points. There are therefore four passing grades (Matrícula de Honor, Sobresaliente, Notable, Aprobado) plus one fail grade (Suspenso).

Matrícula de Honor means getting a Sobresaliente plus an special mention. A maximum of one Matrícula de Honor per 20 students registered in a given subjectc an be awarded.

Some activities are graded only on a Pass (Apto) / Fail (No Apto) basis. These subjects do not have a numeric grade and are not taken under consideration when calculating the Grade Point Average (GPA).

No Presentado means that the student has voluntarily dropped the subject,

The grade's distribution of students passing Mathematics' subjects in the last four years has been

	UAM
Aprobado	65,74 %
Notable	23,50 %
Sobresaliente	7,55 %
Matrícula de Honor	3,21 %

The observation "EQ" means that the student has taken an equivalent subject.
The observation "Recognised" means that the student has taken a similar subject and no grade is assigned.

4.5 Overall classification of the qualification *(in original language)*: Grade Point Average 1.9 [si el estudiante obtiene Premio Extraordinario, señalarlo. En este caso añadir también: No more than 1 student out of 125 in each graduating class can be awarded the distinction of "Premio Extraordinario".]
The mean GPA among Licenciados en Matemáticas at Universidad Autónoma de Madrid in the last four years is 1.3.

The Grade Point Average (GPA) is calculated with the following numerical criterion:

Aprobado or Convalidada	1 point
Notable	2 points
Sobresaliente	3 points
Matrícula de Honor	4 points

5 INFORMATION ON THE FUNCTION OF THE QUALIFICATION

5.1 **Access to further study:** A Licenciado en Matemáticas has access to:

- Diploma de Estudios Avanzados (not only in Mathematics). If this is followed by a Research Thesis, the degree of Doctor is awarded.
- Non-doctoral postgraduate studies like Masters and other specialisation degrees in different fields.
- Certificado de Aptitud Pedagógica, required to be a permanent teacher in the public secondary school system.

5.2 **Professional status *(if applicable)*:** Mathematician is not an officially regulated profession. A Licenciado en Matemáticas, as is the case of all Licenciados, can opt to the highest ranks of the Civil Service. The degree of Licenciado en Matemáticas qualifies for the mathematical formulation, analysis, resolution and, if it is the case, computational processing of problems in different interdisciplinary fields of basic science, social and life sciences, engineering, finances, consulting, etc..., in a view to applications, research and/or teaching.

6 ADDITIONAL INFORMATION

6.1 **Additional information:** The degree of Licenciado en Matemáticas at Universidad Autónoma de Madrid is not directed towards specialized training in any branch of Mathematics. It offers a broad selection of optional subjects, so that students can build up a curriculum geared to their future professional interests. It is possible, but not compulsory, to get credit for an internship in a company or industry.

6.2 **Further information sources:**

- About the Spanish education system: www.mecd.es,
- About the Universidad Autónoma de Madrid: www.uam.es,
- About the Degree in Mathematics at UAM: www.uam.es/matem

7 CERTIFICATION OF THE SUPPLEMENT

7.1 **Date:** October 10th, 2001
7.2 **Signature:** [Secretario General IMPRESA] [Decano/Administrador del Centro/... ORIGINAL]
7.3 **Capacity:** Secretario General de la Universidad ... Decano/Administrador de la Facultad ...
7.4 **Official stamp or seal:** SELLO SECO

8 INFORMATION ON THE NATIONAL HIGHER EDUCATION SYSTEM

(N.B. Institutions who intend to issue Diploma Supplements should refer to the explanatory notes that explain how to complete them.)

Anexo 3

Informe y datos obtenidos respecto a la valoración de créditos europeos (ECTS)

ESTUDIO SOBRE LA ASIGNACIÓN DE CRÉDITOS EUROPEOS A LAS DISTINTAS MATERIAS DEL CURRÍCULUM DE MATEMÁTICAS

UNIVERSIDAD AUTÓNOMA DE BARCELONA

UNIVERSIDAD AUTÓNOMA DE MADRID

UNIVERSIDAD DE CANTABRIA

UNIVERSIDAD DE SANTIAGO DE COMPOSTELA

UNIVERSIDAD DE SEVILLA

Octubre 2002

Índice

1. Introducción

En la declaración de Bolonia de 1999, los 29 países firmantes aspiran a la creación de un espacio europeo de educación superior para el año 2010, con el fin de mejorar el empleo y la movilidad de los ciudadanos y de incrementar la competitividad internacional de la educación superior europea.

Para ello es necesario alcanzar un alto grado de compatibilidad y comparabilidad entre los diferentes sistemas de educación superior, a través de los siguientes objetivos:

- Adopción de un sistema de títulos de fácil interpretación y comparación, mediante la implantación de un Suplemento Europeo al Título
- Adopción de un sistema esencialmente basado en dos ciclos principales, grado y postgrado
- Implantación de un sistema de créditos, basado en el sistema ECTS, como medio adecuado para fomentar la movilidad de los estudiantes
- Promoción de la cooperación europea en los procesos de evaluación y acreditación de calidad mediante el desarrollo de metodologías y criterios comparables
- Promoción de una educación superior de dimensión europea.

2. Sistema Europeo de Transferencia de Créditos

Hace diez años se desarrolló el Sistema Europeo de Transferencia de Créditos (ECTS, en sus siglas en inglés), en forma de proyecto piloto en el marco del programa Erasmus, con el fin de facilitar el reconocimiento académico de los estudios cursados en el extranjero. El ECTS se basa en dos elementos básicos: la información sobre los programas de estudios y la utilización de créditos europeos.

El sistema de créditos ECTS proporciona un procedimiento estandarizado de medida y comparación del aprendizaje en diferentes contextos (diferentes programas académicos, diferentes países, ...). Para ello los créditos en el ECTS no deben indicar simplemente el número de horas dedicadas al aprendizaje, sino que deben además aportar información sobre las cualidades de ese aprendizaje (naturaleza, contexto, nivel, ...). De esta manera los créditos serán una herramienta eficaz para conseguir transparencia e integración de los diferentes sistemas europeos de educación superior.

Se valora en 60 créditos europeos el conjunto organizado de materias/asignaturas que un estudiante medio, dedicado a los estudios a tiempo completo, debe superar en un año. Esto supone una definición del aprendizaje posible en un año académico de un estudiante medio a tiempo completo e incluye las horas de clase en aula o laboratorio, las prácticas, las horas de estudio personal y la preparación y realización de exámenes.

Su equivalencia en horas de trabajo para dicho estudiante medio es de aproximadamente1600 horas:

8 horas diarias x 5 días a la semana x 40 semanas al año = 1600 horas.

Un crédito europeo representa entre 25 y 30 horas de trabajo del estudiante. Evidentemente, los estudiantes más dotados requerirán menos horas y los menos dotados más. Tiene la ventaja de que cada persona, estudie a tiempo completo o a tiempo parcial, puede valorar lo que puede asumir en el tiempo de que disponga.

2.1 Asignación de créditos europeos

En el actual sistema español un crédito corresponde a 10 horas presenciales. No cuantifica en horas el trabajo personal del estudiante ni la preparación y realización de exámenes.

A efectos de movilidad en el programa Erasmus se convirtieron los créditos locales a créditos europeos mediante simples reglas de tres y eso ha creado un hábito que hay que romper. El que los créditos europeos tengan sentido y no se reduzcan a una mera adaptación numérica depende del esfuerzo que internamente realicen las universidades y de la voluntad del profesorado para asumir los nuevos criterios diseñando las estrategias para el aprendizaje adecuadas.

La implantación de los créditos europeos supone un cambio en la orientación pedagógica de las enseñanzas, menos centrada en lo que se enseña y más precisa en lo que se aprende y cuánto esfuerzo requiere este aprendizaje. La implantación de los créditos europeos requiere, por tanto, la elaboración de guías de cada titulación en las que en cada materia se detallen las horas necesarias de dedicación de un "estudiante medio" a cada tipo de actividad, los objetivos formativos que se persiguen y las habilidades que el estudiante debe adquirir.

El volumen de trabajo que realiza un estudiante medio para superar una asignatura depende de la titulación, el programa de estudios, el nivel de exigencia, etc., por lo que cada universidad y, en ella, cada titulación, deberá analizar su caso concreto a la hora de asignar créditos europeos.

La asignación de créditos conviene hacerla siguiendo un procedimiento " top-down" (descendente). Primero se determina el volumen de trabajo de un curso académico completo y se define con 60 créditos. A cada asignatura de ese curso se le asignará un número de créditos según la proporción de trabajo que requiera en relación con el total. Para ello será útil disponer de datos fiables sobre el número de horas que los estudiantes dedican a cada asignatura. En la práctica, el volumen de trabajo varía de año en año, dependiendo de diversos factores, como la utilización de diferentes metodologías docentes, el número de estudiantes por profesor, los conocimientos previos de los estudiantes, ... La única manera de asignar créditos coherentemente es especificando de antemano el volumen de trabajo en una guía docente.

La aplicación del sistema europeo implica un cambio en el diseño de las titulaciones y programas. Primero hay que determinar qué se quiere ofrecer al estudiante (si unos estudios más teóricos o más prácticos, con alguna especialización o no, ...) y fijar unos objetivos, y luego cuánto tiempo necesita el estudiante para alcanzar esos objetivos; y no a la inversa, como se suele hacer actualmente, donde primero se fija la duración de los estudios y después se "rellenan" los cursos con asignaturas.

No hay ninguna institución que controle la aplicación del ECTS. Esto tiene la ventaja de que no se interfiere en la autonomía de las universidades ni de los sistemas educativos pero, por otro lado, las disparidades en los criterios de asignación de créditos influyen negativamente en la comparabilidad.

Si el ECTS se aplica correctamente es posible comparar, desde un punto de vista cuantitativo, estudios en diferentes universidades. Pero la comparabilidad y consigo la transferibilidad de resultados académicos depende también de criterios cualitativos como los contenidos, el grado de dificultad, la calidad de la enseñanza, el tipo de examen y de puntuación.

3. Encuesta

Se ha realizado una encuesta para obtener datos que orienten y faciliten la asignación de créditos ECTS a las distintas materias del currículum de Matemáticas.

3.1 Objetivos

- Obtener una estimación general de las horas de trabajo personal dedicadas por los estudiantes a cada asignatura
- Contrastar el trabajo realizado por los estudiantes con el exigido por los profesores

3.2 Ámbito y diseño de la encuesta

La encuesta se ha realizado en los cursos de primer ciclo de la Licenciatura de Matemáticas de las cinco universidades participantes, así como en los de segundo ciclo de la Universidad Autónoma de Barcelona y de la Universidad de Santiago.

En algunos casos la encuesta se realizó dentro del horario lectivo (se destinó tiempo de clase para que los estudiantes la completaran) y en otros no (los estudiantes debían completarla en casa), siendo la participación notablemente superior en los primeros.

En la Universidad Autónoma de Madrid la encuesta se extendió también a los profesores que impartían docencia en los cursos de primer ciclo.

En la Universidad de Sevilla, la respuesta de los estudiantes fue tan reducida, que no se han incluído los resultados en este estudio

A modo de ejemplo se adjuntan en el Anexo ECTS-1 los dos formularios de la UAM.

Al redactar los formularios para los estudiantes surgió la duda de qué ocurre con aquellos que repiten una asignatura, ¿deben o no contabilizar en el tiempo invertido a esa asignatura el que dedicaron en las convocatorias anteriores? Se creyó que lo correcto era tener en cuenta todas las horas que el estudiante necesitara hasta superarla (¿es realmente lo correcto?) y así se explicó al distribuir el formulario entre los estudiantes. Para controlar este aspecto se preguntó la convocatoria en la que el estudiante se encuentra. En la práctica no se sabe cómo ha sido interpretado este aspecto por los estudiantes ni si lo tuvieron en cuenta. Es algo que habría que mejorar en encuestas posteriores.

Aspectos que habría que mejorar:

- controlar si en el número de horas que los estudiantes repetidores dicen dedicar a una asignatura incluyen las convocatorias anteriores.
- aumentar la participación de los estudiantes (mayor número de respuestas).

3.3 Resultados (ver Anexo ECTS-2)

En general el número de respuestas obtenidas es bajo, lo cual limita la significación de los datos. A pesar de ello es posible extraer algunas características generales, en las que cabe destacar lo siguiente:

- Es frecuente la discrepancia entre las horas de trabajo estimadas por los profesores y las dedicadas por los estudiantes.

- Hay asignaturas que parecen requerir un volumen de trabajo personal de los estudiantes superior al resto.

  En ocasiones, esto podría explicarse por tener la asignatura alguna característica específica, como puede ser la necesidad de utilizar el ordenador. Sería el caso de las asignaturas de Análisis Numérico de varias de las Universidades, y estas características específicas deberían tomarse en cuenta a la hora de asignar créditos ECTS a dichas asignaturas.

  Pero otras veces, la necesidad de dedicar más horas a una asignatura parece deberse más bien a algún error de diseño (contenidos excesivos y/o no adaptados al nivel de los estudiantes,...). Ejemplos de esta situación parecerían ser Geometría I en la UAM, Análisis de varias variables reales en la UC o Ecuaciones diferenciales ordinarias en la USC.

3.4 Ejemplos de asignación de créditos para Matemáticas en la UAM

Se presentan a continuación dos métodos para llevar a cabo dicha asignación. El primero de ellos, que no consideramos aconsejable, está basado directamente en la estimación del volumen total de trabajo del estudiante, tal y como se refleja en las encuestas. El segundo, basado en la guía docente, asigna factores a cada materia que relacionan el número de horas presenciales y el número de horas de trabajo personal del estudiante.

Aunque no se aprecian diferencias notables entre la media de horas de estudio del total de los estudiantes (incluidos quienes suspenden) y la de los que superan la asignatura, nuestra referencia van a ser los datos de los estudiantes que superan cada asignatura de la licenciatura de Matemáticas en la Universidad Autónoma de Madrid.

Primer curso

Asignatura	Créditos españoles	Horas presenciales	Media horas estudio aprobados	Horas totales (presencial + estudio)	Horas estudio por hora presencial	Horas totales por crédito europeo
Cálculo I	12	84	144	228	**1,7**	**24**
Conjuntos y Números	12	84	142	226	**1,7**	**24**
Álgebra Lineal	12	84	143	227	**1,7**	**24**
Cálculo II	12	84	127	211	**1,5**	**22**
Geometría I	12	84	245	329	**2,9**	**34**
Informática	10	98	137	235	**1,4**	**29**
Libre elección	5	50	50	100	**1**	**25**
Total	75	568		**1556**		

Segundo Curso

Asignatura	Créditos españoles	Horas presenciales	Media horas estudio aprobados	Horas totales (presencial + estudio)	Horas estudio por hora presencial	Horas totales por crédito europeo
Cálculo III	8	56	140	196	**2,5**	**31**
Cálculo Numérico I	10	98	240	338	**2,5**	**42**
Probabilidad I	10	77	129	206	**1,7**	**26**
E.D.O.	8	56	158	214	**2,8**	**33**
Topología	8	56	154	210	**2,7**	**33**
Geometría II	8	56	147	203	**2,6**	**32**
Modelización I	8	56	98	154	**1,8**	**24**
Física	8	56	146	202	**2,6**	**32**
Libre elección	7	70	70	140	**1,0**	**25**
Total	75	581		**1862**		

Los resultados de la encuesta en la UAM sugieren que:

- El primer curso, con 7 asignaturas, requiere un volumen razonable de trabajo para el estudiante (entre 1500 y 1600 horas).

- El segundo curso, con 9 asignaturas, requiere excesivo trabajo por parte del estudiante (más de 1700 horas, con una dedicación media de 2’5 horas de estudio por hora presencial en cada asignatura). Habría que disminuir el número de asignaturas (lo que supondría aumentarlo en otro curso) o cambiar la metodología (lo más correcto), sustituyendo clases teóricas por clases de problemas, para no superar las 2 horas de estudio por hora presencial.

- Geometría I, debido a un mal diseño (que se está corrigiendo), es una asignatura a la que los estudiantes dedican muchas más horas que al resto (casi 3 horas de estudio por hora presencial), a menudo a lo largo de varias convocatorias.

- En la asignatura de Informática se imparten más horas presenciales de prácticas con ordenador (56 horas) que teóricas (42 horas), lo que hace que los estudiantes no tengan que dedicarle muchas más horas de estudio fuera del aula (1’4 horas de estudio por hora presencial).

- En Cálculo Numérico I también ocurre que se imparten más horas presenciales de prácticas con ordenador (56 h) que teóricas (42 h), pero en esta asignatura los estudiantes dedican 2’5 horas de estudio por cada hora presencial, excesivas como en otras asignaturas de segundo curso. En Cálculo Numérico I hay tres tipos de trabajo: estudio de la teoría, resolución de problemas y realización de prácticas en ordenador. A diferencia que en Informática, las prácticas apenas están conectadas a la teoría y los problemas (a la materia de examen), de manera que en realidad añaden trabajo extra. Probablemente un cambio de metodología en las clases teóricas como el expresado anteriormente para las asignaturas de segundo curso corregiría el exceso de horas de estudio por hora presencial.

- La estimación de los profesores no puede ser la única referencia que se utilice para determinar el volumen de trabajo de los estudiantes, ya que en la mayoría de los casos sus estimaciones discrepan de las de los alumnos (véase por ejemplo Geometría I, Cálculo Numérico I o Física).

Modelo 1: basado en el volumen total de trabajo del estudiante reflejado en las encuestas (no aconsejable)

En una asignación de créditos basada en las horas totales (horas presenciales más horas de estudio personal) que dedican los estudiantes que superan la asignatura habría que calcular la proporción de trabajo que requiere cada asignatura en relación con el curso completo y multiplicarlo por 60:

$$\frac{\text{volumen de trabajo de aprobados en la asignatura}}{\text{volumen de trabajo del curso completo}} \times 60$$

Esta cantidad es lo que en la tabla aparece como “Proporción”. Para quitar los decimales se redondea a la alta o a la baja según se crea más adecuado.

Primer curso

Asignatura	Créditos españoles	Créditos españoles normalizados a 60	Horas totales aprobados	Proporción	**Propuesta créditos europeos**
Cálculo I	12	9,6	228	8,81	**9**
Conjuntos y Números	12	9,6	226	8,73	**9**
Álgebra Lineal	12	9,6	227	8,75	**9**
Cálculo II	12	9,6	211	8,14	**9**
Geometría I	12	9,6	329	12,67	**11**
Informática	10	8	235	9,05	**9**
Libre elección	5	4	100	3,86	**4**
Total	75	60	1556		60

Segundo curso

Asignatura	Créditos españoles	Créditos españoles normalizados a 60	Horas totales aprobados	Proporción	**Propuesta Créditos europeos**
Cálculo III	8	6,4	196	6,30	**6,5**
Cálculo Numérico I	10	8	338	10,90	**11**
Probabilidad I	10	8	206	6,62	**6,5**
E.D.O.	8	6,4	214	6,89	**7**
Topología	8	6,4	210	6,75	**6,5**
Geometría II	8	6,4	203	6,54	**6,5**
Modelización I	8	6,4	154	4,97	**5**
Física	8	6,4	202	6,52	**6,5**
Libre elección	7	5,6	140	4,51	**4,5**
Total	75	60	1862	60	60

Esta asignación de créditos no es aconsejable ya que no hay razones académicas que justifiquen estas diferencias tan grandes en el volumen de trabajo de las asignaturas. La comparación de las encuestas realizadas en distintas universidades no confirma el hecho de que la dificultad esté en la materia en sí misma, si no en su plasmación concreta en una universidad. Aplicar el resultado de las encuestas directamente no haría más que consolidar las situaciones presuntamente patológicas reflejadas en ella.

Las estimaciones de los estudiantes deben servir para detectar (y corregir) anomalías en asignaturas concretas, pero no para asignar créditos.

Modelo 2: basado en la guía docente

Cada asignatura, según su dificultad y metodología docente adoptada, requerirá diferente volumen de trabajo del estudiante. Las horas presenciales y las horas de trabajo personal del estudiante están relacionadas por un factor, reflejado en el primer cuadro de este apartado, aunque éste no es el mismo para todas las materias.

Las asignaturas de primer curso, excepto Informática, además de cuatro horas semanales de teoría, incluyen dos horas de problemas para facilitar el trabajo a los estudiantes. Esto hace que se necesite alrededor de 1'7 horas de estudio por cada hora presencial. Geometría I, tal y como está diseñada actualmente, requiere mayor dedicación que las otras asignaturas. Creemos que lo razonable sería 1'9 horas de estudio por hora presencial. En la asignatura de Informática, por sus contenidos y objetivos, la teoría y la práctica se complementan y requiere menos horas de trabajo personal por hora presencial (factor 1'5).

En segundo curso, en cambio, se reducen las horas presenciales de cada asignatura (eliminando las clases de problemas) para que los estudiantes dispongan de más tiempo para el estudio personal. La ausencia de clases de resolución de problemas incrementa la dedicación a 2'5 horas de estudio por hora presencial. Como se ha dicho anteriormente creemos necesario un cambio metodológico para que el factor horas estudio / horas presenciales no sea superior a 2. En Cálculo Numérico I y Probabilidad I hay que diferenciar las horas de teoría de las de prácticas, las primeras con un factor 1’7, menor que en las asignaturas sólo teóricas porque el tiempo dedicado a las prácticas ayuda a la comprensión de la teoría, y las segundas con un factor 1’5 como en Informática.

Cálculo I Conjuntos y Números Álgebra Lineal Cálculo II	84 h presenciales x factor **1’7** = 143 h estudio; 143 h estudio + 84 h presenciales = aprox. **225** h totales
Geometría I	84 h presenciales x factor **1’9** = 165 h estudio 165 h estudio + 84 presenciales = aprox. **250** h totales
Informática	98 h presenciales x factor **1'5** = 147 h estudio 147 h estudio + 98 h presenciales = aprox. **250** h totales
Cálculo III E.D.O. Topología Geometría II Física Modelización I	56 h presenciales x factor **2** = 112 h estudio 112 h estudio + 56 h presenciales = aprox. **175** h totales

Cálculo Numérico I	42 h presenciales teoría x factor **1'7** = 72 h estudio teoría 56 h presenciales prácticas x factor **1'5** = 84 h estudio práctica 72 + 84 h estudio + 98 h presenciales = aprox. **250** h totales
Probabilidad I	56 h presenciales teoría x factor **1'7** = 95 h estudio teoría 21 h presenciales práctica x factor **1'5** = 30 h estudio práctica 95 + 30 h estudio + 77 h presenciales = aprox. **210** h totales

Primer curso

Asignatura	Créditos españoles	Horas totales guía docente	Proporción	Propuesta créditos europeos	Horas totales por crédito europeo
Cálculo I	12	225	9	**9**	25
Conjuntos y Números	12	225	9	**9**	25
Álgebra Lineal	12	225	9	**9**	25
Cálculo II	12	225	9	**9**	25
Geometría I	12	250	10	**10**	25
Informática	10	250	10	**10**	25
Libre elección	5	100	4	**4**	25
Total	75	1500		60	

Segundo curso

Asignatura	Créditos españoles	Horas totales guía docente	Proporción	Propuesta créditos europeos	Horas totales por crédito europeo
Cálculo III	8	175	6,5	**6,5**	26,9
Cálculo Numérico I	10	250	9,3	**9**	26,3
Probabilidad I	10	210	7,8	**8**	26,3
E.D.O.	8	175	6,5	**6,5**	26,9
Topología	8	175	6,5	**6,5**	26,9
Geometría II	8	175	6,5	**6,5**	26,9
Modelización I	8	175	6,5	**6,5**	25,0
Física	8	175	6,5	**6,5**	26,9
Libre elección	7	100	3,7	**4,0**	25,0
Total	75	1610	60	60	

4. Guía docente

La aplicación del ECTS implica centrar la formación en el aprendizaje y la adquisición de competencias y destrezas, valorando adecuadamente el esfuerzo requerido y la calidad del aprendizaje de los estudiantes. Para ello es necesario:

Primero, desarrollar una guía docente que describa, para cada materia:
- objetivos
- contenidos mínimos
- destrezas a adquirir
- metodología de enseñanza
- sistema de evaluación del aprendizaje
- tiempo de estudio personal que debe dedicar un estudiante medio para superarla

Segundo, crear mecanismos de seguimiento de la puesta en práctica de la guía docente y que proporcionen información que permita mejorarla.

Uno de los objetivos de la declaración de Bolonia es ***"promover la cooperación europea en garantía de calidad mediante el desarrollo de metodologías y criterios comparables"***. La guía docente es una herramienta para alcanzar este objetivo ya que proporciona transparencia, aportando información cualitativa (además de cuantitativa) sobre la titulación.

La guía docente debe ser pública y conocida por los estudiantes.

4.1 No hay enseñanza si no hay aprendizaje

Parece necesario un cambio hacia una metodología docente que no lleve al estudiante a posiciones quietistas, y un cambio en la actitud del estudiante frente a su aprendizaje. La implantación del Sistema Europeo de Créditos puede ayudar a reorientar pedagógicamente las enseñanzas, al obligar a definir con detalle objetivos y destrezas de las materias, y aportar transparencia.

Las estrategias de aprendizaje no sólo deben centrarse en el desarrollo de destrezas matemáticas básicas, como el razonamiento abstracto, la deducción lógica y la resolución de problemas, es igualmente importante desarrollar otros aspectos como la capacidad de aprender de forma independiente, de transferir conocimientos de un contexto a otro o de comunicar resultados de manera clara, por mencionar algunos.

Las clases destinadas a la resolución de problemas son una ocasión inmejorable para fomentar la participación de los estudiantes y el intercambio de ideas entre ellos.

4.2 La importancia de la evaluación del aprendizaje

La actitud del estudiante frente a su aprendizaje depende en gran parte de la evaluación. En los sistemas de evaluación del aprendizaje basados exclusivamente en un único examen a final del cuatrimestre los estudiantes suelen posponer el trabajo hasta los últimos días antes del examen mientras que, cuando la evaluación es más continua, tienen que realizar un trabajo más constante.

Siguiendo la línea de la formación centrada en el aprendizaje, y en consonancia con lo que se hace en algunos países europeos, nos atrevemos a sugerir una reflexión sobre la posibilidad de introducir algunas novedades en los sistemas tradicionales de evaluación.

Por una parte, se podría pensar en un sistema de evaluación basado en controles frecuentes (como que el estudiante tenga que resolver problemas propuestos por el profesor y entregarlos periódicamente, realizar trabajos o prácticas en ordenador, individuales o en grupo, preparar y exponer temas, exámenes cortos, etc.) que cuenten para la nota final pero que no supongan excesivo trabajo para el profesor. Si esta evaluación continua es superada, entonces se permitiría la realización del examen final.

Por otra, sería interesante meditar sobre la conveniencia de sustituir los exámenes independientes de asignaturas por exámenes de bloques de asignaturas (por ejemplo, un solo examen para las asignaturas de cada cuatrimestre, en el que el estudiante tenga que relacionar los conceptos y destrezas adquiridos a lo largo del cuatrimestre) y en los cuales sería posible la compensación entre asignaturas. Así se superaría o suspendería todo el bloque, aunque en cada asignatura se obtendría una nota individual.

La capacidad de expresión oral y escrita debe reforzarse a lo largo de la carrera, y debe formar parte del sistema de evaluación del aprendizaje en los últimos cursos.

4.3 Ejemplo: posible planificación de una asignatura de "Ecuaciones Diferenciales Ordinarias".

Asignatura
Ecuaciones Diferenciales Ordinarias

Horas presenciales teóricas: 42
de problemas: 14
de prácticas con ordenador: 7 x 1'5 = 10'5

Horas no presenciales: 110 (7 h a la semana, 3 h de teoría y 4 h de problemas/ prácticas, + 12 h preparación del examen final)

Horas de evaluación: 1'5 h x 4 controles periódicos = 6 horas

Total volumen de trabajo : 182'5 horas

Objetivos de la asignatura
- Conocer y saber utilizar los conceptos y los resultados clásicos relacionados con las Ecuaciones Diferenciales Ordinarias, con especial énfasis en el caso lineal.
- Comprender la imposibilidad de resolver de manera exacta (mediante fórmulas) todas las ecuaciones diferenciales ordinarias (EDO) y la necesidad de utilizar métodos numéricos y/o enfoques cualitativos (teóricos) para su resolución.
- Conocer la relación entre los problemas reales y su modelo matemático en términos de EDO.

Contenidos
- Interpretación de la derivada de una función en sentido geométrico (pendiente de una curva) y en sentido físico (velocidad).
- Ecuaciones diferenciales ordinarias (EDO). Problema de Cauchy: ejemplos con solución única, sin solución y con infinitas soluciones.
- EDO lineales de primer orden. EDO homogénea. EDO no homogénea: método de variación de constantes y método de coeficientes indeterminados.

- EDO no lineales de primer orden. EDO reducibles a lineales: Ecuaciones de Bernoulli y Riccati. Ecuaciones de variables separadas y reducibles a ellas. Ecuaciones exactas. Factores integrantes.
- EDO lineales de segundo orden. Espacio vectorial de las soluciones de la EDO homogénea: soluciones linealmente independientes (wronskiano). EDO con coeficientes constantes: polinomio característico. EDO con coeficientes variables: método de serie de potencias. Reducción de orden.
- Sistemas de EDO lineales de primer orden. Espacio vectorial de las soluciones del sistema homogéneo. Sistemas con coeficientes constantes: exponencial de una matriz. Sistemas no homogéneos: método de variación de constantes y método de coeficientes indeterminados.
- Teoremas de existencia y unicidad de solución para problemas de Cauchy. Condición de Lipschitz. Soluciones aproximadas: Iterantes de Picard.
- Sistemas autónomos. Plano de fases.
- Aplicaciones de las EDO: modelos de población de Malthus y logístico, problemas de enfriamiento, desintegración radioactiva, vibraciones en sistemas mecánicos, modelos en biología (ecuación de Volterra).

Destrezas a adquirir

- Distinguir los diferentes tipos de EDO.
- Conocer los principales métodos para resolver EDO.
- Aplicar correctamente los métodos para resolver EDO sencillas.
- Extraer información cualitativa de las soluciones de una EDO, sin necesidad de resolverla (crecimiento, concavidad, …).
- Utilizar algún software de cálculo simbólico para resolver EDO.
- Utilizar algún software para resolver numéricamente problemas de Cauchy asociados a EDO.
- Analizar teóricamente un problema de Cauchy y concluir que tiene solución única en casos donde se verifica la condición de Lipschitz.
- Traducir algunos problemas “reales” (sacados de la Física, Química, Biología, etc…) en términos de EDO.

Temario (con planificación temporal)

Tema 1. Introducción a las Ecuaciones Diferenciales Ordinarias: Concepto. Interpretación geométrica. Modelos. Ecuaciones lineales/no lineales. Problema de valores iniciales. (1 semana)

Tema 2. Ecuaciones Diferenciales Ordinarias de primer orden: Variables separadas, lineales, homogéneas y reducibles a ellas. Exactas: Factor integrante. (2 semanas)

Tema 3. Ecuaciones diferenciales lineales de orden superior. Caso con coeficientes constantes. Método de los coeficientes indeterminados. Ecuaciones diferenciales lineales de segundo orden con coeficientes variables: Soluciones en forma de serie de potencias. Métodos numéricos: método de Euler y Euler mejorado. (3 semanas)

Tema 4. Teorema de Arzela-Ascoli. Soluciones aproximadas: Poligonales de Euler. Desigualdad de Gronwall. Condición de Lipschitz. Teoremas de existencia y unicidad de solución: Teoremas de Cauchy-Peano y Picard. Iterantes de Picard. Prolongación de soluciones. Soluciones maximales. Dependencia continua y diferenciabilidad respecto de los datos iniciales. (3 semanas)

Tema 5. Sistemas de ecuaciones diferenciales lineales. Métodos matriciales.
(2 semanas)

Tema 6. Sistemas autónomos. Trayectorias. Plano de fases. Concepto de estabilidad de un punto crítico. Método directo de Liapunov: estabilidad, estabilidad asintótica, inestabilidad. (2 semanas)

Bibliografía de referencia

- W. E. Boyce, R. C. DiPrima "Ecuaciones Diferenciales Elementales y Problemas con Valores en la Frontera" Limusa,1998.
- M. Braun "Ecuaciones Diferenciales y sus Aplicaciones" Grupo Editorial Iberoamericano, 1990.
- C. H. Edwards, D. E. Penney "Ecuaciones Diferenciales Elementales con Aplicaciones" Prentice-Hall Hispanoamericana ,1986.
- G. F. Simmons "Ecuaciones Diferenciales" Mc. Graw-Hill, 1993.

Conocimientos previos necesarios
Derivación e integración en una y varias variables. Diagonalización de matrices. Convergencia de series de funciones.

Metodología
3 horas de teoría y 1 hora de problemas a la semana. En las clases de problemas los estudiantes corregirán en la pizarra los problemas propuestos.
Cada dos semanas habrá una sesión de una hora y media en el aula de ordenadores donde se aprenderá a utilizar software de cálculo simbólico aplicado a las ecuaciones diferenciales.

Evaluación del aprendizaje
Cada semana se pondrá a disposición de los estudiantes una hoja con 4 o 5 problemas que deberá ser trabajada (individualmente o en grupos de, a lo sumo, dos personas) y entregada a la semana siguiente. Los problemas entregados por cada estudiante serán corregidos y puntuados, de 0 a 10, por estudiantes de cursos más avanzados (que recibirán créditos de libre elección y beca económica por esta tarea).
Cada cuatro semanas se realizará un control del aprendizaje (examen, exposición de un tema o entrega de un trabajo,...) en el que se evaluará siempre el temario visto desde el primer día de clase. Se hará hincapié en las conexiones entre las asignaturas del cuatrimestre (Cálculo III, Cálculo Numérico I, Probabilidad I y EDO). Todos los controles periódicos contarán igual para la nota final.

Los estudiantes que obtengan al menos el 50% de los puntos en los problemas y un 5 de media en los controles periódicos superarán la asignatura. Para aquellos estudiantes que no alcancen el nivel exigido habrá un sistema de compensación consistente en un examen final conjunto con las asignaturas de Cálculo III, Cálculo Numérico I y Probabilidad I.

5. Conclusiones

El sistema europeo de créditos proporciona un procedimiento estandarizado de medida y comparación del aprendizaje en diferentes contextos, aportando información cualitativa sobre el aprendizaje. Su implantación supone centrar la enseñanza en lo que se aprende y cuánto esfuerzo requiere ese aprendizaje, y hace necesaria la elaboración de guías docentes de cada titulación y cambios en la metodología docente.
Las encuestas dirigidas a estudiantes y profesores aportan información sobre el esfuerzo que requiere el aprendizaje y sirven para detectar anomalías (son un mecanismo de seguimiento), pero no pueden ser la base para la asignación de créditos. Ésta debe hacerse conforme a la guía docente.

El sistema de evaluación del aprendizaje es importante ya que determina en gran parte la actitud del estudiante frente a su aprendizaje y debe planificarse en la guía docente.

6. Referencias

- Declaración de Bolonia, 1999 www.esib.org
- ***El ámbito europeo de la enseñanza superior. Informe de situación y programa de acciones piloto en la Comunidad de Madrid***, Carmen Ruiz-Rivas Hernando, junio 2002
- ***ECTS Guía del usuario***, Comisión Europea, 1998 http://europa.eu.int/comm/education/socrates/usersg.html
- ***The Benchmark document on Mathematics, Statistics and Operational Research***, de la UK Quality Assurance Agency for Higher Education, 2002 www.qaa.ac.uk/cnrtwork/benchmark/phase2/mathematics.pdf
- ***Modalisierung in Hochschulen***, Cuaderno 101, Bund-Länder Kommission für Bildungsplanung und Forschungsförderung (Bund-länder Commission for educational planning and research promotion) www.blk-bonn.de
- Guía para el plan docente de la Diplomatura en Turismo de la Universidad de Barcelona

7. Apéndices

7.1 Apéndice ECTS-1: Formularios encuesta UAM

ENCUESTA ECTS - Estudiantes

El Departamento de Matemáticas de la Universidad Autónoma de Madrid participa en el "Proyecto de Convergencia Europea en Matemáticas", cuyo objetivo es facilitar e impulsar la armonización de la carrera de Matemáticas en las diferentes universidades europeas. Dentro de este proyecto se está trabajando en los créditos **ECTS** (European Credit Transfer System), un sistema desarrollado por la Comisión Europea para medir el aprendizaje de la misma manera en toda la UE y que haga más fácil estudiar en otro país. Los créditos **ECTS** reflejan la carga de trabajo que se exige al estudiante para superar cada asignatura (clases, prácticas, seminarios, **trabajo personal**, exámenes,...). 60 créditos **ECTS** representan un curso académico completo.

Para este proyecto te pedimos una estimación lo más aproximada posible del tiempo total invertido en cada <u>asignatura de primer ciclo de la que te hayas examinado</u> (tanto si la aprobaste como si no). Dentro del tiempo invertido debes contar el tiempo dedicado a las siguientes actividades, siempre que guarden relación con la asignatura:

- estudiar, ya sea de forma individual o en grupo
- pasar apuntes
- hacer problemas
- realizar los exámenes
- preparar trabajos para exponer en clase o para entregar
- hacer las prácticas
- utilizar el ordenador para preparar o para buscar material docente
- hacer gestiones en la biblioteca, en la fotocopiadora, etc.

y, en general, el tiempo dedicado a cualquier actividad relacionada con la asignatura **excepto** la asistencia a clase y los desplazamientos (de casa a la facultad, por ejemplo)

Observación: Un cuatrimestre consta de 14 semanas lectivas. Intenta estimar el número de horas que dedicaste a la asignatura cada semana y luego súmalas. También debes añadir el tiempo que dedicaste a estudiar durante los periodos no lectivos (fines de semana, vacaciones, periodo de exámenes). La encuesta es anónima y sólo se utilizarán los datos promediados.

Asignatura	**Calificación obtenida**	**Convocatoria * (1ª, 2ª,...)**	**Número de horas**
Cálculo I			
Conjuntos y Números			
Álgebra lineal			
Cálculo II			
Geometría I			
Informática			
Cálculo III			
Cálculo Numérico I			
Probabilidad I			
Ec. Diferenciales Ordinarias			
Topología			
Geometría II			
Modelización I			
Física para Matemáticos			

***** Convocatoria en la que aprobaste o en la que te encuentras

Muchas gracias.

Encuesta ECTS - Profesores

Querido/a ... :

El Departamento de Matemáticas participa en el "Proyecto de Convergencia Europea en Matemáticas", cuyo objetivo es facilitar e impulsar la armonización de la carrera de Matemáticas en las diferentes universidades europeas. Dentro de este proyecto se está trabajando en los créditos **ECTS** (European Credit Transfer System), un sistema desarrollado por la Comisión Europea para medir el aprendizaje de la misma manera en toda la UE y que haga más fácil estudiar en otro país. Los créditos **ECTS** reflejan la carga de trabajo que se exige al estudiante para superar cada asignatura (clases, prácticas, seminarios, **trabajo personal**, exámenes,...). 60 créditos **ECTS** representan un curso académico completo.

Para este proyecto te pedimos una estimación del número de horas que un estudiante medio debería dedicar **fuera de clase** a la asignatura que impartes para superarla.

Asignatura: ...

	Número de horas a la semana	Número de horas en todo el cuatrimestre
Estudiar la teoría		
Hacer los problemas		
Preparar prácticas/trabajos obligatorios		
Preparar prácticas/trabajos opcionales		
Otros (pasar apuntes, gestiones en la biblioteca, en la fotocopiadora,...)		

Observación: Un cuatrimestre consta de 14 semanas lectivas. Indica el número de horas a la semana o en todo el cuatrimestre según sea más fácil de estimar. No olvides tener en cuenta el tiempo que se debería dedicar a la asignatura durante los periodos no lectivos (fines de semana, vacaciones, periodo de exámenes).

Muchas gracias.

7.2 Apéndice ECTS-2: Resultados encuesta

UNIVERSIDAD AUTÓNOMA DE BARCELONA

Primer Curso (1er semestre)

Asignatura	Horas aula	Media horas estudio alumnos		Total horas aula + estudio alumnos
		todos	Sólo aprob	
Introducción al Álgebra Lineal	75	57,46	58,16	132,46
Cálculo Infinitesimal	75	76,15	77,19	151,15
Matemática Discreta	60	59,55	62,50	119,55
Informática	60	41,59	43,28	101,59
Prácticas Integradas	30	16,90	13,63	46,90
Total	300	251,65		**551,65**

Segundo Curso

Asignatura	Horas aula	Media horas estudio alumnos		Total horas aula + estudio alumnos
		todos	Sólo aprob	
Análisis Matemático I	90	82,59	60,75	172,59
Geometría Lineal	75	57,46	58,77	132,46
Elementos de Física	75	45,15	46,25	120,15
Métodos Numéricos (1er sem.)	60	43,29	55,63	103,29
Análisis Matemático II	90	91,82	78,46	181,82
Geometría Proyectiva	75			75
Fundamentos de Álgebra	90	103,18	93,89	193,18
Total	555	423,48		**978,48**

Tercer Curso

Asignatura	Horas aula	Media horas estudio alumnos		Total horas aula + estudio alumno
		todos	Sólo aprob	
Análisis vectorial	60			
Probabilidad	75	86,00	91,32	161,00
Topología I	75	87,26	89,40	162,26
Modelos con Ec. Diferenciales	75			
Geometría diferencial	75			
Estadística	90			
Análisis complejo	75			
Ecuaciones diferenciales	75	95,75	112,00	170,75

Cuarto y Quinto Curso

Asignatura	Horas aula	Media horas estudio alumnos		Total horas aula + estudio alumno
		todos	sólo aprob	
Análisis Real y Funcional	90	103,08	106,90	193,08
Topología II	60	86,53	91,38	146,53
Álgebra	90			
Cálculo Numérico	90	173,70	184,11	263,70
Análisis de Fourier y EDP	90	106,25	106,25	196,25
Geometría de variedades	60			

TIEMPO MEDIO DE ESTUDIO POR ASIGNATURA Y CALIFICACIÓN

Primer Curso (1er semestre)

Asignatura \ Calificación	Suspenso		Aprobado		Not / Sobr / MH	
	nº resp	horas	nº resp	horas	nº resp	horas
Introducción al Álgebra Lineal	16	56,63	14	59,64	5	54,00
Cálculo Infinitesimal	28	75,18	20	78,50	6	72,83
Matemática Discreta	16	56,59	8	64,13	8	60,88
Informática	7	35,57	20	45,35	5	35,00
Prácticas Integradas	20	22,15	23	14,52	9	11,33

Segundo Curso

Asignatura \ Calificación	Suspenso		Aprobado		Not / Sobr / MH	
	nº resp	horas	nº resp	horas	nº resp	horas
Análisis Matemático I	13	89,31	4	60,75	0	
Geometría Lineal	1	43,00	0		11	58,77
Elementos de Física	13	43,46	16	49,31	4	34,00
Métodos Numéricos (1er sem.)	6	26,83	4	52,50	4	58,75
Análisis Matemático II	9	111,11	11	74,55	2	100,00
Geometría Proyectiva						
Fundamentos de Álgebra	2	145,00	4	108,50	5	82,20

Tercer Curso

Asignatura \ Calificación	Suspenso		Aprobado		Not / Sobr / MH	
	nº resp	horas	nº resp	horas	nº resp	horas
Análisis vectorial						
Probabilidad	8	69,38	12	95,50	13	87,46
Topología I	4	79,25	11	78,45	4	119,50
Modelos con Ec. Diferenciales						
Geometría diferencial						
Estadística						
Análisis complejo						
Ecuaciones diferenciales	7	65,57	10	110,60	3	116,67

Cuarto y Quinto Curso

Asignatura \ Calificación	Suspenso		Aprobado		Not / Sobr / MH	
	nº resp	horas	nº resp	horas	nº resp	horas
Análisis Real y Funcional	14	100,36	7	112,71	3	93,33
Topología II	3	60,67	11	101,27	5	69,60
Álgebra						
Cálculo Numérico	1	80,00	5	199,00	4	165,50
Análisis de Fourier y EDP	0		3	101,67	1	120,00
Geometría de variedades						

RESUMEN DE LOS RESULTADOS DE LOS ALUMNOS QUE SUPERAN LA ASIGNATURA

Primer curso (1er semestre)

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas Estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Introducción al Álgebra Lineal	7,5	75	19	58,16	133,16	**18**
Cálculo Infinitesimal	7,5	75	26	77,19	152,19	**20**
Matemática Discreta	6	60	16	62,50	122,50	**20**
Informática	6	60	25	43,28	103,28	**17**
Prácticas Integradas	3	30	32	13,63	43,63	**15**

Segundo curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas Estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Análisis Matemático I	9	90	4	60,75	150,75	**17**
Geometría Lineal	7,5	75	11	58,77	133,77	**18**
Elementos de Física	7,5	75	20	46,25	121,25	**16**
Métodos Numéricos (1er sem.)	6	60	8	55,63	115,63	**19**
Análisis Matemático II	9	90	13	78,46	168,46	**19**
Geometría Proyectiva	7,5	75	0			
Fundamentos de Álgebra	9	90	9	93,89	183,89	**20**

Tercer curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas Estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Análisis vectorial	6	60	0			
Probabilidad	7,5	75	25	91,32	166,32	**22**
Topología I	7,5	75	15	89,4	164,4	**22**
Modelos con Ec. Diferenciales	7,5	75	0			
Geometría diferencial	7,5	75	0			
Estadística	9	90	0			
Análisis complejo	7,5	75	0			
Ecuaciones diferenciales	7,5	75	13	112	187	**25**

Cuarto y Quinto curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas Estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Análisis Real y Funcional	9	90	10	106,90	196,90	**22**
Topología II	6	60	16	91,38	151,38	**25**
Álgebra	9	90	0			
Cálculo Numérico	9	90	9	184,11	274,11	**30**
Análisis de Fourier y EDP	9	90	4	106,25	196,25	**22**
Geometría de variedades	6	60	0			

UNIVERSIDAD AUTÓNOMA DE MADRID

Primer curso

Además habría que añadir 5 créditos de libre elección

Asignatura	Horas aula Teoría	Horas aula problemas/ práct	Media horas estudio alumnos		Horas estudio profesores	Total horas aula + estudio alumno
			todos	Sólo aprob		
Cálculo I	56	28	144,1	144,4	266	228,1
Conjuntos y Números	56	28	142,4	142,4	140	226,4
Álgebra Lineal	56	28	152,7	142,9	115,5	236,7
Cálculo II	56	28	120,7	127,1	117,5	204,7
Geometría I	56	28	238,3	244,5	142	322,3
Informática	42	56	136,8	136,8	98	234,8
Total	322	196	935		879	**1453**

Segundo curso

Además habría que añadir 7 créditos de libre elección

Asignatura	Horas aula teoría	Horas aula problemas/ práct	Media horas estudio alumnos		Horas estudio profesores	Total horas aula + estudio alumno
			todos	Sólo aprob		
Cálculo III	56	0	139,5	139,5	100,5	195,5
Cálculo Numérico I	42	56	240,4	240,4	140	338,4
Probabilidad I	56	21	120,5	128,5	105,5	197,5
E.D.O.	56	0	152	157,8	210	208
Topología	56	0	153,1	153,6	147,5	209,1
Geometría II	56	0	146,9	146,9	89	202,9
Modelización I	56	0	98,3	98,3	110	154,3
Física	56	0	146,4	146,4	60,5	202,4
Total	434	77	1197,1		963	**1708,1**

TIEMPO MEDIO DE ESTUDIO POR ASIGNATURA Y CALIFICACIÓN

Primer Curso

Asignatura \ Calificación	Suspenso		Aprobado		Notable / Sobresal / MH	
	nº respuestas	Tiempo medio	nº respuestas	Tiempo medio	nº respuestas	tiempo medio
Cálculo I	1	140	6	132,5	8	153,3
Conjuntos y Números	0	0	5	112	9	159,2
Álgebra Lineal	1	280	5	132	8	149,8
Cálculo II	1	50	8	123,5	3	136,7
Geometría I	1	170	5	118	6	350
Informática	0	0	9	140	2	122,5

Segundo Curso

Asignatura \ Calificación	Suspenso		Aprobado		Notable / Sobresal / MH	
	nº respuestas	Tiempo medio	nº respuestas	Tiempo medio	nº respuestas	tiempo medio
Cálculo III	0	0	6	125,8	4	160
Cálculo Numérico I	0	0	5	202	7	267,9
Probabilidad I	1	40	8	134,4	2	105
E.D.O.	1	100	6	153,3	3	166,7
Topología	1	150	3	163,3	4	146,3
Geometría II	0	0	6	164,2	2	95
Modelización I	0	0	4	105	5	93
Física	0	0	4	120	3	181,7

RESUMEN DE LOS RESULTADOS DE LOS ALUMNOS QUE SUPERAN LA ASIGNATURA

Primer curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Cálculo I	12	84	14	144,36	228,36	**19**
Conjuntos y Números	12	84	14	142,36	226,36	**19**
Álgebra Lineal	12	84	13	142,92	226,92	**19**
Cálculo II	12	84	11	127,09	211,09	**18**
Geometría I	12	84	11	244,55	328,55	**27**
Informática	10	98	11	136,82	234,82	**23**

Segundo Curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Cálculo III	8	56	10	139,50	195,50	**24**
Cálculo Numérico I	8	98	12	240,42	338,42	**42**
Probabilidad I	8	77	10	128,50	205,50	**26**
E.D.O.	8	56	9	157,78	213,78	**27**
Topología	8	56	7	153,57	209,57	**26**
Geometría II	8	56	8	146,88	202,88	**25**
Modelización I	8	56	9	98,33	154,33	**19**
Física	8	56	7	146,43	202,43	**25**

UNIVERSIDAD DE CANTABRIA

Primer Curso

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Álgebra Básica 1	70	104,83	123,92	66,67	174,83
Álgebra Básica 2	70	107,22	152	51,25	177,22
Análisis de una variable real	70	101,16	99,67	106,75	171,16
Ampliación de análisis de una variable real	84	120,00	120,00		204,00
Geometría básica	56	87,32	88,50	84,00	143,32
Informática	84	71,32	91,92	26,67	155,32
Estadística básica	56	54,63	60,29	15,00	110,63
Álgebra Lineal 1	70	167,22	185,00	25,00	237,22
Total	588	813,69			**1401,69**

Segundo Curso

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Cálculo Numérico 1	56	91,25	143,50	39,00	147,25
Álgebra Lineal 2	70	133,33	210,00	56,67	203,33
Cálculo de Probabilidad	70	83,88	83,88		153,88
Análisis de varias variables reales	84	254,29	228,00	320,00	338,29
Teoría de grupos					
Ampliación de An. V. Var. Reales					
Topología					
Inferencia estadística					

TIEMPO MEDIO DE ESTUDIO POR ASIGNATURA Y CALIFICACIÓN

Primer Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobresal / MH	
	nº respuestas	horas	nº respuestas	Horas
Álgebra Básica 1	6	66,67	12	123,92
Álgebra Básica 2	4	51,25	5	152
Análisis de una variable real	4	106,75	15	99,67
Ampliación de análisis de una variable real	0		10	120,00
Geometría básica	5	84,00	14	88,50
Informática	6	26,67	13	91,92
Estadística básica	1	15,00	7	60,29
Álgebra Lineal 1	1	25,00	8	185,00

Segundo Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobresal / MH	
	nº respuestas	horas	nº respuestas	Horas
Cálculo Numérico 1	4	39,00	4	143,5
Álgebra Lineal 2	3	56,67	3	210
Cálculo de Probabilidad	0		8	83,88
Análisis de varias variables reales	2	320,00	5	228

RESUMEN DE LOS RESULTADOS DE LOS ALUMNOS QUE SUPERAN LA ASIGNATURA

Asignatura	Créditos asignatura	Horas de contacto	Número de Encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Álgebra Básica 1	7,5	70	12	123,92	193,92	**26**
Álgebra Básica 2	7,5	70	5	152	222,00	**30**
Análisis de una variable real	7,5	70	15	99,67	169,67	**23**
Ampliación de análisis de una variable real	9	84	10	120,00	204,00	**23**
Geometría básica	6	56	14	88,50	144,50	**24**
Informática	12	84	13	91,92	175,92	**15**
Estadística básica	6	56	7	60,29	116,29	**19**
Álgebra Lineal 1	7,5	70	8	185,00	255,00	**34**
Cálculo Numérico 1	6	56	4	143,5	199,50	**33**
Álgebra Lineal 2	7,5	70	3	210	280,00	**37**
Cálculo de Probabilidad	7,5	70	8	83,88	153,88	**21**
Análisis de varias variables reales	9	84	5	228	312,00	**35**

UNIVERSIDAD DE SANTIAGO DE COMPOSTELA

Primer Curso

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Introducción á análise matemática	90	85,83	67	180	175,83
Xeometría métrica	90	127,77	129,28	90	217,77
Álxebra lineal e multilineal	90	118,13	122,14	90	208,13
Cálculo diferencial e integral	90	121,96	116,26	140	211,96
Informática	90	35	45	25	125
Introducción ó cálculo numérico	75	125,26	123,89	150	200,26
Topoloxía dos espacios euclidianos	75	81,19	80,25	100	156,19
Total	600	695,14			**1295,14**

Segundo Curso Habría que añadir 7,5 créditos libre elección

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Análise numérica matricial	60	174,80	163,87	207,6	234,80
Difer. de func. de vv. vv. reais	75	82,64	79,56	86,33	157,64
Integ. de func. de vv. vv. reais	75	134,47	140,92	92,5	209,47
Int. ás ecuacións diferen. ordinarias	75	121,90	130,62	105,71	196,90
Int. ó cálculo de probabilidades	60	93,25	94,12	88,33	153,25
Xeometría afín e proxectiva	90	121,53	129,12	114,74	211,53
Topoloxía	90	121,23	107,5	129,9	211,23
Total	525	849,82			**1374,82**

Tercer Curso Habría que añadir 6 créditos libre elección

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Curvas e superficies	90	129,75	134,68	111	219,75
Elementos de variable complexa	60	91,53	91,6	91,43	151,53
Inferencia estatística	75	148,79	150,23	130	223,79
Introducción á álxebra	75	157,95	162,57	155,64	232,95
Métodos numéricos	60	213,4	212,71	215	273,4
Series de Fourier e int. ás EDPs	45	93,3	99,05	79,9	138,3
Teoría global de superficies	75	151,25	151,36	150	226,25
Vectores aleatorios	60	97,14	104,29	90	157,14
Total	540	1083,11			**1623,11**

Cuarto Curso Habría que añadir 4,5 cr optativos y 7,5 cr libre elección

Asignatura	Horas aula	Media horas estudio alumnos			Total horas aula + estudio alumnos (todos)
		todos	sólo aprob	sólo susp	
Xeometría e topoloxía	95	167,03	169,36	157,7	262,03
Teoría da medida	60	144,84	147,89	140	204,84
Álxebra	95	192,71	209,78	173,5	287,71
Análise func. en espacios de Banach	75	179,38	240	143	254,38
Cálculo numérico	90	232,31	228,18	255	322,31
Ecuacións diferenciais ordinarias	60	171,5	250	162,78	231,5
Total	475	1087,77			**1562,77**

Variable complexa	50	110	103,75	120	160

TIEMPO MEDIO DE ESTUDIO POR ASIGNATURA Y CALIFICACIÓN

Primer Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobr / MH	
	nº resp	Horas	nº resp	horas
Introducción á análise matemática	1	180	5	67
Xeometría métrica	1	90	25	129,28
Álxebra lineal e multilineal	1	90	7	122,14
Cálculo diferencial e integral	6	140	19	116,26
Informática	4	25	4	45
Introducción ó cálculo numérico	1	150	18	123,89
Topoloxía dos espacios euclidianos	1	100	20	80,25

Segundo Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobr / MH	
	nº resp	Horas	nº resp	horas
Análise numérica matricial	5	207,6	15	163,87
Difer. de func. de vv. vv. reais	15	86,33	18	79,56
Integ. de func. de vv. vv. reais	2	92,5	13	140,92
Int. ás ecuacións diferen. ordinarias	7	105,71	13	130,62
Int. ó cálculo de probabilidades	3	88,33	17	94,12
Xeometría afín e proxectiva	19	114,74	17	129,12
Topoloxía	19	129,9	12	107,5

Tercer Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobr / MH	
	nº resp	horas	nº resp	horas
Curvas e superficies	5	111	19	134,68
Elementos de variable complexa	7	91,43	10	91,6
Inferencia estatística	2	130	26	150,23
Introducción á álxebra	14	155,64	7	162,57
Métodos numéricos	6	215	14	212,71
Series de Fourier e int. ás EDPs	9	79,9	21	99,05
Teoría global de superficies	2	150	22	151,36
Vectores aleatorios	7	90	7	104,29

Cuarto Curso

Asignatura \ Calificación	Suspenso		Aprob / Not / Sobr / MH	
	nº resp	horas	nº resp	horas
Xeometría e topoloxía	7	157,7	28	169,36
Teoría da medida	12	140	19	147,89
Álxebra	8	173,5	9	209,78
Análise func. en espacios de Banach	5	143	3	240
Cálculo numérico	2	255	11	228,18
Ecuacións diferenciais ordinarias	9	162,78	1	250

Variable complexa	5	120	8	103,75

RESUMEN DE LOS RESULTADOS DE LOS ALUMNOS QUE SUPERAN LA ASIGNATURA

Primer Curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Introducción á análise matemática	9	90	5	67	157	**17**
Xeometría métrica	9	90	25	129,28	219,28	**24**
Álxebra lineal e multilineal	9	90	7	122,14	212,14	**24**
Cálculo diferencial e integral	9	90	19	116,26	206,26	**23**
Informática	9	90	4	45	135	**15**
Introducción ó cálculo numérico	7,5	75	18	123,89	198,89	**27**
Topoloxía dos espacios euclidianos	7,5	75	20	80,25	155,25	**21**

Segundo Curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Análise numérica matricial	6	60	15	163,87	223,87	**37**
Difer. de func. de vv. vv. reais	7,5	75	18	79,56	154,56	**21**
Integ. de func. de vv. vv. reais	7,5	75	13	140,92	215,92	**29**
Int. ás ecuacións diferen. ordinarias	7,5	75	13	130,62	205,62	**27**
Int. ó cálculo de probabilidades	6	60	17	94,12	154,12	**26**
Xeometría afín e proxectiva	9	90	17	129,12	219,12	**24**
Topoloxía	9	90	12	107,5	197,5	**22**

Tercer Curso

Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Curvas e superficies	9	90	19	134,68	224,68	**25**
Elementos de variable complexa	6	60	10	91,6	151,6	**25**
Inferencia estatística	7,5	75	26	150,23	225,23	**30**
Introducción á álxebra	7,5	75	7	162,57	237,57	**32**
Métodos numéricos	6	60	14	212,71	272,71	**45**
Series de Fourier e int. ás EDPs	4,5	45	21	99,05	144,05	**32**
Teoría global de superficies	7,5	75	22	151,36	226,36	**30**
Vectores aleatorios	6	60	7	104,29	164,29	**27**

Cuarto Curso

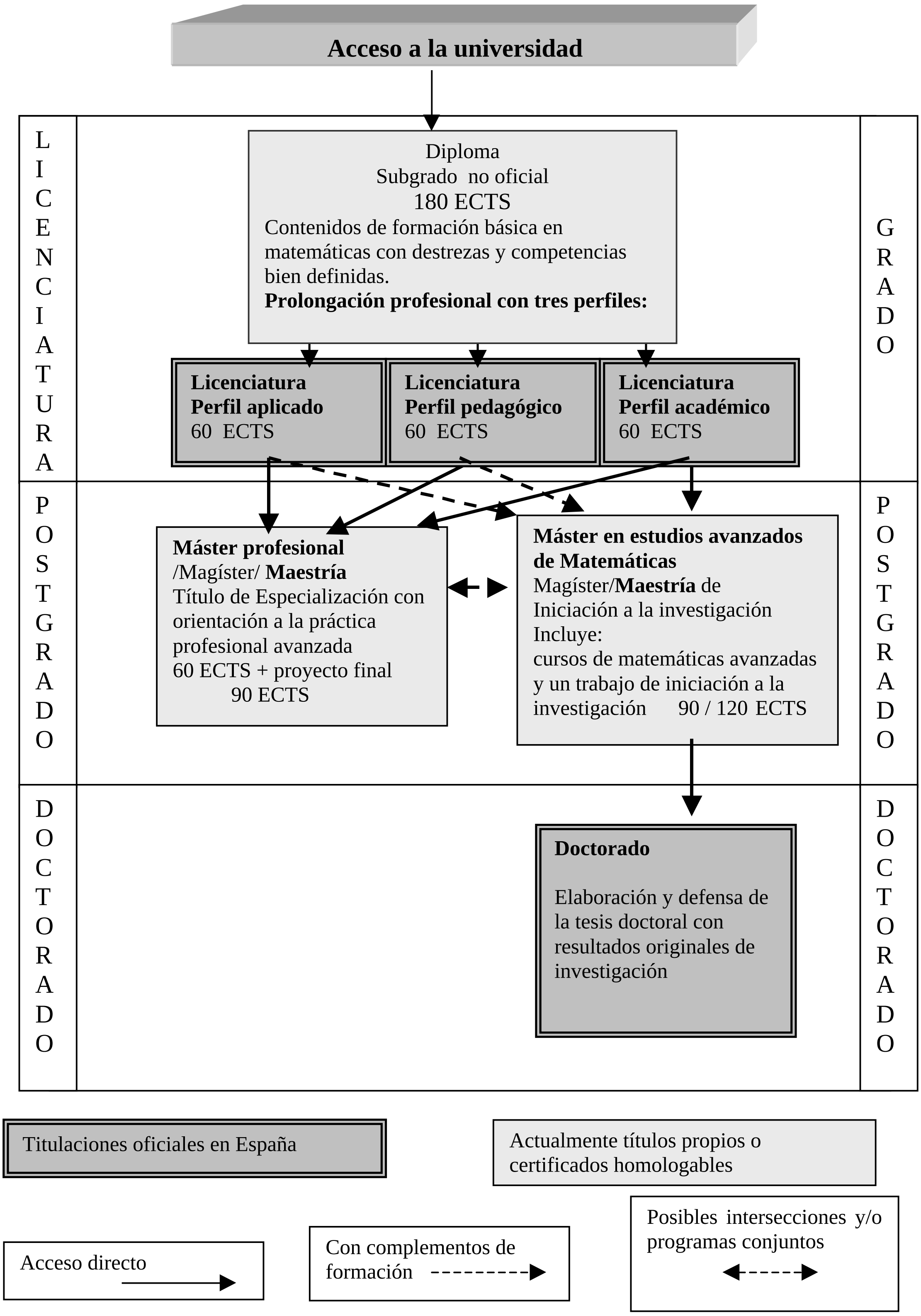
Asignatura	Créditos asignatura	Horas de contacto	Número de encuestados	Media horas estudio aprobados	Horas totales (contacto+estudio)	Horas totales por crédito
Xeometría e topoloxía	9,5	95	28	169,36	264,36	**28**
Teoría da medida	6	60	19	147,89	207,89	**35**
Álxebra	9,5	95	9	209,78	304,78	**32**
Análise func. en espacios de Banach	7,5	75	3	240	315	**42**
Cálculo numérico	9	90	11	228,18	318,18	**35**
Ecuacións diferenciais ordinarias	6	60	1	250	310	**52**

Variable complexa	5	50	8	103,75	153,75	**31**

Anexo 4

Propuesta de esquema general de estructura de los estudios

Propuesta de
esquema general para los estudios de matemáticas



El Grado en Matemáticas
Licenciatura en Matemáticas

El Grado en Matemáticas debe posibilitar el acceso directo al mercado de trabajo en puestos con un nivel alto de responsabilidad. Las administraciones públicas deben aceptar que el Grado en Matemáticas dé acceso al grupo A en la función pública.

Para garantizar ambas condiciones, sin olvidar la tradición española, el nombre del Grado debería ser **“Licenciado en Matemáticas”**.

El título de Licenciado en Matemáticas debe cualificar para la formulación matemática, análisis, resolución y, en su caso, tratamiento informático de problemas en diversos campos interdisciplinares de las ciencias básicas, ciencias sociales y de la vida, ingeniería, finanzas, consultoría, etc..., con vistas a las aplicaciones, los desarrollos científicos y/o la docencia.

El documento completo y detallado con los objetivos generales, contenidos básicos y destrezas a adquirir se encuentra en el anexo 5. A continuación se resumen algunos aspectos.

Contenidos

Quienes obtengan el grado de Licenciado en Matemáticas deben conocer y entender los métodos y técnicas básicas de las matemáticas a un nivel que les permita utilizarlas con eficacia para realizar tareas con contenido matemático en su vida laboral.

Todos los programas deben incluir como bases comunes: calculo en una y varias variables reales y álgebra lineal. Además, los graduados deben estar familiarizados con las principales áreas de las matemáticas, no sólo las que han guiado históricamente la actividad matemática, sino también las de origen más moderno. Por tanto, todos los graduados en matemáticas deben conocer las ideas básicas de:

- ecuaciones diferenciales
- funciones de variable compleja
- probabilidad
- estadística
- métodos numéricos
- geometría de curvas y superficies
- estructuras algebraicas
- matemática discreta

El conocimiento de otros métodos, técnicas y contenidos puede depender en gran medida de las características concretas del programa y del perfil profesional elegido pero es importante que todos los graduados hayan alcanzado un nivel más elevado en algún área.

Es necesario que todos los graduados hayan tratado al menos un campo en el que las matemáticas se apliquen de una manera que se considere esencial para la comprensión del campo en cuestión y su aplicación profesional.

Destrezas

La gran variedad de salidas profesionales que se ofrecen hoy en día a los graduados en matemáticas son, en gran medida, consecuencia del valor que los empleadores otorgan a la capacidad para el razonamiento riguroso, el análisis cuantitativo y la resolución de problemas, que caracterizan a los licenciados en matemáticas.

Las tres destrezas clave que deberían adquirirse durante este periodo son por tanto:

- la capacidad para idear una demostración,
- la habilidad para modelar una situación,
- la facilidad para resolver problemas, incluida la búsqueda de soluciones numéricas.

Es evidente que, hoy en día, encontrar soluciones numéricas a un problema requiere conocimientos sólidos de programación y algoritmos.

No es necesario recordar que estas destrezas se desarrollan progresivamente. No se empiezan los estudios de matemáticas con un curso llamado “Cómo hacer una demostración” y otro llamado “Cómo construir un modelo”, de modo que ambas cosas se aprendan inmediatamente. Por el contrario, todos los cursos del grado deben dirigirse a desarrollar las tres destrezas básicas.

Duración y organización

Hay en principio y dentro de los esquemas europeos, dos alternativas para la duración del Grado en Matemáticas: 3 años (180 créditos ECTS) y 4 años (240 créditos ECTS). Ambas tienen ventajas e inconvenientes y, sobre todo, responden a concepciones que, aunque parecidas, no son idénticas.

Se podría optar por **3 años** si se desease dar en el grado únicamente la formación básica. Las ventajas de esta alternativa son que permite una más rápida inserción en el mercado de trabajo (aunque sin un perfil definido) y que los alumnos desencantados por uno u otro motivo podrían dejar antes sus estudios con un título oficial. Una desventaja evidente en el contexto español sería la dificultad para el reconocimiento de un título alcanzado en tres años como correspondiente a una Licenciatura.

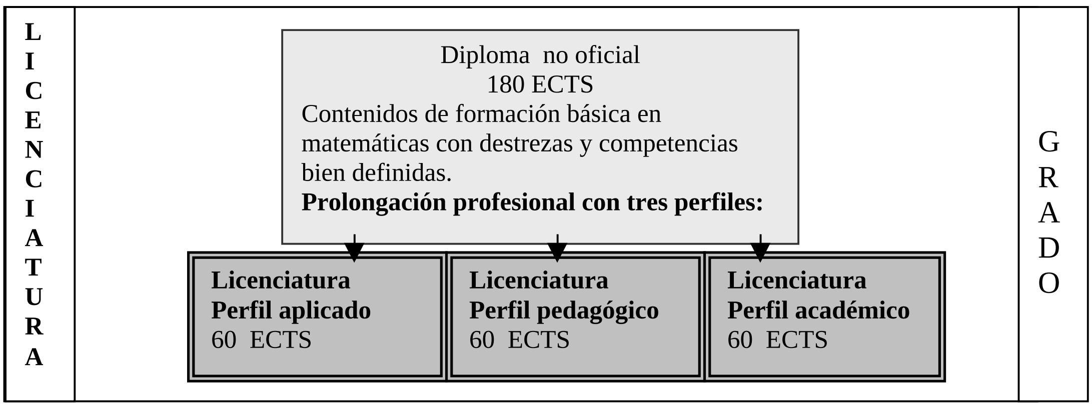
La opción de **4 años,** que corresponde a nuestra propuesta, permite completar la formación básica con una formación específica, que dependería de las aspiraciones profesionales futuras del estudiante, en el cuarto año. Se podrían diseñar, al menos, tres perfiles profesionales correspondientes a las tres actividades mayoritarias de los licenciados en matemáticas:

- Académico: dirigido a quienes deseen seguir una carrera como investigador o profesor universitario. Estaría centrada sobre todo en la matemática más pura o "teórica" (en todas las áreas, incluidas las asociadas con las aplicaciones).
- Didáctico: dirigida a los futuros profesores de enseñanza secundaria. El cuarto año debería incluir también la obtención del Certificado oficial de Capacitación Pedagógica (o equivalente) y por tanto ser el requisito para optar a la enseñanza pública de las matemáticas en la secundaria.
- Aplicado: para quienes quieran encaminarse a la industria o la empresa. Sus contenidos serían claramente aplicados y deberían complementarse con prácticas concretas.

Cada universidad, según su capacidad e intereses podría ofrecer los tres perfiles o sólo alguno de ellos; también podría diseñar programas conjuntos orientados a que el estudiante pueda realizar varias opciones.

Consideramos que esta opción es más realista y adecuada para nuestra tradición. Sin embargo creemos interesante mantener el diseño de un primer diploma, no necesariamente oficial, de 3 años que recoja los contenidos básicos y las destrezas correspondientes a una formación generalista en matemáticas y que pueda capacitar para, mediante las oportunas pasarelas, acceder a otros estudios.

La propuesta, por tanto se estructura de la siguiente forma:



El anexo 5 detalla la propuesta sobre los objetivos contenidos y destrezas correspondientes a la formación básica del graduado y los tres perfiles profesionales.

El postgrado en Matemáticas
Máster y Doctorado

El postgrado previo a la elaboración de una tesis doctoral de investigación debe denominarse **Máster** como en la actualidad en la mayoría de los países europeos incorporados al proceso abierto en Bolonia. En el anexo 6 se encuentran algunos ejemplos.

Nuestra propuesta contempla dos tipos de máster:

Máster profesional (90 créditos ECTS)
- Acceso directo desde cualquier perfil de los establecidos en los estudios de licenciatura.
- Prueba de acceso o, en su caso, complementos para estudiantes con otro tipo de estudios previos de grado.
- Orientación más aplicada o profesional.
- Consiste en cursos de formación profesional avanzada en ámbitos concretos y debería finalizar con un “proyecto fin de máster” (un trabajo que requiera una apreciable cantidad de trabajo personal) o con prácticas avanzadas en la industria o la empresa.
- Ejemplos: matematicas de la ingeniería, matemáticas de las finanzas, estadística aplicada, formación avanzada en enseñanza de las matemáticas,…

Máster científico / Estudios Avanzados (90/120 créditos ECTS)
- Acceso directo desde el perfil académico de la licenciatura y con complementos para los otros perfiles que deben establecerse de manera específica al presentar la programación de máster científico concreto
- Orientación más académica de iniciación a la investigación.
- Otorga la suficiencia investigadora y puede prolongarse con la elaboración de una tesis doctoral.
- Consiste en cursos que completen la formación académica avanzada en uno o varios campos de la matemática más un trabajo de iniciación a la investigación que puede servir de punto de partida para una tesis doctoral.

Sólo el máster científico de iniciación a la investigación otorga la suficiencia investigadora y por tanto el acceso a la elaboración de la tesis conducente al grado de Doctor. Consideramos posible y , en algunos casos, recomendable la organización de programas conjuntos con una amplia intersección común entre los dos tipos de máster, de forma que se pueda obtener un perfil de investigador adaptado a las actividades de I+D de industrias y empresas.

Doctorado
El título de **Doctor** se obtendrá tras la defensa y aprobación de una tesis doctoral con resultados originales de investigación. Será posterior a la obtención del título de máster científico.

El anexo 6 presenta varios ejemplos de postgrado de diversos tipos.

Anexo 5

Propuesta de contenidos básicos y destrezas a adquirir para la obtención del grado de Licenciado en Matemáticas: Formación generalista + tres perfiles profesionales

CONTENIDOS BÁSICOS Y DESTREZAS A ADQUIRIR EN LOS TRES PRIMEROS CURSOS DE LA LICENCIATURA EN MATEMÁTICAS

1. Objetivos y destrezas de carácter general

Objetivos:

Desarrollar las capacidades analíticas y el pensamiento lógico y riguroso de los alumnos a través del estudio de las matemáticas.

Adquirir la capacidad de utilizar los conocimientos teóricos y prácticos aprendidos, en la definición de problemas y en la búsqueda de soluciones en contextos académicos o empresariales.

Preparar para posteriores estudios especializados, tanto en una disciplina matemática como en cualquiera de las ciencias que requieran buenos fundamentos matemáticos

Destrezas:

1. Destrezas teóricas

- Asimilar la definición de un nuevo objeto matemático, en términos de otros ya conocidos, y ser capaz de utilizar este objeto con corrección y exactitud.

- Reconocer razonamientos correctos en demostraciones sobre objetos matemáticos e identificar falacias o errores en razonamientos incorrectos

- Comprender y utilizar con soltura el lenguaje matemático y conocer los distintos tipos de demostración.

- Conocer demostraciones rigurosas de algunos resultados clásicos en distintos campos matemáticos. Poder idear demostraciones nuevas de enunciados que se le proporcionen

2. Resolución de problemas

- Conocer métodos para plantear la resolución de un problema dado, en diversos campos matemáticos para llevarlo a una solución exacta o a una solución numérica aproximada.

- Aplicar métodos de aproximación adecuados al problema y a las herramientas de que se disponga

- Trabajar en la solución de problemas con restricciones de tiempo y recursos.

3. **Modelización**

- Proponer modelos de situaciones reales sencillas, utilizando las herramientas matemáticas más adecuadas a los fines que se persigan.
- Seleccionar definiciones y objetivos para una modelización
- Interpretar resultados obtenidos a través de un modelo

2. Objetivos, contenidos mínimos y destrezas por materias

(no es necesario identificar *materia* con *asignatura*)

Álgebra Lineal

Objetivos:

Asimilar y manejar con toda fluidez los principales conceptos del Álgebra Lineal, incluyendo en este las Geometrías Afín y Euclídea.

Proporcionar la capacidad de realizar la traducción (y la correspondiente resolución), en términos de matrices, de todos aquellos problemas que surgen en la manipulación de los espacios vectoriales y de las aplicaciones lineales.

Contenidos mínimos:

- Espacios vectoriales y aplicaciones lineales
- Sistemas de ecuaciones lineales y matrices.
- Autovalores y autovectores. Forma Canónica de Jordan.
- Aplicaciones bilineales y formas cuadráticas. Diagonalización.
- Espacios afines y euclideos. Transformaciones. Clasificación de cónicas y cuádricas.

Destrezas:

- Operar con vectores, bases, subespacios y aplicaciones lineales.
- Resolver sistemas de ecuaciones lineales.
- Decidir si una matriz (o endomorfismo) es o no diagonalizable (sobre **R** o **C**) calculando la forma diagonal.
- Calcular la Forma Canónica de Jordan de una matriz.
- Diagonalizar una forma cuadrática y calcular su signatura (en el caso real).
- Decidir si dos matrices dadas son equivalentes, semejantes o congruentes.
- Uso del método Gram-Schmidt para calcular bases ortonormales,

subespacios ortogonales y proyecciones ortogonales.

- Operar con puntos y vectores en un espacio afín así como con los sistemas de referencia afines, los subespacios afines y sus problemas de intersección y las transformaciones afines siendo capaz de calcular los elementos característicos de las traslaciones, homotecias, simetrías y proyecciones.

- Operar con puntos, vectores, distancias y ángulos en un espacio euclídeo. Efectuar cambios de sistemas de referencia rectangulares.

- Clasificar una isometría del plano o del espacio determinando su tipo y elementos característicos.

- Clasificar cónicas y cuádricas.

Cálculo Diferencial e Integral

Objetivos:

Conocer y saber utilizar los conceptos y los resultados fundamentales del Cálculo Diferencial e Integral para funciones de un número finito de variables reales, así como del Cálculo Vectorial clásico.

Conocer y comprender demostraciones de algunos de los teoremas más importantes.

Contenidos mínimos:

- Estructura del cuerpo ordenado de los números reales. Topología de $\mathbf{R}^n$.

- Convergencia de sucesiones en $\mathbf{R}$ y en $\mathbf{R}^n$. Series numéricas.

- Continuidad de funciones reales de una variable real y de funciones vectoriales de variable vectorial.

- Diferenciación de funciones reales de una variable. Desarrollos en serie. Extremos de funciones.

- Convergencia de sucesiones de funciones. Series de potencias.

- Funciones elementales.

- Derivadas parciales. Teorema de la función inversa. Teorema de la función implícita. Extremos locales y extremos condicionados.

- La integral definida como área. Teorema Fundamental del Cálculo. Técnicas elementales de integración.

- Integral de Riemann para una función de varias variables.

- Representación paramétrica de curvas y superficies. Longitud y área. Integrales de línea y de superficie. Teoremas de Green-Gauss, de Stokes y de la Divergencia.

Destrezas:

- Manipular desigualdades, analizar y dibujar funciones, deducir propiedades de una función a partir de su gráfica, comprender y trabajar intuitiva, geométrica y formalmente con las nociones de límite, derivada, integral.
- Calcular integrales de una variable utilizando la Regla de Barrow y con ayuda de cambios de variable y de la integración por partes, incluyendo al menos funciones racionales y trigonométricas.
- Utilizar algún programa de cálculo simbólico para obtener (e interpretar) límites, sumas de series, derivadas e integrales.
- Plantear adecuadamente y resolver integrales de varias variables, integrales curvilíneas e integrales de superficie.
- Utilizar correctamente en aplicaciones a otros campos los conceptos asociados a las derivadas parciales, a las integrales de línea y de superficie, y a las integrales de dos o tres variables.
- Conocer y saber utilizar métodos de aproximación numérica en el cálculo de integrales y de aproximación de funciones.
- Resolver problemas que impliquen el planteamiento de integrales.
- Resolver problemas de optimización.
- Conocer definiciones formalmente correctas de los conceptos más importantes (convergencia, continuidad, integrabilidad, etc.)

Matemática Discreta

Objetivos:

Plantear y resolver problemas de optimización lineal.

Conocer y manejar los conceptos y resultados básicos de teoría de grafos y combinatoria enumerativa.

Contenidos mínimos:

- Programación lineal.
- Teoría elemental de grafos.
- Combinatoria y métodos de enumeración.

Destrezas:

- Saber utilizar el método del simplex para resolver problemas de optimización lineal
- Ser capaz de determinar en grafos razonablemente pequeños los diferentes conceptos de teoría de grafos
- Modelar problemas de redes, geometría, etc., en términos de grafos e interpretar el significado de los conceptos de teoría de grafos en dichos contextos.

- Ser capaz de aplicar los principios de doble conteo y de inclusión-exclusión en diversos contextos.
- Identificar objetos que se pueden contar con números binomiales y/o multinomiales.

Informática

Objetivos:

Iniciar al alumno en algún lenguaje de programación científica.

Conocer los conceptos fundamentales de la algorítmica.

Objetivos de carácter instrumental:

Conocer, a nivel de usuario, las herramientas básicas de los ordenadores.

Contenidos mínimos:

- Conceptos básicos sobre ordenadores y sus componentes, sistemas operativos y lenguajes de programación.
- Lenguaje de programación científica.
- Introducción al diseño y análisis de algoritmos.
- Contenidos de carácter instrumental:
  - Herramientas básicas: edición de textos, hojas de cálculo, internet.
  - Edición de textos científicos.

Destrezas:

- Utilizar con soltura algún sistema operativo.
- Conocer los conceptos básicos del hardware y software del ordenador.
- Programar algoritmos para resolver problemas científicos y técnicos.
- Destrezas de carácter instrumental:
  - Manejar las herramientas básicas de comunicaciones.
  - Crear y manipular ficheros de texto y realizar operaciones elementales con hojas de cálculo.
  - Editar textos con fórmulas matemáticas.

Estructuras Algebraicas

Objetivos:

Conocer las propiedades de las estructuras correspondientes a los conjuntos de números enteros, racionales, reales y complejos, de los polinomios en una y varias variables y manejar con soltura todo tipo de expresiones algebraicas.

Manejar con soltura las nociones básicas de la teoría de conjuntos y aplicaciones, las propiedades elementales de las estructuras algebraicas básicas así como de las correspondientes subestructuras y cocientes y conocer ejemplos de todas ellas.

Conocer algunos casos de clasificación de objetos en una misma estructura algebraica mediante el uso de la noción de isomorfismo y la búsqueda de invariantes o características que permitan decidir cuando, por ejemplo, dos grupos no son isomorfos.

Contenidos mínimos:

- Estructuras algebraicas elementales: **Z, Q**, **R, C** y polinomios en una y varias variables
- Conjuntos y aplicaciones.
- Relaciones de equivalencia y orden. Conjuntos cociente.
- Grupos y homomorfismos de grupos
- Subgrupo normal y grupo cociente. Teorema de Lagrange. Teorema de Cayley
- Clasificación de grupos abelianos finitamente generados.
- Anillos e ideales. Homomorfismos de anillos.
- Divisibilidad y factorización.
- Números algebraicos y transcendentes.

Destrezas:

- Manejar con precisión el lenguaje proposicional, siendo capaz de traducir a éste la veracidad o falsedad de cualquier afirmación sobre conjuntos y aplicaciones.
- Utilizar el Algoritmo de Euclides para el cálculo del mcd de números enteros y polinomios así como para determinar los coeficientes en la Identidad de Bézout.
- Determinar la factorización en primos (resp. irreducibles) de números enteros (resp. polinomios) en casos sencillos.
- Descomponer en fracciones simples una fracción algebraica.
- Estudiar la existencia de elementos de orden dado en un grupo simétrico de grado n. Determinar subgrupos cíclicos, diédricos o abelianos sencillos en un grupo simétrico de grado n.
- Determinar si un subgrupo dado es normal o no y, en caso afirmativo, calcular el correspondiente grupo cociente.
- Saber definir homomorfismos sencillos para estudiar sus propiedades y analizar si dos grupos dados son isomorfos, si uno es isomorfo a un subgrupo de otro o para expresar un grupo como cociente de otro.
- Operar de forma correcta en el anillo cociente y cálculo de inversos modulares.
- Operar de forma correcta con los cocientes de anillos de polinomios o en anillos de la forma **Z**[a] con a en **C** algebraico prestando especial atención a los aspectos de factorización.
- Manipular de forma precisa expresiones algebraicas involucrando elementos algebraicos y transcendentes.

Geometría

Objetivos:

Conocer y saber utilizar los conceptos básicos de la Geometría del plano y el espacio euclídeo.

Conocer y saber utilizar los conceptos básicos de la Geometría Diferencial de Curvas y Superficies.

Desarrollar la capacidad de interpretar geométricamente enunciados y propiedades presentados analíticamente, y, recíprocamente, de formular en términos matemáticos situaciones de índole geométrica.

Contenidos mínimos:

- Geometría del triángulo y la circunferencia. Lugares geométricos.
- Movimientos en el plano y el espacio.
- Polígonos y poliedros regulares.
- Geometría vectorial de $\mathbf{R}^2$ y $\mathbf{R}^3$.
- Curvas. Longitud de arco. Triedro de Frenet.
- Curvas notables: hélices, evolutas y evolventes.
- Superficies. La primera y la segunda formas fundamentales. Teorema egregio de Gauss.
- Superficies notables: de revolución, regladas, desarrollables y minimales.

Destrezas:

- Saber resolver problemas geométricos del plano, y saber elegir los métodos sintético y analítico según convenga más en cada caso.
- Saber resolver problemas geométricos en el plano que involucren puntos y rectas.
- Saber formular la ecuación de un lugar geométrico y conocer una buena colección de ellos.
- Conocer las propiedades geométricas de una colección grande de curvas y superficies de $\mathbf{R}^3$.
- Reconocer la naturaleza de los puntos de una superficie de $\mathbf{R}^3$, sabiendo calcular los objetos geométricos asociados al punto (curvatura de Gauss,...)
- Reconocer las propiedades intrínsecas de una superficie.
- Conocer la existencia de las geometrías no euclídeas. Conocer la existencia de los problemas clásicos griegos: cuadratura del círculo, trisección del ángulo, duplicación del cubo.
- Reconocer la geometría subyacente en la Naturaleza y en el Arte.

Probabilidades y Estadística

Objetivos:

Desarrollar la intuición sobre fenómenos aleatorios.

Comprensión y manejo de los principios básicos del Cálculo de Probabilidades.

Conocimiento de los teoremas fundamentales del Cálculo de Probabilidades incluyendo su demostración, al menos en situaciones sencillas.

Comprensión de los principios y conceptos básicos de la Estadística Matemática así como su aplicación para la solución de problemas reales.

Contenidos mínimos:

- Espacios probabilísticos continuos y discretos. Funciones de densidad y de distribución.

- Variables aleatorias y sus distribuciones. Esperanza matemática.
- Independencia. Leyes de los Grandes Números y Teorema Central del Límite.
- Estadística descriptiva y análisis de datos.
- El método estadístico. Estimación puntual y por intervalo.
- Contrastes de hipótesis.
- El modelo lineal.

Destrezas:

- Saber calcular probabilidades en espacios discretos y probabilidades geométricas.
- Manejar variables aleatorias discretas y continuas.
- Comprensión y utilización del concepto de independencia.
- Saber utilizar las aplicaciones estadísticas del teorema central del límite.
- Saber analizar e interpretar un conjunto de datos.
- Conocer las técnicas de generación de números pseudoaleatorios.
- Manejo de los métodos de máxima verosimilitud, de Bayes y de mínimos cuadrados para la construcción de estimadores.
- Comprensión de las propiedades básicas de los estimadores puntuales y de intervalo.
- Saber plantear problemas de contraste de hipótesis en una o dos poblaciones.
- Comprensión del fundamento del modelo lineal incluyendo los modelos más simples del Análisis de la Varianza y de los problemas de regresión y correlación.
- Utilización de software estadístico.

Ecuaciones Diferenciales

Objetivos:

Conocer la relación entre los problemas reales y su modelo matemático en términos de Ecuaciones Diferenciales Ordinarias.

Conocer y saber utilizar los conceptos y los resultados clásicos relacionados con las EDO, con especial énfasis en el caso lineal.

Comprender la imposibilidad de resolver de manera exacta todas las EDO y la necesidad de utilizar métodos numéricos y enfoques cualitativos para su resolución.

Contenidos mínimos:

- Ecuaciones diferenciales ordinarias (EDO). Problema de Cauchy.

- Métodos elementales de resolución de ecuaciones de primer orden. Aplicaciones.

- EDO lineales de segundo orden. EDO con coeficientes constantes. Método de serie de potencias. Modelos y aplicaciones.

- Sistemas lineales de ecuaciones diferenciales de primer orden. Sistemas con coeficientes constantes: exponencial de una matriz.

- Teoremas de existencia y unicidad de solución para problemas de Cauchy.

- Dependencia de condiciones iniciales y parámetros.
- Sistemas autónomos. Plano de fases. Aplicaciones.
- Introducción a las ecuaciones en derivadas parciales.

Destrezas:

- Distinguir los diferentes tipos de ecuaciones diferenciales.
- Traducir algunos problemas reales en términos de EDO.
- Conocer y aplicar los principales métodos para resolver EDO sencillas.
- Extraer información cualitativa de las soluciones de una EDO, sin necesidad de resolverla (crecimiento, concavidad, …)
- Utilizar algún software para resolver EDO.

Variable compleja

Objetivos:

Conocer los fundamentos de la teoría de funciones de una variable compleja.

Contenidos mínimos:

- El plano complejo.
- Funciones derivables. Series de potencias.
- Fórmula de la Integral de Cauchy.
- Desarrollo de funciones analíticas en serie de potencias.
- Funciones enteras y sus propiedades.
- Propiedades básicas de las funciones analíticas.
- Desarrollos de Laurent y Teorema de Cauchy del Residuo.

Destrezas:

- Identificar las diferencias básicas entre las propiedades de las funciones de variable real y de variable compleja.
- Conocer y saber utilizar las propiedades fundamentales de las funciones analíticas.
- Calcular residuos e integrales reales por este método.

Cálculo Numérico

Objetivos:

Analizar, programar e implantar en ordenador algunos de los algoritmos o métodos constructivos de soluciones de problemas.

Conocer las implicaciones de la presencia de los errores de redondeo y aproximación.

Valorar la importancia del coste operativo y de memoria-ordenador de los métodos y su eficacia así como el equilibrio entre complejidad, precisión y rapidez.

Contenidos mínimos:

- Representación de números en el ordenador y errores en el cálculo numéricos.
- Métodos de resolución de ecuaciones no lineales
- Resolución de sistemas lineales: métodos directos e iterativos.
- Cálculo de autovalores y autovectores
- Interpolación y ajuste: polinomial, splines y trigonométrica.
- Integración aproximada.
- Métodos de cálculo de mínimos de funciones reales.

Destrezas:

- Manejar software de cálculo numérico.
- Analizar la conveniencia de uno u otro método numérico para un problema concreto.
- Programar e implantar en el ordenador los métodos numéricos y aplicarlos de manera efectiva.
- Evaluar los resultados obtenidos y obtener conclusiones después de un proceso de cómputo.
- Conocer y saber aplicar algoritmos para resolver numéricamente sistemas de ecuaciones lineales de orden medio y alto.
- Localizar y calcular numéricamente raíces de ecuaciones no lineales.
- Localizar y calcular autovalores y autovectores: métodos de la potencia.
- Conocer y saber aplicar los principales algoritmos de interpolación y de ajuste.
- Conocer y saber utilizar fórmulas de cuadratura para integración aproximada sobre intervalos reales y sobre dominios sencillos del plano y del espacio (triángulos, rectángulos, ...).
- Conocer y utilizar algoritmos sencillos de búsqueda de mínimos para resolver numéricamente algunos problemas de optimización.

Perfil aplicado

Objetivos

Que el licenciado sea capaz de plantear modelos matemáticos de cierta complejidad en diversos ámbitos.
Que conozca herramientas informáticas y numéricas de resolución de problemas.
Que conozca técnicas estadísticas y de optimización.

Contenidos mínimos

En este perfil el licenciado debe de cursar al menos 4 de las siguientes materias:

- Ecuaciones en derivadas parciales
- Cálculo científico y programación avanzada.
- Modelos matemáticos
- Matemática discreta
- Probabilidad
- Métodos estadísticos
- Investigación operativa y optimización.

Destrezas

Saber interpretar en términos matemáticos situaciones expresadas en el lenguaje de otras disciplinas.
Saber realizar análisis de datos y extraer conclusiones.
Plantear matemáticamente problemas reales e identificar o idear métodos de aproximación y resolución numérica en ordenador.
Manejar las herramientas básicas del cálculo científico.

Rango profesional

Este perfil de la licenciatura de matemáticas capacita profesionalmente para la inserción de los matemáticos en equipos interdisciplinares de empresas, industrias y consultorías así como en unidades de i+I+D.

Perfil didáctico

Objetivos

Que el licenciado conozca y domine, desde un punto de vista superior, los contenidos de los programas de matemáticas en secundaria
Que sepa planificarlos, estructurarlos y comunicarlos al nivel adecuado
Que conozca las distintas facetas del conocimiento matemático (conexión con otras disciplinas y con el entorno, carácter formativo, aplicaciones...)

Contenidos mínimos

- Fundamentos teóricos de las matemáticas elementales.
- Historia de las matemáticas.
- Teorías del aprendizaje en la educación matemática.
- Diseño curricular en matemáticas
- Metodologías, materiales y recursos para la enseñanza de las matemáticas.
- Prácticas de enseñanza en las aulas de secundaria.

Destrezas

Saber conectar los conceptos matemáticos del currículo de la secundaria con los fenómenos que los originan.
Reconocerlos en situaciones cotidianas y ámbitos multidisciplinarios.
Dominar de técnicas de comunicación y transmisión de conocimientos matemáticos.
Secuenciar y estructurar los puntos centrales de un tema matemático.
Diagnosticar errores y dificultades de aprendizaje.
Capacidad para elaborar instrumentos de seguimiento y evaluación.
Aplicar los nuevos recursos tecnológicos en los procesos de enseñanza / aprendizaje de las matemáticas en secundaria.
Adquirir autonomía en la utilización de material bibliográfico

Rango profesional

Este perfil de la licenciatura de matemáticas capacita profesionalmente para la enseñanza de las matemáticas en secundaria.

Perfil académico

Objetivos

Completar la formación matemática general de los licenciados.
Afianzar las destrezas adquiridas en los tres primeros cursos.
Formar licenciados con vocación de incorporarse a un postgrado de estudios avanzados en Matemáticas.

Contenidos mínimos

En este perfil el licenciado debe de cursar al menos 4 de las siguientes materias:

- Análisis Funcional
- Geometría Diferencial
- Ecuaciones en Derivadas Parciales
- Estructuras Algebraicas
- Topología
- Teoría de la medida y probabilidad
- Estadística matemática
- Análisis numérico

Destrezas

- Manipular objetos matemáticos con alto nivel de abstracción y complejidad.
- Profundizar en la capacidad de idear demostraciones.
- Escribir Matemáticas con un nivel de rigor aceptable.
- Adquirir autonomía en la utilización de material bibliográfico

Rango profesional

La obtención del título de Licenciado en Matemáticas con perfil “académico” debe dar acceso directo a cualquier postgrado de estudios avanzados en Matemáticas en cualquier universidad.

Anexo 6

Ejemplos de posibles postgrados (másters)

1. **Máster en “Matemáticas de la Ingeniería”**
2. **Máster en “Matemáticas de las T.I.C”**
3. **Máster en “Matemáticas de la Industria”**
4. **Máster en “Matemáticas de la Economía y la Empresa”**
5. **Máster en “Matemáticas de las Finanzas”**
6. **Máster en “Estadística Aplicada”**
7. **Máster en “Estudios Avanzados de Matemáticas”**

1. MÁSTER EN "MATEMÁTICAS DE LA INGENIERÍA"

Perfil

Se trata de ofrecer una formación especializada en modelización matemática y simulación numérica en ordenador para matemáticos/ingenieros destinados a trabajar en empresas de alta tecnología y equipos interdisciplinares de I+D de las universidades, centros de investigación o industrias de ingeniería civil, químico-farmacéuticas, medioambientales, salud, ... como expertos en los modelos matemáticos y herramientas informáticas que se utilizan en estos sectores.

Objetivos

- Estudio riguroso de los modelos más importantes de la mecánica de medios continuos desde su motivación física y su correcta formulación matemática hasta la resolución numérica con vistas a la simulación en ordenador de dispositivos y procesos de la ingeniería.
- Profundizar en los conocimientos en ecuaciones en derivadas parciales, métodos numéricos e informática que permitan el planteamiento y la búsqueda de soluciones en problemas de contexto industrial.
- Adquirir una formación especializada en cálculo científico y simulación numérica en física e ingeniería que permita la inserción en los equipos de I+D de los sectores industriales mencionados.

Destrezas:

- Conocer a fondo la motivación física y el planteamiento matemático de los modelos más importantes de la mecánica de medios continuos: elasticidad, fluidos, transferencia de calor, electromagnetismo, reacciones químicas.
- Ser capaz de plantear matemáticamente problemas industriales o semi-industriales de cierta complejidad en los campos anteriores, identificar o idear métodos de aproximación y resolución numérica en ordenador.
- Conocer las herramientas informáticas, de programación y de diseño y análisis asistido por ordenador (CAD/CAE) para implantar en ordenador dichos métodos, visualizar gráficamente e interpretar los resultados, validar los modelos y los métodos, obtener conclusiones sobre el modelo y el posible control del proceso que ayude en la toma de decisiones.

Contenidos fundamentales:

- Modelización matemática de problemas industriales o semi-industriales en ingeniería (sólidos, fluidos, térmica, electromagnetismo, reacciones químicas)
- Integración numérica de ecuaciones en derivadas parciales (diferencias finitas, elementos finitos).
- Métodos numéricos avanzados (métodos numéricos en optimización, grandes sistemas lineales con matriz dispersa)
- Ingeniería del software de cálculo científico
- Paquetes comerciales de simulación en ingeniería.
- Proyecto fin de Máster: simulación numérica de un dispositivo o proceso característico en los sectores industriales mencionados.
- ....

Contenidos optativos:

- Bases de datos y redes de comunicación
- Cálculo vectorial y paralelo
- Criptografía y seguridad computacional
- Procesamiento de imágenes
- Sistemas expertos
- Gráficos por ordenador
- Teoria de la señal
- Control óptimo
- ......

2. MÁSTER EN “MATEMÁTICAS DE LAS T.I.C.”

Perfil

Se trata de ofrecer una formación especializada en matemáticas e informática para matemáticos/ingenieros/informáticos destinados a trabajar en empresas de alta tecnología y equipos interdisciplinares de I+D en las universidades, centros de investigación o industrias de las tecnologías de la información y comunicación (informática, telecomunicación, electrónica, bases de datos, industrias del *software*, programación, redes transmisión de la señal, ...) como expertos en los métodos matemáticos y herramientas informáticas que se utilizan en estos sectores.

Objetivos

- Estudio avanzado de los modelos y métodos matemáticos propios de las ingenierías y sectores relacionados con las T.I.C.
- Profundizar en los conocimientos de matemáticas (matemática discreta, métodos computacionales del álgebra y la geometría, análisis de Fourier, cálculo numérico, investigación operativa) y programación avanzada que permitan el planteamiento y la búsqueda de soluciones en problemas de contexto industrial en las T.I.C.
- Adquirir una formación práctica especializada que permita la inserción en los equipos de I+D de los sectores industriales y empresariales mencionados .

Destrezas:

- Ser capaz de plantear matemáticamente problemas industriales o semi-industriales de cierta complejidad en el campo de las T.I.C., identificar o idear métodos matemáticos teórico-computacionales para su resolución.
- Conocer las herramientas informáticas, de programación y de diseño y análisis asistido por ordenador (CAD/CAE) para desarrollar dichos métodos, interpretar los resultados, validar los modelos y obtener conclusiones, en situaciones de restricción de tiempo y recursos.

Contenidos fundamentales

- Modelización matemática de sistemas y problemas industriales o semi-industriales en las T.I.C. (redes, teoría de la señal, teoría de colas, cadenas de Markov, redes de Petri, ...).
- Matemática discreta.
- Métodos numéricos avanzados (métodos numéricos en optimización, grandes sistemas lineales con matriz dispersa, ...).
- Programación avanzada (programación orientada a objetos, programación distribuida, ...).
- Bases de datos y redes de comunicación (bases de datos relacionales, introducción a las redes locales y recursos compartidos: www, e-mail, FTP, ...).
- Proyecto: Planteamiento/resolución de un problema o simulación de un proceso característicos de los sectores industriales de las T.I.C.

Contenidos optativos

- Sistemas operativos.
- Métodos computacionales del álgebra y la geometría.
- Teoría y métodos numéricos en ecuaciones diferenciales.
- Cálculo vectorial y paralelo.
- Criptografía y seguridad computacional.
- Tratamiento y procesado de imágenes.
- Sistemas expertos.
- Gráficos por ordenador.
- Inteligencia artificial.
- Grafos y complejidad.
- Transmisión de la información
- Introducción a la economía.
- Visión por ordenador.
- ....

3. MÁSTER EN “MATEMÁTICAS DE LA INDUSTRIA”

Perfil.

Se trata de ofrecer una formación suplementaria en Informática y Matemática Industrial que potencie la inserción de los matemáticos en empresas de alta tecnologia, equipos R+D, consultorias, etc. como expertos en los modelos matemáticos y las herramientas informáticas que se usan en estos sectores industriales y empresariales próximos a las ingenierias.

Prerequisitos

- Ecuaciones en Derivadas Parciales.
- Investigación Operativa.
- Modelización Matemática.
- Análisis de Fourier.

Materias Obligatorias:

- Programación Avanzada (Programación orientada a objetos: Java, C++,...)
- Integración Numérica de Ecuaciones en Derivadas Parciales.
- Bases de datos y redes de comunicación (Bases de datos relacionales, introducción a las redes locales y compartición de recursos: www, e-mail, FTP, news, ...)
- Planificación de la Producción (Introducción a los problemas de planificación y utilización de técnicas experimentales basadas en la simulación digital).
- Introducción a la Economía.
- Proyecto fin de máster

Materias Optativas

- Criptografia y Seguridad Computacional.
- Procesamiento de imágenes (Tratamiento y análisis de imágenes digitales por ordenador. Eliminación del ruido, distorsiones geométricas, etc.)
- Visión por Computador.
- Inteligencia Artificial.
- Grafos y Complejidad (Aplicación de los grafos a problemas de optimización)
- Tratamiento y transmisión de señales (Representación y tratamiento de señales discretas y estudios de filtros).
- Economía y organización industrial.
- Sistemas expertos.
- Gráficos por ordenador (Representación y visualización de gráficos 2D y 3D).
- ….

4. MÁSTER EN
“MATEMÁTICAS DE LA ECONOMÍA Y LA EMPRESA”

Perfil.

Se trata de ofrecer una formación suplementaria en Economía y Empresa que potencie la inserción de los matemáticos en empresas y instituciones como expertos en técnicas y métodos de previsión, organización empresarial y finanzas.

Prerequisitos:

- Modelos lineales.
- Series temporales.
- Investigación operativa.
- Análisis multivariante.

Materias obligatorias :

- Programación avanzada (programación orientada a objetos, programacón distribuida, ...).
- Microeconomía.
- Macroeconomía.
- Econometria.
- Teoria de juegos (teoria de la elección individual, juegos cooperativos y no cooperativos, equilibrio de Nash).
- Teoría y métodos numéricos en ecuaciones diferenciales
- Proyecto fin de máster.

Materias optativas:

- Microeconomía avanzada.
- Macroeconomía avanzada.
- Marketing e investigación comercial (aplicación de las técnicas de análisis multivariante a la investigación de mercados).
- Análisis de mercados financieros y gestión de carteras (estudio de los mercados financieros de títulos primitivos y títulos derivados, estrategias financieras).
- Valoración de productos finacieros derivados
- Organización industrial.
- Control de calidad.

5. MÁSTER EN “MATEMÁTICAS DE LAS FINANZAS”

Perfil.

El objetivo del máster es formar especialistas en matemáticas financieras, básicamente en el ámbito de los productos derivados (valoración de productos derivados, y su cobertura) y de gestión del riesgo (evaluación y control de riesgos), con los conocimientos necesarios de procesos estocásticos, ecuaciones en derivadas parciales, métodos numéricos y simulación, para gestionar los productos financieros existentes y desarrollar otros nuevos según las necesidades del mercado. Dichos especialistas podrían incorporarse en los servicios de valoración de opciones o de gestión del riesgo de las entidades bancarias.

Prerequisitos

- Modelos Lineales.
- Series Temporales.
- Investigación Operativa.
- Ecuaciones en derivadas parciales.
- Métodos numéricos

Materias Obligatorias :

- Conceptos básicos de Finanzas
- Macroeconomía.
- Econometría.
- Series temporales avanzadas
- Medida y control de riesgos
- Modelos estocásticos en Finanzas (valoración de opciones, fórmula de Black-Scholes).
- Ecuaciones en derivadas parciales en Finanzas
- Métodos numéricos y simulación
- Proyecto fin de máster

Materias Optativas:

- Valoración de carteras.
- Técnicas actuariales
- Microeconomía
- Análisis multivariante
- ….

6. MÁSTER EN “ESTADÍSTICA APLICADA”

Perfil

Se trata de ofrecer una formación especializada en el dominio y correcta utilización de los métodos estadísticos, para poder interpretar, valorar y extraer toda la información a los datos procedentes de distintos ámbitos. Este máster está especialmente dirigido a matemáticos interesados en trabajar como expertos en estadística en empresas o equipos interdisciplinares de I+D en los sectores medioambiental, sanitario, financiero, etc.

Objetivos

- Introducir a los alumnos en métodos estadísticos con un enfoque eminentemente aplicado y multidisciplinar.
- Estudio riguroso de la metodología estadística, capacitando a los alumnos para diseñar y analizar modelos teóricos apropiados.
- Conseguir que los alumnos alcancen una madurez superior en los métodos más recientes de Estadística e Investigación Operativa, haciendo posible la futura investigación en dichos campos.
- Adquirir una formación especializada en software estadístico.

Destrezas

- Ser capaz de plantear y analizar modelos estadísticos apropiados en distintos campos científicos.
- Conocer con soltura software estadístico, así como las herramientas informáticas y de programación para implantar en ordenador dichos modelos, visualizar gráficamente e interpretar los resultados.

Contenidos fundamentales

- Modelización estocástica
- Modelos de regresión
- Series de tiempo
- Simulación y métodos de computación
- Técnicas de muestreo
- Estimación no paramétrica de curvas
- Técnicas de estadística espacial
- Modelos de la investigación operativa
- Modelos de análisis multivariante
- Modelos de tipo biosanitario
- Control estadístico de la calidad
- Proyecto fin de máster.

7. MÁSTER DE
"ESTUDIOS AVANZADOS EN MATEMÁTICAS"

CONSIDERACIONES PREVIAS

La función del postgrado en estudios avanzados en Matemáticas debe ser proporcionar al alumno los contenidos y herramientas necesarios para que esté en condiciones de elaborar una tesis doctoral en Matemáticas, y dedicarse profesionalmente a la investigación y la docencia universitaria.

Esta función debería cumplirse teniendo en cuenta que las Universidades determinarán autónomamente una parte importante de los contenidos de las Licenciaturas en Matemáticas, y que el acceso al Doctorado en Matemáticas va a estar abierto tanto a personas que hayan obtenido la licenciatura en Matemáticas en otros países, como a licenciados en otras disciplinas. Entendemos que existen una serie de materias que quienes vayan a dedicarse a la investigación en Matemáticas deberían conocer con mayor o menor detalle. Entre ellas, podríamos incluir

1. **Análisis Funcional** (espacios de Hilbert, sistemas ortonormales, bases hilbertianas, espacios de Banach, teoremas fundamentales, dualidad, ...)
2. **Introducción a las Ecuaciones en Derivadas Parciales** (EDPs de primer orden, ecuaciones elementales de segundo orden, separación de variables, principio del máximo, modelos y aplicaciones)
3. **Introducción a las estructuras algebraicas** (números algebraicos, complementos de teoría de grupos, anillos, ideales y cuerpos, extensiones finitas)
4. **Variedades diferenciales** (Teoría local de curvas planas y alabeadas. Introducción a la teoría de superficies. Primera forma fundamental. Geometría intrínseca y extrínseca de superficies. Teorema fundamental de superficies).
5. **Introducción a la teoría de funciones de variable compleja** (funciones holomorfas, teorema de Cauchy-Riemann, representación conforme, teorema de Cauchy, representación integral de Cauchy, teorema de los residuos)
6. **Introducción a la Optimización** (geometría de los conjuntos poliédricos, programación lineal, convexidad, condiciones de optimalidad en programación no lineal, métodos numéricos en optimización, aplicaciones)
7. **Análisis Numérico de las Ecuaciones Diferenciales** (Métodos elementales de resolución de problemas de Cauchy, método de diferencias finitas para problemas de contorno, implementación, aplicaciones)
8. **Estadística Matemática** (variables aleatorias y modelos, regresión y correlación, distribuciones multidimensionales, convergencias, métodos de estimación puntual paramétrica, regiones de confianza y contraste de hipótesis)

ESTRUCTURA GENERAL Y ACCESO

Para acceder al Máster en Estudios Avanzados se requerirá un título de Licenciado, así como acreditar haber cursado (en grado o postgrado) un cierto número de créditos (a determinar) del total de materias antes mencionadas. Para acceder al Doctorado en Matemáticas se requerirá un título de Máster.

De 90 a 120 créditos ECTS de duración. En él se ofertarían cursos de dos tipos: **Generales** (de carácter introductorio sobre grandes áreas temáticas, por ejemplo, de las anteriormente citadas) y **Específicos** (cursos especializados sobre nuevas áreas de investigación, de tipo instrumental, o cursos encuadrables en programas de Máster de otras materias). Además el alumno deberá realizar un **Proyecto final de máster** consistente en la realización de un trabajo de iniciación a la investigación de un mínimo de 30 créditos ECTS.

Sería recomendable que el alumno visitara (para seguir cursos o bien preparar el trabajo de investigación) durante un semestre otro centro nacional o extranjero, usando los programas de becas basados en acuerdos de movilidad nacionales e internacionales.

Superadas las pruebas pertinentes y defendido el trabajo de investigación, el alumno recibe el título de **Máster de Estudios Avanzados en Matemáticas.**

Informe sobre la investigación matemática en España en el período 1990-1999

Elaborado por iniciativa del
Comité Español para el **Año Mundial** de las **Matemáticas (CEAMM)**

Coordinadores:

Carlos Andradas
y
Enrique Zuazua

PRÓLOGO

La celebración en España del año 2000 como Año Mundial de las Matemáticas ha supuesto una creciente vertebración y colaboración entre las distintas sociedades matemáticas y ha propiciado que los matemáticos reflexionemos conjuntamente sobre el estado de nuestra ciencia. Con el fin de celebrar adecuadamente el Año Mundial de las Matemáticas, en España se constituyó en 1998 un Comité CEAMM2000 en el que se integraban todas las sociedades matemáticas españolas, además de instituciones tales como el Consejo Superior de Investigaciones Científicas, la Real Academia de Ciencias Exactas, Físicas y Naturales y el propio Ministerio de Educación y Cultura[1]. Entre las actividades que este Comité acordó figuraba la realización de un informe en el que se presentase una panorámica de la producción matemática española en el ámbito de la investigación.

Tras un año de trabajo nos complace completar la tarea entonces emprendida presentando este informe en el que recogemos los datos más relevantes de la producción científica matemática española en la década de los 90.

La primera constatación es que la investigación matemática española ha experimentado un crecimiento extraordinario en los últimos años tanto en intensidad como en calidad e impacto. Se trata de un cambio espectacular, paralelo al desarrollo general del país, que en Matemáticas ha sido especialmente significativo. España ha pasado de tener pocos matemáticos activos en investigación, a tener una amplia base de investigadores matemáticos, razonablemente financiados, con especialistas en casi todas las áreas de las Matemáticas, incluyendo las más punteras.

Este estudio pretende aportar datos precisos y objetivos que justifiquen las afirmaciones anteriores. Para su realización nos hemos encontrado con diversas dificultades que nos han obligado a hacer opciones, muchas de ellas discutibles, pero necesarias para lograr sacar adelante el informe.

La primera de ellas era delimitar lo que entendemos por producción matemática. Como se trataba de huir en lo posible de concepciones subjetivas, hemos tomado como definición de artículo de matemáticas todo aquel que aparece en la base de datos "MathSciNet" de la AMS (American Mathematical Society). A partir de ella hemos seleccionado la producción española quedándonos con todos los documentos en los que alguno de los firmantes incluía España o alguna institución española en el campo "Institución" de dicha base de datos. Vayan por delante nuestras disculpas a aquellos que han realizado o realizan su actividad investigadora en centros extranjeros. De este modo hemos creado lo que podríamos llamar el espacio muestral del estudio, que contiene un total de 11.813 documentos repartidos a lo largo de toda la década y que han pasado de suponer el 1,7% de la producción matemática mundial en el año 1990 al 3,2% en el 1999. En la memoria se recogen también algunos datos de la distribución de estos trabajos por códigos de la MSC (Mathematics Subject Classification) y el peso relativo de dicho código a nivel mundial que pueden resultar de interés.

[1] Comité Español del Año Mundial de las Matemáticas, CEAMM2000. http://dulcinea.uc3m.es/ceamm

Para el estudio de calidad, hemos cruzado y filtrado esta base de datos siguiendo el listado de revistas de la clasificación del ISI[2] por índices de impacto, es decir hemos refinado la base anterior quedándonos únicamente con los documentos que aparecen publicados en alguna de las revistas que aparecen en el ISI. Somos conscientes que esto no siempre es un criterio fiable del impacto ni la calidad de un artículo y que excelentes trabajos se publican en ocasiones en revistas que no aparecen en estos listados, pero se trata de un parámetro objetivo de evaluación de calidad cada vez más usado, y en la medida que la producción global es muy significativa, hemos considerado que cumplía el objetivo de indicador de calidad. Finalmente, a la vista del elevado número de artículos que aparecían en áreas fronterizas, especialmente Física, y que distorsionaban extraordinariamente los datos, hemos procedido a un filtrado manual, llevado a cabo por expertos, de los documentos que aparecían en dichas áreas, hasta donde nos ha sido posible. Como resultado de este proceso hemos obtenido una segunda base de datos que podríamos denominar de "documentos de calidad" que consta de 6.220 artículos y que supone el 52,65% de la cuantitativa y que ha sido la base usada para todo el estudio restante del informe. Comparándola con la producción total recogida por el ISI la contribución española ha experimentado una evolución desde un 1,7% en 1990 hasta un 3,9% en el 1999. Esta evolución ha seguido en aumento y según los datos del ISI de este año se encuentra ahora mismo en el 4,18%.

Con el fin de acercar más los datos a la realidad española hemos intentado catalogar los epígrafes de la clasificación AMS en las distintas áreas de conocimiento de la LRU, lo cual no siempre ha sido fácil a causa de las numerosas áreas fronterizas. Esta catalogación ha sido realizada (con las pertinentes consultas) por nosotros, por lo que debe achacársenos la responsabilidad de los errores cometidos y sobre todo tomarse con las reservas de que está basada en nuestros criterios personales. Ocurre también que investigadores que realizan su actividad investigadora en unos campos de la clasificación AMS pueden en realidad estar adscritos a otras áreas de conocimiento, cuestión a tener presente al interpretar las tablas comparativas de producción y personal adscrito a áreas de conocimiento.

Tal y como se constata a través de este informe, la producción matemática española figura en el grupo de los diez países más importantes a nivel mundial. Pero queda pendiente la tarea de conseguir que este hecho sea reconocido adecuadamente en los distintos foros internacionales, si bien en este aspecto la progresión es también claramente positiva. Recientemente hemos conocido que Madrid ha sido propuesta por el Comité Ejecutivo de la UMI como sede para el Congreso Internacional de Matemáticos del año 2006, nominación que deberá aprobarse en la Asamblea General de la UMI en Pekín en Agosto del 2002. Esta propuesta está relacionada, sin duda, con el creciente reconocimiento al que aludimos. Confiamos en que este evento, en el caso de que la candidatura sea finalmente aprobada, supondrá un estímulo adicional a la investigación matemática. Hay otros muchos síntomas de la salud de nuestro tejido investigador. En efecto, aunque de manera aún insuficiente, es cada vez más frecuente encontrar nombres de investigadores españoles como conferenciantes invitados y miembros de Comités Científicos en Congresos y encuentros del máximo nivel, así como en los Comités Editoriales de prestigiosas publicaciones internacionales. A este respecto conviene también señalar que en la actualidad algunas de las revistas españolas

[2] ISI: Institute for Scientific Information. Realiza estudios bibliométricos de la producción científica mundial en todos los campos. Confecciona una lista de revistas con mayor impacto, el SCI: Science Citation Index. Constituye la referencia más objetiva y más usada en ciencia y tecnología en todo el mundo. Página web: http://www.isinet.com/

figuran ya en los listados del ISI (Revista Matemática Iberoamericana, Test) y en posiciones encomiables.

A pesar de todo ello, tal y como ha quedado de manifiesto en numerosos foros y encuentros celebrados en torno al AMM 2000, algunas sombras acechan también a nuestra investigación matemática. En primer lugar las carencias de nuestro sistema educativo tanto a nivel primario como secundario que muchas veces hacen difícil la transmisión del entusiasmo y los conocimientos matemáticos incluso a los alumnos más interesados en ellos. En segundo lugar, somos conscientes de las dificultades que atraviesan las Licenciaturas de Matemáticas, con un decrecimiento del número de alumnos a pesar de la relevancia cada vez mayor del conocimiento matemático en el mundo tecnológico y las buenas perspectivas laborales de que gozan los licenciados en matemáticas. En tercer lugar, la incapacidad de nuestro sistema universitario para acoger a los jóvenes investigadores de los últimos años ha contribuido a un "envejecimiento" de la masa crítica de investigadores que pone en peligro el mantenimiento de la tendencia creciente que este informe muestra y ya hemos comentado anteriormente. Por último, es bien conocido que las estructuras, condiciones y medios que el investigador español encuentra en sus Centros no son siempre las adecuadas para realizar una actividad investigadora intensa y de calidad. Ni que decir tiene, los más jóvenes encuentran dificultades muchas veces insalvables para desarrollar una carrera investigadora en condiciones razonables en un entorno donde el trabajo realizado sea adecuadamente evaluado, valorado y estimulado. Estos temas forman parte de la preocupación de todos los investigadores e incluso están siendo objeto de iniciativas legislativas. Confiemos que, entre todos, consigamos no sólo mantener este nivel de investigación sino mejorarlo dotándonos de las herramientas y estructuras necesarias para ello, de modo que, dentro de diez años, la nueva realidad sea aún más alentadora.

La realización del informe no ha sido fácil y hubiese resultado del todo imposible sin la gran labor realizada por Gema Villacián cuya contratación ha sido posible a través de la Acción Especial APC 1999-0265 "Informe sobre la investigación matemática española en la década de los 90" de la Secretaría de Estado de Educación, Universidades, Investigación y Desarrollo. La colaboración de los firmantes de esta Acción en nombre del CEAMM: Manuel de León, José Luis Fernández y Juan Luis Vázquez, ha resultado decisiva para el diseño final del estudio y para que éste saliera adelante de algunos momentos de estancamiento. Finalmente, debemos agradecer también al CINDOC, al CSIC y a la UCM por facilitar el uso de sus instalaciones y de las bases de datos de MathSciNet y el ISI.

Como decíamos antes, este trabajo no podría haber sido realizado sin la inestimable ayuda de Gema a la que queremos agradecer aquí su extraordinaria labor y dedicación. Por supuesto, la responsabilidad última de las lagunas que este informe pueda presentar sólo es atribuible a nosotros.

Madrid, Mayo del 2001

Carlos Andradas y Enrique Zuazua, Universidad Complutense de Madrid

ÍNDICE

1. INTRODUCCIÓN

La idea inicial de realizar este trabajo surgió con la iniciativa de solicitar el paso de España como miembro de la Unión Matemática Internacional del nivel 3 al 4, para cuya aceptación es preciso "justificar" que la relevancia de nuestra producción matemática está acorde a lo solicitado. Así nació la idea de hacer un estudio de la evolución del número de publicaciones en el pasado reciente. Hay un convencimiento generalizado entre los matemáticos de que la producción matemática española ha mejorado ostensiblemente en los últimos años, pero era preciso contrastar esta creencia con datos concretos. Más aún, tan importante como constatar el crecimiento es analizar si se trata de un crecimiento sostenido a lo largo del tiempo, si se da por igual en todas las áreas matemáticas, detectar las desviaciones que pudieran surgir, examinar la calidad de nuestra producción, los patrones de colaboración, etc.

Pronto nos dimos cuenta de que hacer un estudio que no quedara en un mero análisis cuantitativo de las publicaciones, sino que además hiciera un análisis de la calidad de las mismas y que pudiera ser útil a la hora de dibujar un mapa de la situación de la investigación matemática por Comunidades Autónomas, Universidades, Áreas de Conocimiento, etc., requería una dedicación y una infraestructura específica. Afortunadamente el paraguas del Año Mundial de las Matemáticas hizo posible la petición de una Acción Especial al Ministerio para la realización del informe, cuyo resultado son las páginas que tienen ahora entre sus manos.

El planteamiento general es la siguiente: entresacar de la base de datos MathSciNet de la AMS todos las entradas de la década 90-99 correspondientes a investigadores españoles, para formar una base de datos de la producción matemática española en dicha década. A partir de esta base se construye una subbase con los documentos de aquella que aparecen en la base de datos del ISI (Institut for Science Information) y que constituirá la base de datos para los distintos análisis de calidad, siempre con los parámetros, sin duda discutibles pero cada vez más generalizados, del índice de impacto elaborado por dicho instituto. ¿Por qué el período 1990-1999? En primer lugar porque se trataba de hacer un estudio suficientemente reciente para que revelara información de la situación española actual, es decir tras varios años de incorporación a los foros internacionales de investigación. Al mismo tiempo, el retraso en la publicación de los artículos y su posterior inclusión en la base de datos del MathSci obliga a terminar el estudio en el año 99 para que los datos de este año sean razonablemente fiables. Llamamos la atención sobre el hecho de que si estimamos un período medio de 18 meses desde el envío inicial del trabajo por el autor hasta su publicación en la revista, estamos hablando en realidad de investigación realizada entre 1988 y 1998.

La estructura del presente informe es la siguiente: en el Capítulo 2 se dan las explicaciones técnicas de la metodología seguida: bases de datos utilizadas, estructura de las mismas, campos e indicadores usados, modo de tratamiento de los datos, etc. Se incluye también la clasificación MSC (Mathematics Subject Classification) y su adaptación a nuestras áreas de conocimiento. En el Capítulo 3 se muestran los resultados de la base de datos general de producción matemática española, esto es, los datos del MathSciNet, tanto globales como desglosados por códigos de la MSC y la comparación porcentual del peso de cada una de ellas en la producción total. Creemos que este dato proporciona una primera panorámica global de la distribución de nuestra producción matemática y de las áreas en las que existe tanto una sub como una sobre-representación con respecto a la media mundial. Digamos que, dentro de la base datos del MathSciNet, la producción española ha pasado de ser el 1,7% de la mundial en 1990 al 3,2% en 1999.

En el Capítulo 4 se presenta la base de datos de "calidad", esto es, la subbase de la anterior consistente en todas los documentos de la misma que han aparecido publicados en revistas incluidas en la relación del ISI para la elaboración del índice de impacto y a las que nos referiremos como revistas ISI. Esta base de datos es la que se usa en el resto del informe. Un primer indicador de calidad es el tamaño relativo de esta subbase con respecto a la inicial: el 62,80%, mientras que a nivel mundial el tamaño de la subbase ISI es sólo del 52,88% de la MathSci. De hecho la aportación española a la base de datos del ISI es del 4,18%, superior por tanto a la aportación a la base total del MathSciNet señalada en el párrafo anterior. Se presentan también datos comparativos de la producción y la situación de las matemáticas en el conjunto de la producción científica nacional y de otros países de nuestro entorno. Es de destacar que las Matemáticas son ahora mismo en España la disciplina científica que ocupa a nivel nacional el tercer lugar en cuanto a su contribución mundial, superada sólamente por Francia, país con una tradición matemática secular.

En el Capítulo 5 se hace un estudio de la producción matemática y su evolución por Comunidades Autónomas, Universidades y Centros de Investigación, Áreas de Conocimiento y códigos de la MSC, relativizando la producción por número de profesores y por cada 10.000 habitantes. A primera vista resulta una ratio de 2 artículos ISI por profesor en toda la década, porcentaje ciertamente superable, teniendo en cuenta además que dentro del colectivo de profesores sólo se han contabilizado los profesores funcionarios, por lo que en la ratio anterior se les atribuye a ellos la producción matemática realizada por los profesores contratados y becarios.

En el Capítulo 6 se presenta un estudio más detallado de la "calidad" de la investigación, mostrando la distribución de la misma por cuartiles dentro de la clasificación de las revistas por índice de impacto. Esta distribución se hace también por Universidades y Áreas de Conocimiento. Asimismo se presentan datos de la revistas matemáticas con mayor volumen de publicaciones españolas con su asignación al cuartil correspondiente. Finalmente en el capítulo 7 se presenta un estudio del patrón de colaboración o coautoría de realización de los trabajos en matemáticas que muestra la generalización progresiva de la tendencia a firmar los trabajos en equipo, posiblemente acentuada por el uso de Internet.

En definitiva, creemos que el Informe contiene gran cantidad de información que puede ser de mucha utilidad a la hora de tener un mapa completo de la distribución de la investigación matemática en España, sus carencias y sus grupos o áreas más fuertes. Por otra parte muestra la buena salud de las Matemáticas dentro del conjunto nacional, lo que debe cargarnos de argumentos a la hora de reivindicar y negociar con nuestras autoridades un mejor tratamiento de las Matemáticas en todos lo niveles educativos. No obstante evidencia también que aun queda mucho camino por recorrer hasta incorporar a todos los profesores universitarios a la actividad investigadora habitual. Finalmente la base de datos elaborada puede ser un instrumento útil para estudios posteriores más pormenorizados.

2. METODOLOGÍA

La primera dificultad a la hora de realizar el estudio consistió en delimitar qué se entiende por artículo de Matemáticas. Se optó por considerar artículos matemáticos todos los que aparecieran en la base de datos MathSci, por lo que se procedió a identificar los documentos firmados por autores de una institución española que hayan sido publicados durante los años 1990-1999. A partir de estos documentos se realizó una segunda selección, obteniendo exclusivamente los documentos que fueron publicados en revistas que durante el periodo 1990-1999 fueron recogidas por las bases de datos SCI o SSCI. Esta segunda selección de documentos se hizo con el objeto de poder trabajar con el indicador bibliométrico “factor de impacto”. Una última fase consistió en una depuración manual de la base de datos para eliminar artículos, que aunque cumplen la primera definición, es decir aparecen en la base de datos MathSci, lo hacen en áreas fronterizas (principalmente Física) y quedaban en cuanto a contenido claramente fuera de lo que el colectivo de matemáticos considera matemáticas.

2.1. Fuentes de datos

El estudio se ha realizado utilizando como primera fuente de información la base de datos multidisciplinar MathSci, que recoge recensiones sobre un millón de artículos de revistas de matemáticas y sus aplicaciones y refleja los contenidos de las revistas de referencia "Mathematical Reviews" y "Current Mathematical Publications", publicadas por la AMS. Fundamentalmente comprende matemática clásica (álgebra, geometría, topología, análisis matemático, etc.), estadística, mecánica, informática, teoría cuántica, relatividad, astronomía, astrofísica y geofísica. Los datos proceden de tres discos de la versión en CD-ROM, correspondientes a los periodos 1988-1992, 1993-1997 y 1998-2001 (febrero).

Aunque se han recogido los datos que aparecen en el disco que comprende la producción hasta febrero de 2001, teniendo en cuenta el desfase temporal desde la publicación del artículos hasta su inclusión en la base de datos, es posible que en posteriores discos todavía puedan aparecer artículos correspondientes al período de estudio, especialmente al último año, 1999, lo que podría alterar ligeramente los resultados que aquí presentamos.

A continuación se muestra como ejemplo un registro tipo de la base de MathSci con sus principales campos:

- **Nº Inst:** 1
- **Nº Autores:** 4
- **Título:** On the use of divergence statistics to make inferences about three habitats.
- **Año de publicación:** 1995
- **Revista:** Kybernetes [Kybernetes.-The-International-Journal-of-Systems-and- Cybernetics] 24 (1995), no. 1, 2, 44--54.
- **Tipo de documento:** Journal
- **ISSN:** 0368-492X
- **Autores:** Esteban,-M.-D., (E-MADC)
  Pardo,-J.-A. [Pardo-Llorente,-Julio- Angel], (E-MADC)
  Pardo,-M.-C. [Pardo,-Maria-del-Carmen], (E-MADC)
  Vicente,-M.-L. [Vicente-Hernanz,-Ma.-Lina], (E-MADC)
- **Instituciones:** (E-MADC), Department of Mathematics, Universidad Complutense de Madrid, 28040 Madrid, Spain
- **MSC:** 62B10, 62B, 62

Las bases de datos multidisciplinares Science Citation Index y Social Sciences Citation Index recogen más de 5.000 revistas, mayoritariamente en lengua inglesa. Las revistas recogidas por las bases de datos ISI son las más representativas de la actividad científica internacional. El Science Citation Index recoge 3.500 revistas sobre ciencia y tecnología que cubren más de 150 disciplinas, mientras que el Social Sciences Citation Index recoge 1.700 revistas que cubren más de 50 disciplinas.
Un defecto de las bases de datos ISI es el sesgo lingüístico y geográfico que presentan a favor de las revistas en inglés, por tanto a favor de las publicadas en EEUU y el Reino Unido.

2.2. Estrategia de búsqueda

La producción científica matemática en España se obtuvo seleccionando en la base de datos MathSci todos aquellos documentos en los que apareciese “Spain” en el campo “Institution” y correspondientes a cualquier año comprendido entre 1990 y 1999 en el campo “Publication Year”.

2.3. Tipo de documento

De los cinco tipos de documentos que contempla la base de datos MathSci: “journal”, “journal-translation”, “book”, “book proceedings” y “proceedings-paper”, se han seleccionado exclusivamente los de tipo “journal” que corresponden a los artículos. Esta elección viene motivada además que éste el tipo de documento en la base del ISI.

2.4.Clasificación temática MSC y Áreas de Conocimiento

La clasificación temática utilizada es la proporcionada por la base de datos MathSci en el campo “Primary Classification Codes” correspondiente a la “Mathematics Subject Classification 1991”. A partir del año 2000 existe una actualización de la clasificación MSC que, aunque en principio no debería afectar al presente estudio, ha sido utilizada para clasificar un porcentaje menor del 0.05 % de los documentos. La nueva clasificación no ha supuesto grandes cambios al nivel con el que trabajamos, siendo el más significativo la creación de tres nuevos códigos: los números 37, 74 y 91, que aparecen en cursiva en la siguiente lista.

Para relacionar los artículos con las Áreas de Conocimiento, se ha asignado cada código MSC a una o varias Áreas de Conocimiento. Esta asignación se ha realizado por los encargados del Informe, con las consultas pertinentes en los casos más dudosos. En cualquier caso hemos optado siempre por asignar cada código MSC a todas la Áreas de Conocimiento a las que pudiera pertenecer de modo natural. Esto implica que los documentos de un código MSC son contabilizados en cada una de las Áreas a las que ha sido adscrito, por lo que en las tablas correspondientes a Áreas de Conocimiento se obtienen sumatorios totales superiores al número total real de documentos.

La siguiente tabla muestra la clasificación MSC traducida al castellano, y la asignación a Áreas de Conocimiento. En el Apéndice se incluye la clasificación MSC original (en inglés).

Clasificación MSC por áreas de conocimiento:

00	General	Todas
01	Historia y biografías	Todas
03	Lógica y fundamentos	Álgebra, Cc. de la Computación e IA
04	Teoría de conjuntos	Álgebra
05	Combinatoria	Álgebra, Estadística e IO
06	Retículos, estructuras algebraicas ordenadas	Álgebra
08	Sistemas matemáticos generales	Todas
11	Teoría de números	Álgebra
12	Teoría de cuerpos y polinomios	Álgebra
13	Anillos conmutativos y álgebras	Álgebra
14	Geometría algebraica	Álgebra, Geometría y Topología
15	Álgebra lineal y multilineal, teoría de matrices	Álgebra
16	Anillos y álgebras asociativos	Álgebra
17	Anillos y álgebras no asociativos	Álgebra
18	Teoría de categorías, álgebra homológica	Álgebra
19	K-teoría	Álgebra, Geometría y Topología
20	Teoría de grupos y generalizaciones	Álgebra
22	Grupos topológicos, grupos de Lie	Álgebra, Geometría y Topología
26	Funciones reales	Análisis matemático
28	Medida e integración	Análisis matemático
30	Funciones de una variable compleja	Análisis matemático
31	Teoría de potencial	Análisis matemático, Matemática Aplicada
32	Varias variables complejas y espacios analíticos	Análisis matemático, Geom. y Topología
33	Funciones especiales	Análisis matemático
34	Ecuaciones diferenciales ordinarias	Análisis matemático, Matemática Aplicada
35	Ecuaciones en derivadas parciales	Matemática Aplicada
37	*Sistemas dinámicos y teoría ergódica*	*Mat. Aplicada, Geom. y Topología, Análisis Mat.*
39	Ecuaciones de diferencias finitas y funcionales	Análisis matemático, Matemática Aplicada
40	Sucesiones, series, sumabilidad	Análisis matemático, Matemática Aplicada
41	Aproximaciones y expansiones	Matemática Aplicada
42	Análisis de Fourier	Análisis matemático, Matemática Aplicada
43	Análisis armónico abstracto	Análisis matemático
44	Transformaciones integrales, cálculo operacional	Matemática Aplicada
45	Ecuaciones integrales	Análisis Mat., Mat. Aplicada, Geom. y Topología
46	Análisis funcional	Análisis matemático
47	Teoría de operadores	Análisis matemático, Matemática Aplicada
49	Cálculo de variaciones, optimización	Matemática Aplicada, Geometría y Topología
51	Geometría	Álgebra, Geometría y Topología
52	Geometría convexa y discreta	Álgebra, Geometría y Topología, Mat. Aplicada
53	Geometría diferencial	Geometría y Topología
54	Topología general	Geometría y Topología
55	Topología algebraica	Álgebra, Geometría y Topología
57	Variedades y complejos celulares	Geometría y Topología
58	Análisis global, análisis en variedades	Análisis Mat., Mat. Aplicada, Geom. y Topología
60	Teoría de la probabilidad y procesos estocásticos	Estadística e IO, Análisis Mat., Mat. Aplicada
62	Estadística	Estadística e IO
65	Análisis numérico	Matemática Aplicada
68	Ciencias de la computación	Cc. de la Computación e IA
70	Mecánica de sistemas y partículas	Geometría y Topología, Matemática Aplicada
73	Mecánica de sólidos	Geometría y Topología, Matemática Aplicada
74	*Mecánico de sólidos deformables*	*Geometría y Topología, Matemática Aplicada*
76	Mecánica de fluidos	Matemática Aplicada
78	Óptica, electromagnetismo	Matemática Aplicada
80	Termodinámica clásica, transmisión del calor	Matemática Aplicada
81	Teoría cuántica	Geometría y Topología, Matemática Aplicada
82	Mecánica estadística, estructura de la materia	Geom. y Top., Mat. Aplicada, Estadística e IO
83	Relatividad y teoría gravitatoria	Geometría y Topología, Matemática Aplicada

85	Astrofísica y astronomía	Geometría y Topología, Matemática Aplicada
86	Geofísica	Matemática Aplicada
90	Economía, investigación operativa, programación, juegos	Estadística e IO
91	*Teoría de juegos, economía, ciencias sociales y del comportamiento*	*Estadística e IO*
92	Biología y otras ciencias naturales, ciencias del comportamiento	Matemática Aplicada
93	Teorías del control y sistema	Matemática Aplicada
94	Información y comunicaciones, circuitos	Matemática Aplicada, Cc. de la Comput. e IA

A la hora de contabilizar el personal investigador, se han utilizado las fuentes del Consejo de Universidades, por lo que sólo se tienen en cuenta los datos referentes a profesores numerarios (titulares y catedráticos universidad y de escuelas universitarias) pertenecientes a las áreas de conocimiento referidas a matemáticas: “Análisis Matemático”, “Ciencias de la Computación e Inteligencia Artificial”, “Geometría y Topología”, “Matemática Aplicada” y “Estadística e Investigación Operativa”.
Obviamente, ello supone que no se ha contabilizado los profesores no numerarios, a los que presentamos nuestras disculpas, pero nos resultaba extremadamente complicado tener datos fiables sobre los mismos.

2.5. Instituciones

Para determinar las instituciones que han colaborado en la producción de un documento hemos recurrido al campo “Institution” de la base de datos MathSci, campo que incluye el lugar de trabajo de cada uno de los autores firmantes de un documento. Esta información permite estudiar la productividad de las instituciones y la colaboración matemática entre las mismas.

Hay que señalar que esta información no está normalizada, apareciendo unas veces información relativa a departamentos, otras a facultades y otras a universidades. Dadas las diferencias existentes en la organización de las distintas universidades respecto a facultades y departamentos, en los estudios pormenorizados se ha optado por descender únicamente hasta el nivel de desagregación por Universidades.

Para el estudio de la actividad matemática de las instituciones a un nivel general, los centros se agrupan en los siguientes sectores institucionales: Universidad, Consejo Superior de Investigaciones Científicas (CSIC), centros mixtos CSIC-Universidad y otros centros. Como sólo aparecen dos centros ajenos a la universidad y al CSIC en la base de datos, se ha optado por dejar que aparezcan los datos referentes a cada uno de ellos.

2.6. Autores

El estudio de los autores firmantes de un documento se ha realizado a través del campo “Author” de la base de datos MathSci, donde se incluye el nombre (generalmente en forma de nombre y un apellido) y el lugar de trabajo de cada uno de los autores del documento.

2.7. Adscripción de documentos

En este estudio se ha utilizado el sistema de recuento total, según el cual se asigna cada documento completo a cada uno de los autores y por consiguiente, a cada una de las instituciones firmantes del mismo. Se ha preferido este método frente al recuento fraccionado de documentos, en el que cada documento escrito por varios autores se divide por el número de ellos, o al recuento por primer autor, ya que en Matemáticas es tradicional que el orden de firma sea generalmente el alfabético. El sistema de recuento completo permite cuantificar la participación de las distintas instituciones en los trabajos, ofrece una visión más completa que el recuento por primer autor, y su fiabilidad ha sido repetidamente comprobada. El inconveniente que presenta este método es la multiplicación de documentos en los recuentos, que hace que los sumatorios alcancen valores superiores al total real de documentos.

2.8. Tratamiento de datos

Los datos procedentes de la base de datos MathSci se descargaron en una base de datos relacional. Esta base de datos consta de una serie de ficheros de datos:

- Documentos
- Instituciones
- Autores
- Clasificación MSC
- Revistas SCI

2.9. Indicadores bibliométricos

a) Factor de impacto (FI)

Como indicador de visibilidad o difusión de los resultados de la investigación se ha utilizado el factor de impacto de las revistas de publicación, tal como aparece en los Journal Citation Reports de los años comprendidos en el periodo 1990-1999. El factor de impacto de una revista representa el número medio de citas recibidas por artículo en un periodo de tiempo. Así, el factor de impacto de 1998 de la revista X se calcula dividiendo las citas que, en 1998, las revistas fuente del SCI, SSCI y A&HCI (Arts and Humanities Citation Index) han hecho a los artículos de la revista X de los años 1996 y 1997, dividido por el total de ítems citables publicados por la revista X en esos dos años.

El factor de impacto de una revista se utiliza como una medida indirecta de su calidad, pero en realidad se limita a valorar su impacto o influencia sobre la comunidad científica. Aunque la validez del factor de impacto como indicador de visibilidad es un hecho ampliamente aceptado, hay que tener en cuenta ciertas limitaciones en su uso. Por ejemplo, en su cálculo solamente se recogen las citas recibidas a muy corto plazo, lo que perjudica especialmente a las áreas de evolución más lenta y a las revistas que no cumplen las fechas de publicación previstas. Por otro lado, no es posible comparar factores de impacto correspondientes a diferentes temas, por depender, entre otras causas, del tamaño de la comunidad científica, de sus hábitos de publicación y del carácter básico o aplicado del campo. Esto obliga a manejar por separado el factor de impacto de cada disciplina.

En el presente estudio se emplea el FI de una revista como "factor de impacto esperado" de todos los documentos publicados en la misma, es decir, se considera que todos los artículos de una revista tienen la misma probabilidad de ser citados.

b) Índice de actividad

El índice de actividad es un indicador del grado de actividad en un determinado tema por parte de un centro o área geográfica, señalando si la actividad es mayor o menor que el promedio nacional.
Se calcula como el cociente entre el porcentaje de la producción de un centro o área geográfica dedicada a un determinado tema y el porcentaje que representa el tema en la producción nacional.

c) Posición normalizada

La imposibilidad de comparar factores de impacto correspondientes a revistas de distintas disciplinas ISI hace necesaria otra medida que sí lo permita. La posición normalizada se calcula como el complemento a 1 del cociente que resulta de dividir la posición de la revista dentro de la disciplina ISI por el número total de revistas dentro de esa disciplina. En este estudio, cuando una revista pertenece a varias disciplinas ISI, se ha optado por tener en cuenta únicamente la posición normalizada más elevada.

d) Colaboración

En el estudio de la colaboración puede distinguirse entre varios tipos: cuando en el campo "Institution" figura una dirección extranjera, la colaboración es considerada *internacional* y cuando figuran más de una institución española, se considera documento en *colaboración nacional*. Dentro de ésta se distinguen dos tipos: colaboración *nacional extramuros*, cuando se da entre distintos centros de investigación nacionales, y colaboración *nacional intramuros*, cuando se da entre diferentes departamentos de un mismo centro. Cuando se trata de universidades, la colaboración extramuros se refiere a la colaboración interfacultativa y la colaboración intramuros a la colaboración interdepartamental. Cuando se trata de centros del CSIC la colaboración extramuros se refiere a la colaboración entre los distintos centros del CSIC, pero no tiene sentido hablar de la colaboración intramuros por no estar disponible en la base de datos esta información.
Cuando se da la colaboración nacional y la internacional en un mismo artículo, este se contabiliza en ambos tipos de colaboración.

e) Índice de coautoría

Es un indicador de la colaboración entre autores y se calcula como el número medio de autores que participan en un documento.

3. BASE DE DATOS MATHSCI

Producción matemática en la base de datos MathSci

En la tabla 3.1 se muestran los artículos que contiene la base de datos MathSci, en las últimas décadas. Los datos referente a España y la Unión Europea no pueden obtenerse hasta la década de los 80. Para la consecución de los datos, al igual que para el resto de resultados de esta sección, se ha utilizado la versión en línea de la base de datos MathSci.

Década	España	UE	Mundial
1940-1949			32595
1950-1959			73863
1960-1969			135347
1970-1979			279882
1980-1989	3334	45922	349463
1990-1999	11504	104231	481105
Total	**14839**	**150190**	**1352255**

Tabla 3.1. Producción matemática por décadas en MathSci

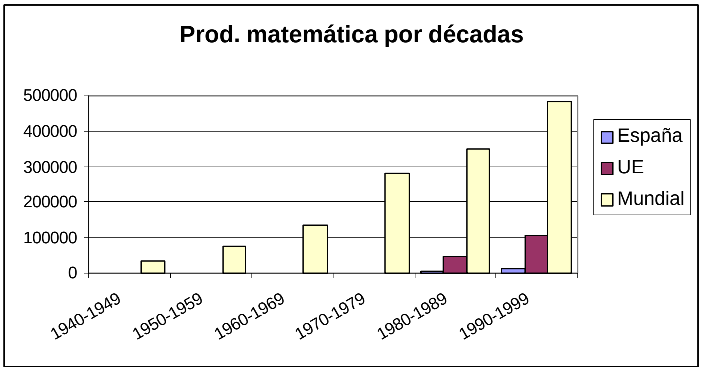


Gráfico 3.1. Producción matemática por décadas en MathSci

Distribución de la producción matemática en el periodo 1990-1999 en la base de datos MathSci

La tabla 3.2 muestra la tipología de la producción matemática correspondiente al período 1990-1999 que aparece en la base de datos MathSci.Tanto en la producción matemática española como en la de la Unión Europea y la mundial, existe un claro predominio del artículo, llamado "journal" en la base de datos, como tipo de documento más utilizado, suponiendo el 78% de la producción española, el 79% de la producción de la UE y el 78% de la producción mundial. Le siguen los proceedings-paper que suponen un 19% de la producción española y de la UE y un 15% de la producción mundial. Las traducciones de artículos y los book-proceedings tienen un peso muy bajo en la producción, al igual que los libros, que suponen un 4% de la producción mundial, un 2% de la producción de la UE y un 1% en la producción española.

La producción matemática española durante la década de los 90 supone el 10,6% de la producción de la Unión Europea y el 2,3% de la producción matemática mundial.

Tipo de documento	España	UE	Mundial	% España respecto UE	% España resp. prod. mundial
Journal	11813	110106	494330	10,7%	2,4%
Proceedings-paper	2862	26978	99341	10,6%	2,9%
Book	118	2532	22701	4,7%	0,5%
Journal-translation	36	365	27032	9,9%	0,1%
Book-proceedings	2	46	9332	4,3%	0,0%
Total	**14831**	**140027**	**652736**	**10,6%**	**2,3%**

Tabla 3.2. Tipología de la producción matemática en 90-99 según MathSci

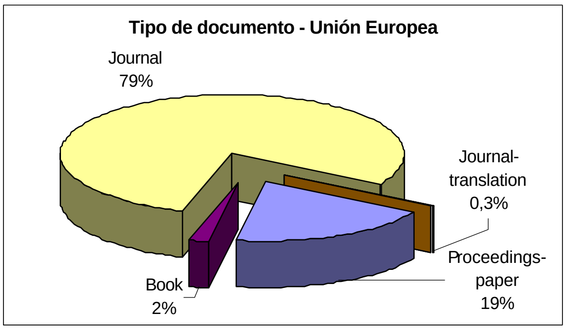


Gráfico 3.2. Tipo de documentos en la UE

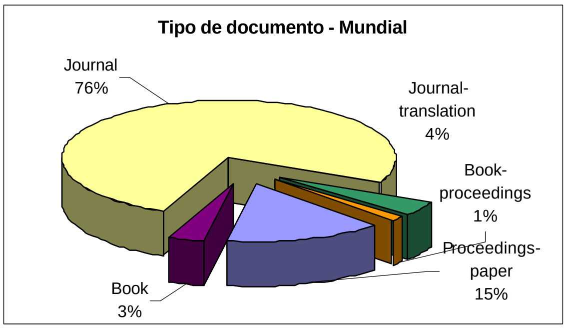


Gráfico 3.3. Tipo de documentos en el mundo

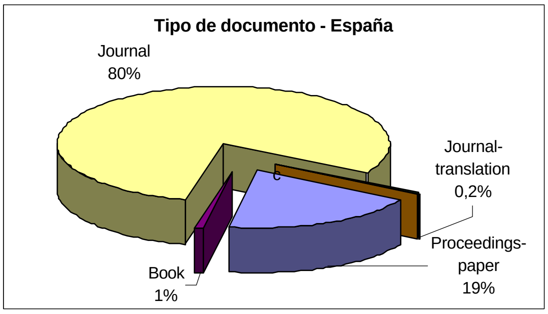


Gráfico 3.4. Tipo de documentos en España

Por ser el artículo el tipo de documento objeto del presente estudio, vamos a comparar su evolución a lo largo de la última década tanto en la base de datos MathSci como, en el próximo capítulo, en la base de datos ISI.

Comparación entre la producción matemática mundial y la española en el periodo 1990-1999 correspondiente a artículos en la base de datos MathSci

En la tabla 3.3 se compara la evolución anual de la producción matemática mundial con la producción española durante el periodo 1990-1999 en la base de datos MathSci. La discrepancia que existe entre los totales que aparecen en esta tabla y en la tabla 3.2 es debida a los artículos que aparecen con una doble fecha de publicación (1992-1993, por ejemplo) en la base de datos MathSci, que aparecen contabilizados doblemente en la tabla 3.3.

Tanto en el mundo como en los ámbitos europeo y español, la década de los 90 se caracteriza por un aumento de la producción matemática recogida en la base de datos MathSci. El incremento real de esta producción matemática desde el año 1990 al año 1999 es de un 27% a nivel mundial, un 58% a nivel europeo y un 133% a nivel español. También en el crecimiento anual se puede observar que la producción española crece a un nivel mayor que la del resto del mundo.

Resultan curiosos los descensos en la producción de los años 1993 y 1998 que se producen tanto a nivel mundial como estatal, y que en parte pueden ser debidos a los altos crecimientos de los años anteriores. No obstante, en nuestro estudio hemos detectado algunas duplicaciones de documentos en la base de datos MathSci, precisamente en los años 1992 y 1997, por lo que el alto crecimiento de estos años puede estar un poco inflado por este fenómeno. Estudiando la serie del porcentaje relativo que la producción española supone respecto a la producción mundial, se observa que ésta ha ido creciendo a lo largo de la década, pasando de ser el 1,7% en el año 1990 al 3,2% en el año 1999. Esto también ocurre si comparamos en el seno de la UE, donde la producción española durante la última década ha pasado de suponer el 8,9% en 1990 a suponer el 13,0% en 1999.

Los datos referidos a los últimos años, sobre todo a los concernientes al año 1999, podrían sufrir alguna pequeña variación por inclusión de información de últimos datos en los próximos discos.

	España		**UE**		**Mundial**		**% relativo**	
Año	**Nº docs**	**Increm**	**Nº docs**	**Increm**	**Nº docs**	**Increm**	**España - Mundo**	**España - UE**
1990	690		7795		40116		**1,7%**	**8,9%**
1991	919	*33,2%*	9954	*27,7%*	47073	*17,3%*	**2,0%**	**9,2%**
1992	1167	*27,0%*	11854	*19,1%*	54078	*14,9%*	**2,2%**	**9,8%**
1993	926	*-20,7%*	8655	*-27,0%*	41576	*-23,1%*	**2,2%**	**10,7%**
1994	901	*-2,7%*	9363	*8,2%*	43620	*4,9%*	**2,1%**	**9,6%**
1995	1097	*21,8%*	10321	*10,2%*	46853	*7,4%*	**2,3%**	**10,6%**
1996	1323	*20,6%*	12585	*21,9%*	56121	*19,8%*	**2,4%**	**10,5%**
1997	1776	*34,2%*	15925	*26,5%*	65653	*17,0%*	**2,7%**	**11,2%**
1998	1428	*-19,6%*	11986	*-24,7%*	50508	*-23,1%*	**2,8%**	**11,9%**
1999	1610	*12,7%*	12344	*3,0%*	50885	*0,7%*	**3,2%**	**13,0%**
Total	**11837**		**110782**		**496483**		**2,4%**	**10,7%**

Nota: Incrementos calculados respecto al año anterior

Tabla 3.3. Comparación de la producción matemática durante 90-99 según MathSci

La siguiente gráfica compara las evoluciones anuales de la producción matemática. Con el fin de homogeneizar los recorridos de valores y poder así compararlas, las gráficas correspondientes a España y la UE se han multiplicado por el cociente entre el total de documentos mundial y el total de documentos respectivos de España y la UE.

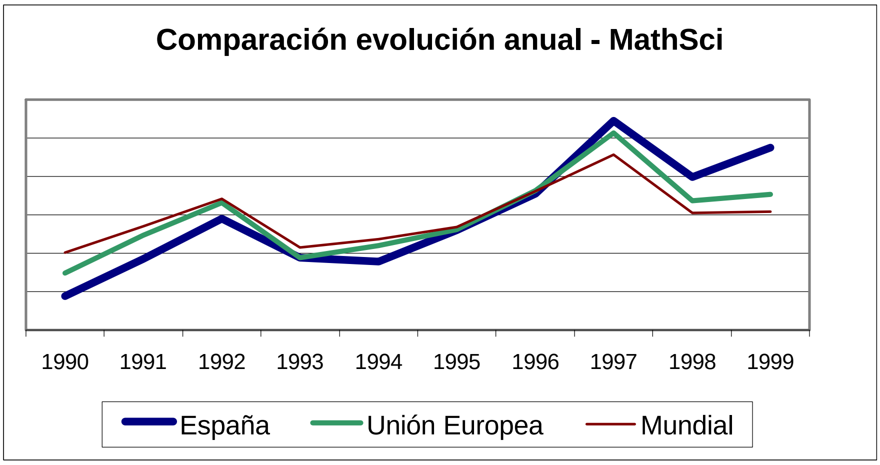


Gráfico 3.5. Evolución del número de documentos en la MathSci

Distribución de la producción de MathSci por clasificación MSC, mostrando el índice de actividad de España

En la tabla 3.4 se presenta la producción española y mundial de la década 1990-1999 clasificada por su código principal MSC, y los porcentajes que el código representa dentro de la producción total. La última columna muestra el índice de actividad de España, calculado como el cociente entre el porcentaje de producción española en el área y el porcentaje mundial en la misma. Un índice de actividad mayor que uno indica una actividad mayor que la media mundial en el área.
Como el lector comprobará trabajamos aquí con el total real de los 11.813 artículos correspondientes a la producción española en la base de datos MathSci durante la década 1990-1999 y los 494.330 artículos correspondientes a la producción mundial, que figuraban en la tabla 3.2. Las pequeñas diferencias con los totales que aparecen en la tabla se deben a que algunos documentos no incluían la información del campo MSC, por lo que no ha podido ser catalogados.
Se observa que los porcentajes de España respecto a la producción total están muy por encima de los mundiales en los códigos:

44: Transformaciones integrales, cálculo operacional
46: Análisis funcional
04: Teoría de conjuntos
53: Geometría diferencial
42: Análisis de Fourier

En todos ellos, el porcentaje que la producción española representa frente a la mundial está muy por encima del porcentaje medio que representa la producción española en la producción mundial de toda la década, que habíamos visto que alcanzaba el 2,4% (tabla 3.3).
Por el contrario, la aportación española resulta mínima en temas como

51: Geometría (general)
11: Teoría de números
05: Combinatoria

Estos datos hay que tomarlos con las cautelas necesarias ya que, por ejemplo, muchas de las publicaciones españolas en Teoría de Números van a parar al área de Geometría Algebraica y, similarmente, las publicaciones de Geometría se suelen encaminar hacia códigos más específicos. Los datos referidos a los nuevos temas de la clasificación MSC no se tienen en cuenta por no resultar representativos.

Código MSC	Publicaciones en el periodo 1990-1999	Porcentaje respecto al total	Publicaciones en España	Porcentaje respecto a la actividad española	Porcentaje respecto al código	Indice de actividad
00	1174	0,2	37	0,3	3,15	**1,32**
01	6915	1,4	129	1,1	1,87	0,78
03	11012	2,2	280	2,4	2,54	**1,06**
04	986	0,2	53	0,4	5,38	**2,24**
05	19418	3,9	128	1,1	0,66	0,28
06	3369	0,7	30	0,3	0,89	0,37
08	982	0,2	9	0,1	0,92	0,38
11	16214	3,3	127	1,1	0,78	0,33
12	1202	0,2	37	0,3	3,08	**1,29**
13	3599	0,7	141	1,2	3,92	**1,64**
14	6732	1,4	228	1,9	3,39	**1,41**
15	4955	1,0	88	0,7	1,78	0,74
16	7490	1,5	237	2,0	3,16	**1,32**
17	4719	1,0	224	1,9	4,75	**1,98**
18	1332	0,3	53	0,4	3,98	**1,66**
19	603	0,1	7	0,1	1,16	0,48
20	11812	2,4	249	2,1	2,11	0,88
22	3060	0,6	29	0,2	0,95	0,40
26	3747	0,8	37	0,3	0,99	0,41
28	3430	0,7	104	0,9	3,03	**1,27**
30	7554	1,5	138	1,2	1,83	0,76
31	1292	0,3	21	0,2	1,63	0,68
32	5600	1,1	135	1,1	2,41	**1,01**
33	3441	0,7	120	1,0	3,49	**1,46**
34	15826	3,2	326	2,8	2,06	0,86
35	25191	5,1	587	5,0	2,33	0,97
37*	*1525*	*0,3*	*82*	*0,7*	*5,38*	*2,25*
39	2797	0,6	63	0,5	2,25	0,94
41	5236	1,1	140	1,2	2,67	**1,12**
42	4926	1,0	249	2,1	5,05	**2,11**
43	1114	0,2	9	0,1	0,81	0,34
44	931	0,2	90	0,8	9,67	**4,04**
45	1693	0,3	23	0,2	1,36	0,57
46	12639	2,6	1082	9,2	8,56	**3,57**
47	11758	2,4	262	2,2	2,23	0,93
49	6382	1,3	88	0,7	1,38	0,58
51	3322	0,7	10	0,1	0,30	0,13
52	3618	0,7	45	0,4	1,24	0,52
53	10330	2,1	529	4,5	5,12	**2,14**
54	8583	1,7	228	1,9	2,66	**1,11**
55	2575	0,5	119	1,0	4,62	**1,93**
57	5427	1,1	96	0,8	1,77	0,74
58	18537	3,8	583	4,9	3,15	**1,31**
60	19036	3,9	273	2,3	1,43	0,60
62	27034	5,5	644	5,5	2,38	0,99
65	23848	4,8	499	4,2	2,09	0,87
68	18825	3,8	259	2,2	1,38	0,57
70	3552	0,7	152	1,3	4,28	**1,79**
73	8401	1,7	133	1,1	1,58	0,66
74*	*1043*	*0,2*	*23*	*0,2*	*2,21*	*0,92*
76	12985	2,6	189	1,6	1,46	0,61
78	2494	0,5	38	0,3	1,52	0,64
80	1312	0,3	18	0,2	1,37	0,57
81	29527	6,0	840	7,1	2,84	**1,19**
82	9999	2,0	160	1,4	1,60	0,67

83	11605	2,4	378	3,2	3,26	**1,36**
85	416	0,1	9	0,1	2,16	0,90
86	1133	0,2	12	0,1	1,06	0,44
90	21311	4,3	499	4,2	2,34	0,98
91*	*1037*	*0,2*	*67*	*0,6*	*6,46*	*2,70*
92	4372	0,9	64	0,5	1,46	0,61
93	17166	3,5	233	2,0	1,36	0,57
94	5065	1,0	69	0,6	1,36	0,57
Total	**493209**		**11811**			

* Proceden de MSC 2000

Tabla 3.4. Distribución de la producción MathSci según la MSC

4. BASE DE DATOS ISI

En este capítulo presentamos los datos de la producción matemática española de la década que aparecen en la base de datos del ISI. Para ello se ha utilizado como fuente de datos el SCISearch, versión en línea de la base de datos SCI.

4.1 Comparación entre la producción matemática mundial y la española en el periodo 1990-1999 correspondiente a artículos en la base de datos ISI

En la tabla 4.1 se compara la evolución anual de la producción matemática mundial con la producción española durante el periodo 1990-1999 en la base de datos ISI. Se han tenido en consideración los documentos que recoge la base de datos ISI durante los años 1990-1999 correspondientes a las disciplinas matemáticas ISI: "mathematical methods, biology & medicine", "mathematical methods, physical science", "mathematical methods, social sciences", "mathematics", "mathematics and statistics", "applied mathematics", "general mathematics", "miscellaneous mathematics", "pure mathematics", "mathematics, statistics & probability", "mathematical physics", "mathematical psychology", "social sciences, mathematical methods" y "statistics & probability". Se estima que puede faltar por contabilizar un 10% de los documentos correspondientes al último año, debido a la demora respecto a su fecha de publicación con que se recogen los datos.

Un primer análisis de esta tabla reafirma la conclusión obtenida en el capítulo anterior de que la producción matemática española ha crecido durante la última década muy por encima de lo que lo ha hecho la producción matemática mundial. Mientras que la producción mundial ha ido presentando un ligero aumento durante toda la década, la producción española ha ido siempre creciendo a un ritmo mucho más rápido (con excepción de los años 1994 y 1996, en los que, de nuevo, haya que tener en cuenta el efecto del gran aumento que se había dado en años anteriores).

La aportación española en la base de datos ISI ha pasado de representar el 1,7% de la producción mundial en 1990 al 3,9% en 1999, y ha continuado creciendo hasta situarse en el 4,18% según los últimos datos del ISI del 2001.

	España		Mundial		% relativo
Año	**Nº docs**	**Incremento**	**Nº docs**	**Incremento**	**España - Mundial**
1990	339		20500		1,7%
1991	374	*10,3%*	21386	*4,3%*	1,7%
1992	459	*22,7%*	22081	*3,2%*	2,1%
1993	606	*32,0%*	23651	*7,1%*	2,6%
1994	627	*3,5%*	25126	*6,2%*	2,5%
1995	785	*25,2%*	26917	*7,1%*	2,9%
1996	835	*6,4%*	28133	*4,5%*	3,0%
1997	1010	*21,0%*	30278	*7,6%*	3,3%
1998	1128	*11,7%*	31457	*3,9%*	3,6%
1999	1256	*11,3%*	31883	*1,4%*	3,9%
Total	**7419**		**261412**		**2,8%**

Nota: Incrementos calculados respecto al año anterior

Tabla 4.1. Comparación de la producción matemática durante 90-99 según ISI

La siguiente gráfica compara las evoluciones anuales de la producción matemática. Como antes, al efecto de poder compararlas, la gráfica correspondiente a España se ha multiplicado por el cociente entre el total de documentos mundial y el total de documentos de España.

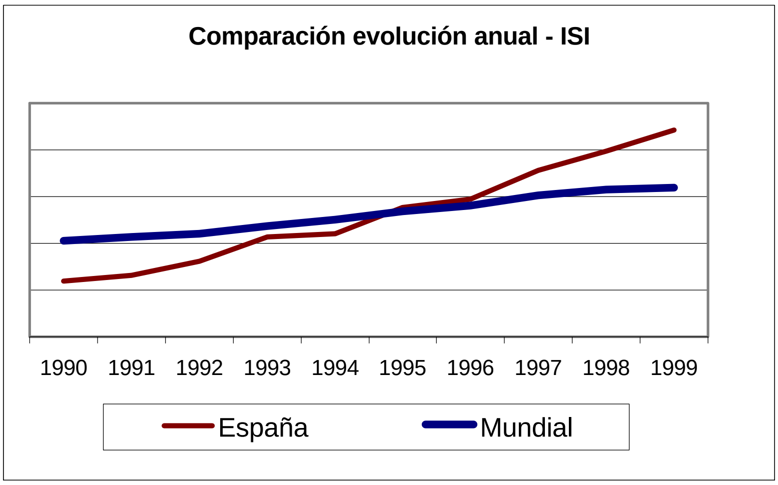


Gráfico 4.1. Evolución del número de documentos en el ISI

4.2. Selección de artículos para el estudio y su distribución en el período 1990-1999

Para hacer estudios más finos de la producción cualitativa española hemos seleccionado de los 11.813 trabajos de la base española MathSci, aquellos publicados en revistas que aparecen en la base de datos ISI. Posteriormente hemos descartado una serie de trabajos que aparecían publicados en epígrafes no propiamente matemáticos dentro de la clasificación usada por el ISI y cuyo contenido fue considerado claramente no matemático por los expertos consultados. Esta filtración ha afectado especialmente a artículos de Física que aparecían en la base de datos MathSci y por consiguiente estaban recogidos entre las 11.813 extraídas de dicha base.

Los documentos finalmente seleccionados y con los que va a realizarse el presente estudio son los 6.220 que se reparten en la década de la siguiente manera:

Año	Nº art.	Incremento
1990	330	
1991	388	17,6%
1992	448	15,5%
1993	520	16,1%
1994	524	0,8%
1995	644	22,9%
1996	672	4,3%
1997	828	23,2%
1998	883	6,6%
1999	983	11,3%
Total	**6220**	

Nota: Incrementos calculados respecto al año anterior
Tabla 4.2. Producción matemática española 90-99

No obstante en la sección 6.5 se ofrecen también algunos datos de la base ISI sin el filtrado posterior, que efectivamente abundan en la idea de la necesidad del mismo.

Hay que tener en cuenta que la base de datos del ISI incluye muy pocas revistas españolas, por lo que trabajamos casi exclusivamente con la producción matemática española publicada en revistas extranjeras. Durante la década 1990-1999, ninguna revista española de Física y tan sólo dos de matemáticas aparecen en el ISI: la *Revista Matemática Iberoamericana* que se incorporó al mismo en 1998, y la revista *Test* en 1999. En el 2000 se ha incorporado *Publicacions Matematiques*, editada en la UAB, y es probable que alguna más se incorpore en el 2001. Todo ello es síntoma también de una mejora en la competitividad de nuestra publicaciones, aunque es sabido que al ser el ISI un organismo privado también hay otros factores (entre ellos la posible rentabilidad) que inciden en la inclusión o no de una revista en dicha base.

4.3. Distribución de la producción matemática en el periodo 1990-1999

El gráfico 4.2 recoge la evolución de la producción española durante la década 1990-1999. La producción matemática ha experimentado un crecimiento continuado y elevado a partir del año 1990. No obstante se observa a partir de 1993, que dicho aumento no es lineal, sino que ofrece un cierto patrón a bienal, siendo muy inferior en los años 94, 96 y 98, y superior en el 95, 97 y 99.

Simplificando, cabe decir que la producción española se ha incrementado en un 300%, mientras que la producción mundial lo ha hecho en menos de la mitad.

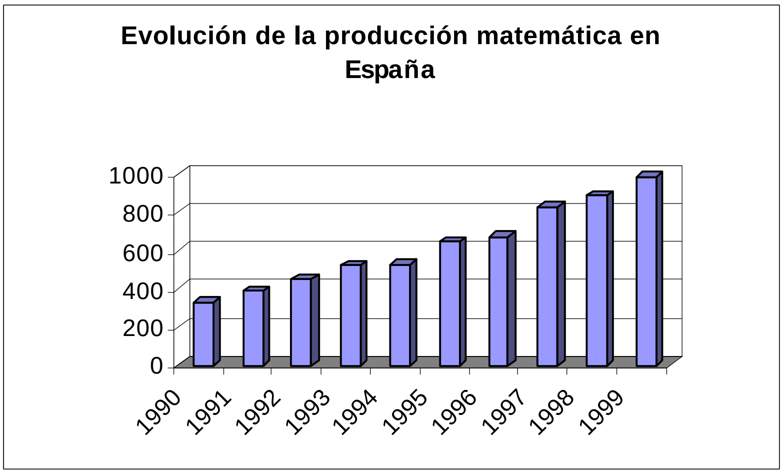


Gráfico 4.2. Evolución de la producción matemática en España

4.4. La investigación matemática en el seno de la investigación nacional

Por último, para tener un panorama de lo que supone la investigación matemática dentro de toda la investigación del país, así como para poder compararla con los países circundantes, hemos recogido en la tabla 4.3, los siguientes tres datos:

a) el porcentaje que la producción matemática española supone respecto a la producción mundial matemática del ISI,
b) el orden en el que el porcentaje anterior sitúa a las matemáticas dentro dentro de los 21 campos científicos que se relacionan a continuación, y
c) el impacto relativo, esto es, la desviación (en tanto por ciento) respecto al número medio de citas por artículo a nivel mundial. Por ejemplo, que España

aparezca con un –16 en esta columna significa que la media de citas que reciben los artículos españoles es un 16% inferior a la media mundial, esto es, que si, digamos, la media de citas de un artículo a nivel mundial es de 10 veces, los artículos españoles son citados un promedio de 8,4 veces.

El tercer lugar de las Matemáticas en España significa que sólo hay dos disciplinas (Astrofísica y Ciencias Agrarias) cuya producción científica nacional tiene un peso en la producción mundial superior al 4,18% que tiene ahora mismo la producción matemática. Los 21 campos considerados por el ISI son: ciencias del espacio, ciencias agrarias, matemáticas, microbiología, química, ciencias de las plantas y animales, ecología y medio ambiente, farmacología, física, biología y bioquímica, inmunología, ciencias materiales, neurociencia, biología molecular, medicina clínica, ciencias geológicas, económicas y empresariales, ingeniería, informática, psicología y psiquiatría y ciencias sociales.

Los datos de España, Japón, Francia y Estados Unidos están referidos al quinquenio 1996-2000, mientras que el resto lo son al 1995-1999 y han sido extraídos de la página web del Institute for Scientific Information de Filadelfia, ISI.

Como se ve, en España, la investigación matemática ocupa el tercer lugar de toda la producción científica nacional (a pesar de que los recursos dedicados a ella no pueden compararse con los dedicados a otras disciplinas), aunque la media de citas por artículo está por debajo de la media mundial. Sólo en Francia la investigación matemática ocupa un lugar más importante dentro de la producción nacional, ya que se encuentra en primer lugar, pero no debemos olvidar la extraordinaria tradición matemática de Francia. Tras España, son Alemania e Italia los países que siguen, quedando los demás países con órdenes muy inferiores. Los países de tamaño menor: Dinamarca, Bélgica y Noruega son los países cuya producción tiene un mayor impacto relativo: el número medio de citas por artículo está muy por encima de la media mundial.

País	**%**	**Orden**	**Impacto relativo**
España	4,18%	3º	-16
Alemania	9,88%	4º	2
Australia	2,78%	12º	15
Bélgica	1,24%	11º	46
Dinamarca	0,79%	15º	52
Estados Unidos	35,43%	15º	29
Francia	12,25%	1º	2
Holanda	1,79%	21º	14
Italia	4,80%	6º	-2
Japón	5,26%	16º	-21
Noruega	0,49%	16º	35
Reino Unido	6,72%	21º	26
Suecia	1,30%	20º	-2
Suiza	1,25%	16º	31

Tabla 4.3. Comparación mundial de la investigación matemática

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5. ESTUDIO DE LA PRODUCCIÓN MATEMÁTICA

Todos los resultados con los que se presentan a continuación son el resultado de analizar los 6.220 artículos finalmente seleccionados.

5.1. Datos generales por Comunidades Autónomas

Distribución de la producción matemática de España por Comunidades Autónomas

El análisis de la producción matemática española por Comunidades Autónomas que recoge la tabla 5.1 por cifras absolutas pone de manifiesto la gran concentración de la investigación existente en Madrid y Cataluña. Son seguidas por Andalucía y, a cierta distancia, por la Comunidad Valenciana, acaparando entre estas cuatro comunidades autónomas casi el 70% de la producción matemática española.
Evidentemente, esto se debe a la gran concentración de universidades y profesores en estas comunidades, sobre todo Madrid y Cataluña, por lo que el dato de la proporción de artículos por profesor resulta importante. Así Aragón, que ocupa el quinto lugar por número de artículos, pasa a ser la comunidad autónoma con un mayor ratio de artículos por profesor seguida por Cantabria, mientras que las cuatro comunidades con mayor producción pasan a ocupar puestos más bajos, siendo Cataluña la comunidad que mejor mantiene su posición, pasando de ser la segunda con mayor producción a ser la tercera en cuanto a número de artículos por profesor.

La UNED se mantiene separada por no ser posible adscribir su producción a ninguna comunidad autónoma.

No hay que olvidar al utilizar estos datos, que sólo se ha tenido en cuenta el profesorado numerario, obtenido de la base de datos del año 2000 del Consejo de Universidades. Así, las Comunidades con mayor índice de "funcionarización" pueden verse perjudicadas en cuanto a la ratio, frente a las que mantienen un mayor número de profesores asociados y ayudantes que realicen investigación. Hay que recordar también, sobre todo a la hora de analizar los datos de Comunidades cuyas Universidades son de reciente creación, que las publicaciones pueden llevar un retraso bastante grande hasta su aparición final, y que la atribución del trabajo a una Comunidad Autónoma se hace por la dirección consignada en el artículo, que en la mayoría de los casos es la del momento del envío y no de su aparición.

Otro factor a considerar es la heterogeneidad de las CCAA en cuanto al número de universidades que comprenden. Madrid, Cataluña y Andalucía agrupan un gran número de universidades con comportamientos muy hetereogéneos, por lo que unas Universidades pueden "penalizar" a otras al promediarse los datos globales de la CCAA, mientras que, por ejemplo, Cantabria o La Rioja tienen una única universidad y por lo tanto los datos de la CCAA coinciden con los de ésta. La tabla 5.2 muestra las universidades por Comunidad Autónoma presentes en el estudio.

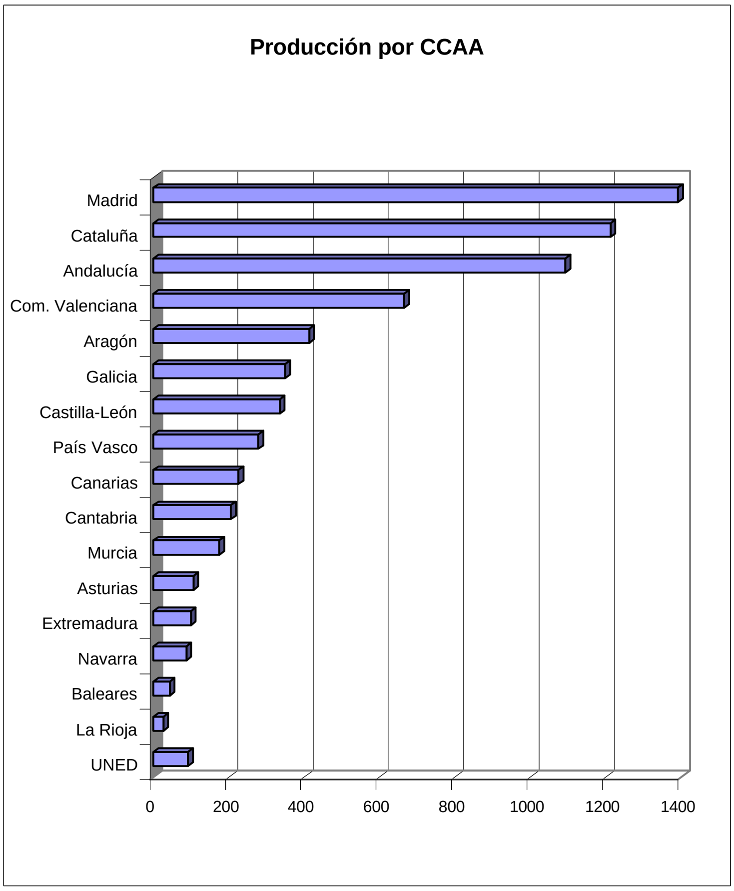


Gráfico 5.1. Producción por Comunidades Autónomas

CCAA	Nº art.	%	Nº art./profesor
Aragón	413	6,6%	3,82
Cantabria	205	3,3%	3,66
Cataluña	1212	19,5%	3,16
Extremadura	99	1,6%	2,75
Murcia	174	2,8%	2,49
Navarra	87	1,4%	2,42
Madrid	1391	22,4%	2,39
Valencia	665	10,7%	2,25
País Vasco	278	4,5%	2,09
Andalucía	1092	17,6%	1,92
Castilla-León	335	5,4%	1,84
Canarias	225	3,6%	1,73
Galicia	349	5,6%	1,65
La Rioja	26	0,4%	1,18
Asturias	106	1,7%	1,05
Baleares	43	0,7%	1,05
Total real	**6220**		**2,22**

Tabla 5.1. Producción matemática por comunidades autónomas

Tabla 5.2. Universidades por CCAA:

- Andalucía
  - Universidad de Almería
  - Universidad de Cádiz
  - Universidad de Córdoba
  - Universidad de Granada
  - Universidad de Jaén
  - Universidad de Málaga
  - Universidad de Sevilla
- Aragón
  - Universidad de Zaragoza
- Asturias
  - Universidad de Oviedo
- Baleares
  - Universidad de las Islas Baleares
- Canarias
  - Universidad de La Laguna
  - Universidad de Las Palmas de Gran Canaria
- Cantabria
  - Universidad de Cantabria
- Castilla-León
  - Universidad de Burgos
  - Universidad de Salamanca
  - Universidad de Valladolid
- Castilla la Mancha
  - Universidad de Castilla la Mancha
- Cataluña
  - Universidad Autónoma de Barcelona
  - Universidad de Barcelona
  - Universidad de Lleida
  - Universidad Jaume I
  - Universidad Politécnica de Cataluña
  - Universidad Pompeu Fabra
- Comunidad Valenciana
  - Universidad de Alicante
  - Universidad de Valencia
  - Universidad Politécnica de Valencia
- Extremadura
  - Universidad de Extremadura
- Galicia
  - Universidad de La Coruña
  - Universidad de Santiago de Compostela
  - Universidad de Vigo
- La Rioja
  - Universidad de La Rioja
- Madrid
  - Universidad Autónoma de Madrid
  - Universidad Carlos III de Madrid
  - Universidad Complutense de Madrid
  - Universidad de Alcalá de Henares
  - Universidad Politécnica de Madrid
- Murcia
  - Universidad de Murcia
- Navarra
  - Universidad de Navarra
  - Universidad Pública de Navarra
- País Vasco
  - Universidad del País Vasco

Evolución de la producción matemática por Comunidades Autónomas y año

En la tabla 5.3 se muestra la evolución de la producción por CCAA y el incremento experimentado, tomando como base el primer bienio estudiado, o en su defecto, el primer bienio en que la Comunidad Autónoma tuvo alguna publicación.

Las Comunidades con una mayor producción matemática, sobre todo Madrid y Cataluña, experimentan un crecimiento por debajo del incremento medio. Se observa, pues, una tendencia hacia la descentralización de la investigación. Asturias y Murcia son las comunidades con mayor crecimiento en su producción matemática, seguidas por Andalucía, Navarra y Canarias. El enorme incremento de La Rioja no resulta representativo por ser en un único bienio y contar con un único artículo en su primer año de producción matemática. Aún así, es destacable su rápido crecimiento. El País Vasco es la comunidad con un menor incremento en su producción.

	90-91	**92-93**	**94-95**	**96-97**	**98-99**	**Total**	**Incr**
Madrid	166	242	268	337	378	1391	128%
Cataluña	159	187	233	287	346	1212	118%
Andalucía	93	134	196	276	393	1092	323%
Com. Valenciana	83	99	132	154	197	665	137%
Aragón	56	80	91	85	101	413	80%
Galicia	36	49	56	84	124	349	244%
Castilla-León	49	54	59	72	101	335	106%
País Vasco	50	43	50	58	77	278	54%
Canarias	16	24	44	70	71	225	344%
Cantabria	23	38	38	55	51	205	122%
Murcia	19	23	35	37	60	174	216%
Asturias	5	12	21	33	35	106	600%
Extremadura	11	19	18	21	30	99	173%
Navarra		10	16	27	34	87	240%
Baleares	4	9	5	14	11	43	175%
La Rioja				1	25	26	2400%
UNED	*8*	*12*	*19*	*34*	*19*	*92*	*138%*
Total real	718	968	1168	1500	1866	6220	198%

Tabla 5.3. Evolución de la producción matemática por comunidades autónomas

En el siguiente gráfico se estudia la evolución de las cinco Comunidades Autónomas con mayor producción en la década, aquellas que con más de 400 artículos.

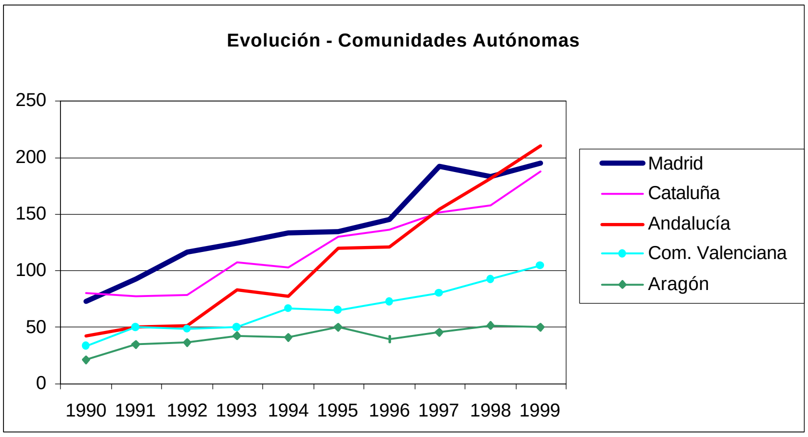


Gráfico 5.2. Evolución de la producción por CCAA

5.2. Datos generales por Sectores Institucionales

Distribución de la producción matemática de España por sectores institucionales

En la tabla 5.4 se estudia la producción matemática por sectores institucionales. Los porcentajes se refieren a la participación de los distintos sectores en la producción matemática. Al producirse colaboraciones de distintos sectores institucionales en un mismo trabajo, por ejemplo la Universidad y el CSIC, dicho trabajo figura en ambos sectores. Ello hace que las sumas totales superen los 6.220 documentos reales y la suma de porcentajes sea superior a 100.

La Universidad es el sector más productivo y participa en casi la totalidad de la producción española, el 98,6% de los documentos, mientras que el CSIC lo hace en el 2,3%. Un 0.1% de la producción es aportado por centros mixtos universidad-CSIC. Tan sólo dos centros más aparecen como participantes en producción matemática en España: el Instituto de Estudios Catalanes (IEC) y el Banco de España, esté último con tan sólo dos documentos en la década. Cabe llamar la atención sobre la ausencia total del sector privado en la producción matemática española, lo que pone de manifiesto la falta de incorporación de matemáticos al ámbito empresarial en labores de I+D, y el poco a nulo interés de la empresa privada en la investigación.

Sector	Nº art.	%
Universidad	6133	98,6%
CSIC	144	2,3%
IEC *	19	0,3%
Mixto CSIC-Univ	4	0,1%
Banco España	2	0,0%
Total real	6220	

IEC * : Instituto de Estudios Catalanes

Tabla 5.4. Producción por sectores

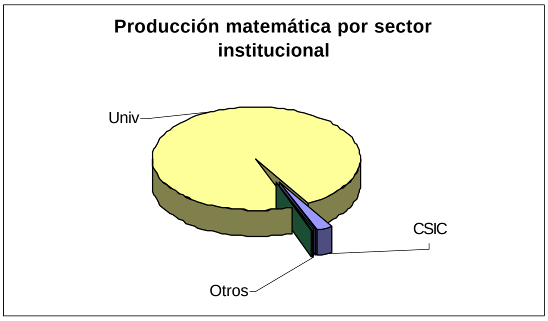


Gráfico 5.3. Producción por sector institucional

Evolución anual de la producción matemática de España por sectores institucionales

La tabla 5.5 nos muestra la evolución de la producción científica en los dos sectores más productivos: la Universidad y el CSIC, y el incremento que ha experimentado esta producción.

Tanto el CSIC como la Universidad experimentan una tendencia ascendente en su producción, aunque es más acusada en el caso del CSIC por la poca producción a principios de la década.

	90-91	92-93	94-95	96-97	98-99	Total	Incr
Universidad	714	950	1154	1474	1841	6133	158%
CSIC	5	21	25	48	45	144	800%

Nota: Incrementos calculados respecto al primer bienio o, en su defecto, respecto al primer bienio con publicación

Tabla 5.5. Evolución de la producción matemática en la Universidad y el CSIC

En el siguiente gráfico se muestra la evolución de los dos sectores institucionales más productivos. Como antes, la gráfica del CSIC se ha multiplicado por 6133/144 con el fin de poder compararla entre sí.

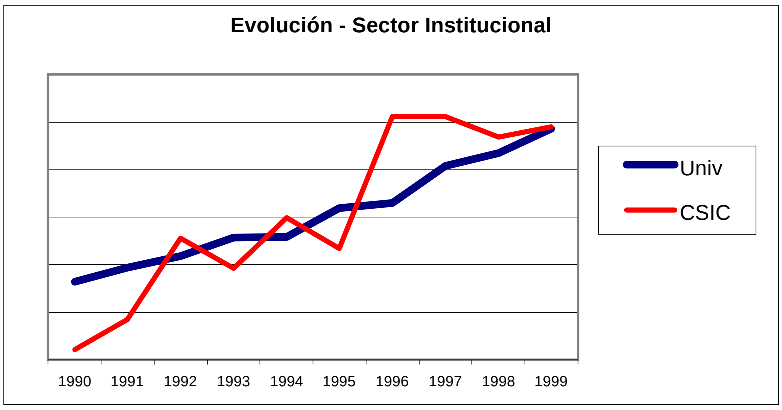


Gráfico 5.4. Evolución de la producción por sector institucional

5.3. Datos generales por Centro de Investigación

Se presenta a continuación la producción por centros de los dos sectores institucionales más importantes: Universidad y CSIC, y de los centros mixtos universidad-CSIC.
Se considera centro universitario a cada una de las universidades españolas.

Distribución de la producción matemática de España por universidades

La tabla 5.6 recoge la distribución de la producción matemática en las distintas universidades españolas, que ya hemos visto que constituyen el principal sector de investigación matemática en nuestro país. En la tabla sólo aparecen las universidades con producción matemática durante el periodo a estudio. Los porcentajes que aparecen en la tabla se refieren al porcentaje que la producción de la universidad representa respecto al total de documentos estudiados.

El análisis de la producción matemática por Universidad permite observar importantes diferencias entre ellas. La Universidad Complutense de Madrid es la que realiza una mayor aportación a la investigación matemática (el 11,4% de la producción total), seguida a una cierta distancia por las Universidades de Granada y Politécnica de Cataluña (8,8% y 7,1% respectivamente). Las Universidades de Burgos, Navarra, Las Palmas, Jaén y Lleida son las universidades de menor producción matemática, con un porcentaje sobre el total menor del 0,15%.

En la tabla se relativiza la producción de cada centro universitario por el número de profesores que hay en ellas y se calcula la ratio de documentos por profesor en los diez años que abarca el estudio. Destacan por su alta proporción de número de documentos por profesor la Universidad de Barcelona y la Universidad Autónoma de Madrid. La Universidad de Barcelona, que ocupa el cuarto puesto en cuanto a número absoluto de documentos, está a la cabeza de la producción por ratio de número de documentos por profesor. La Universidad Complutense de Madrid, que está a la cabeza en cuanto a número absoluto de documentos, pasa al quinto puesto en cuanto a documentos por profesor.

Universidad	Nº art.	%	Nº prof	Nº art./prof
Univ. Barcelona	425	6,8%	78	5,45
Univ. Autónoma de Madrid	299	4,8%	56	5,34
Univ. Autónoma de Barcelona	381	6,1%	78	4,88
Univ. La Laguna	220	3,5%	56	3,93
Univ. Complutense de Madrid	709	11,4%	184	3,85
Univ. Zaragoza	413	6,6%	108	3,82
Univ. Cantabria	205	3,3%	56	3,66
Univ. Valencia (Estudi General)	313	5,0%	89	3,52
Univ. Pompeu Fabra	14	0,2%	4	3,50
Univ. Granada	549	8,8%	160	3,43
Univ. Santiago de Compostela	286	4,6%	92	3,11
Univ. Extremadura	99	1,6%	36	2,75
Univ. Murcia	174	2,8%	70	2,49
Univ. Carlos III de Madrid	93	1,5%	38	2,45
Univ. Publica de Navarra	86	1,4%	36	2,39
UNED	92	1,5%	41	2,24
Univ. Málaga	153	2,5%	70	2,19
Univ. Alicante	107	1,7%	49	2,18
Univ. Politécnica de Cataluña	442	7,1%	211	2,09
Univ. Politécnica de Valencia	253	4,1%	121	2,09
Univ. País Vasco	278	4,5%	133	2,09
Univ. Valladolid	254	4,1%	123	2,07
Univ. Salamanca	81	1,3%	42	1,93
Univ. Sevilla	335	5,4%	178	1,88
Univ. Almería	47	0,8%	32	1,47
Univ. Jaume I de Castellón	46	0,7%	36	1,28
Univ. La Rioja	26	0,4%	22	1,18
Univ. Vigo	67	1,1%	57	1,18
Univ. Oviedo	106	1,7%	101	1,05
Univ. Islas Baleares	43	0,7%	41	1,05
Univ. Politécnica de Madrid	274	4,4%	279	0,98
Univ. Lleida	9	0,1%	12	0,75
Univ. Córdoba	33	0,5%	47	0,70
Univ. Alcalá de Henares	18	0,3%	26	0,69
Univ. A Coruña	25	0,4%	63	0,40
Univ. Cádiz	18	0,3%	59	0,31
Univ. Jaén	7	0,1%	24	0,29
Univ. Las Palmas de Gran Canaria	5	0,1%	74	0,07
Univ. Burgos	1	0,0%	17	0,06
Univ. de Navarra	1	0,0%		
Total real	6133		3124	

Tabla 5.6. Producción matemática por universidades

Evolución de la producción matemática de España por universidad española y por año.

La tabla 5.7 muestra la evolución y el crecimiento de la producción matemática en las universidades a lo largo de la década. Con el fin de facilitar su lectura, los datos se han agrupado en bienios.

Resulta curioso que las universidades con mayor ratio de artículos por profesor están entre las que menos incrementan su producción. Posiblemente esto pueda deberse a dos factores: en primer lugar a que en estos centros ha existido desde el comienzo de la década un planteamiento "moderno" de actividad investigadora y publicaciones, mientras que en el resto de los centros se ha producido una incorporación progresiva a la actividad investigadora y la dinámica de publicaciones a lo largo de los años. En segundo lugar a que las plantillas de estos centros hayan sufrido pocas variaciones por lo que el número de investigadores "reales" en los mismo se haya mantenido constante a lo largo de la década.

Los altos incrementos en algunas Universidades de reciente creación son un indicador del esfuerzo que se está realizando en las Universidades más nuevas y de menor tamaño por llevar a cabo una buena producción matemática. Las Universidades de Córdoba, Jaume I y la Autónoma de Madrid son las que menor incremento han experimentado.

Centro	**90-91**	**92-93**	**94-95**	**96-97**	**98-99**	**Total**
U. Complutense de Madrid	82	128	144	163	192	709
U. de Granada	51	67	116	140	175	549
U. Politécnica de Cataluña	41	65	74	125	137	442
U. de Barcelona	70	80	94	83	98	425
U. de Zaragoza	56	80	91	85	101	413
U. Autónoma de Barcelona	54	62	84	86	95	381
U. de Sevilla	32	42	48	92	121	335
U. de Valencia	38	58	70	78	69	313
U. Autónoma de Madrid	53	57	64	66	59	299
U. de Santiago de Compostela	36	49	56	63	82	286
U. del País Vasco	50	43	50	58	77	278
U. Politécnica de Madrid	28	57	57	62	70	274
U. de Valladolid	34	41	48	53	78	254
U. Politécnica de Valencia	44	44	50	49	66	253
U. de La Laguna	16	24	44	68	68	220
U. de Cantabria	23	38	38	55	51	205
U. de Murcia	19	23	35	37	60	174
U. de Málaga	8	22	30	36	57	153
U. de Alicante	4	4	18	32	49	107
U. de Oviedo	5	12	21	33	35	106
U. de Extremadura	11	19	18	21	30	99
U. Carlos III de Madrid			1	32	60	93
UNED	8	12	19	34	19	92
U. Pública de Navarra		10	16	27	33	86
U. de Salamanca	15	13	11	18	24	81
U. de Vigo				20	47	67
U. de Almería				8	39	47
U. Jaume I				15	31	46
U. de las Islas Baleares	4	9	5	14	11	43
U. de Córdoba	4	6	5	8	10	33
U. de La Rioja				1	25	26
U. de La Coruña				7	18	25
U. de Alcalá de Henares	2	4	0	4	8	18
U. de Cádiz	1	1	0	7	9	18
U. Pompeu Fabra				1	13	14
U. de Lleida					9	9
U. de Jaén				1	6	7
U. de Las Palmas de G.C.				2	3	5
U. de Burgos				1	0	1
U. de Navarra					1	1
Total	714	950	1154	1474	1841	6133

Nota: Incrementos calculados respecto al primer bienio o, en su defecto, respecto al primer bienio con publicación

Tabla 5.7. Evolución de la producción matemática por universidades

En el siguiente gráfico se muestra la evolución de los cinco centros universitarios con mayor producción en la década.

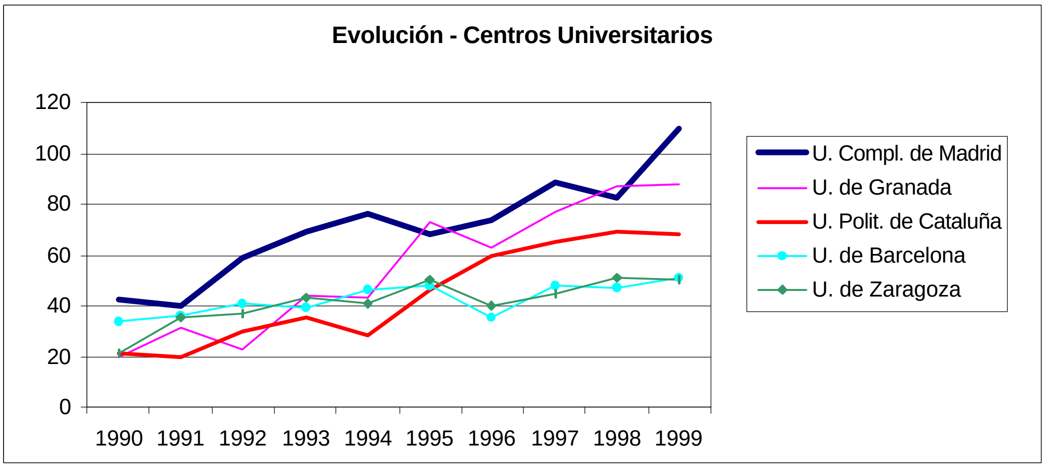


Gráfico 5.5. Evolución de la producción por centros universitarios

Distribución de la producción matemática de España por centros del CSIC y centros mixtos CSIC-Universidad

La tabla 5.8 recoge la distribución de la producción matemática en los distintos centros del CSIC y los centros mixtos Universidad-CSIC (con asterisco en la tabla), con los porcentajes que su producción representa respecto al total de documentos a estudio.

Centro	Nº art.	% resp prod. total
Centro de Física Miguel A. Catalán (CFMAC)	93	1,5%
Centro de Tecnologías Físicas L. Torres Quevedo (CETEF)	21	0,3%
Instituto de Análisis Económico (IAE)	17	0,3%
Instituto de Ciencia de Materiales de Madrid (ICMM)	7	0,1%
Instituto Investigación en Inteligencia Artificial (IIIA)	6	0,1%
Instituto de Astrofísica de Andalucía (IAA)	1	0,0%
Instituto Andaluz de Ciencias de la Tierra (IACT) (*)	2	0,0%
Instituto de Estudios Espaciales de Cataluña (IEEC) (*)	1	0,0%
Instituto de Robótica e Informática (IRII) (*)	1	0,0%
Total	148	

Tabla 5.8. Producción matemática por centros del CSIC y centros mixtos

El centro de mayor producción matemática del CSIC es el Centro de Física Miguel A. Catalán, que está integrado por tres institutos: el Instituto de Estructura de la Materia, el Instituto de Matemática y Física Fundamental y el Instituto de Óptica Daza de Valdés. Por consiguiente el peso de la investigación matemática en el CSIC descansa en el Instituto de Matemáticas y Física Fundamental, que cuenta con un reducido pero muy activo grupo de matemáticos (tres investigadores) y cuya producción supone el 80% de la producción matemática del CSIC. Hay también un investigador matemático en el Instituto de Física Aplicada del Centro de Tecnologías Físicas L. Torres Quevedo.

Los tres últimos centros de la tabla son Centros Mixtos ninguno de los cuales es propiamente de matemáticas. El Instituto Andaluz de Ciencias de la Tierra ha cambiado de nombre durante la década siendo anteriormente el Instituto Andaluz Mediterráneo.

La tabla y gráfico siguientes muestran la evolución del Centro de Física Miguel A. Catalán:

Centro	90-91	92-93	94-95	96-97	98-99	Total	Incr
C. Física M. Catalán	3	16	16	36	22	93	200%

Tabla 5.9. Evolución de la producción matemática del Centro de Física Miguel A. Catalán

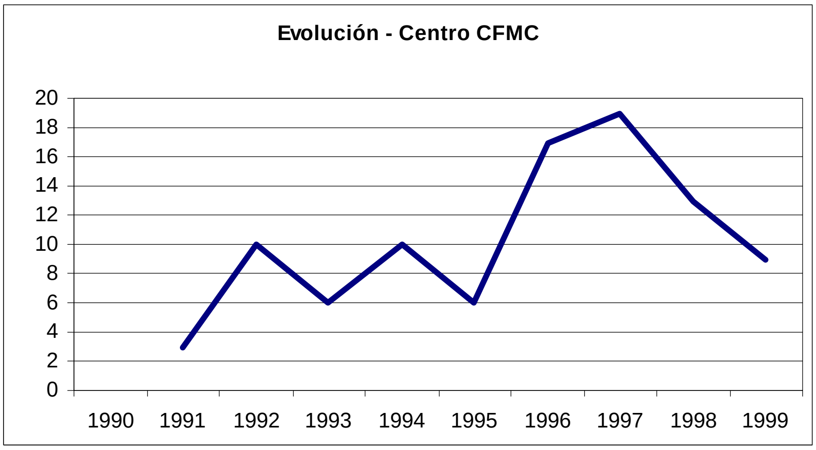


Gráfico 5.6. Evolución de la producción del Centro de Física M.A. Catalán

5.4. Datos generales por Clasificación MSC

En la tabla 5.10 se recoge la distribución de la producción según la clasificación MSC 1991 con la inclusión de los tres nuevos temas de la MSC 2000 ordenada por el número de documentos de cada tema en orden descendente, señalando además el porcentaje de producción.

Más del 50% de la investigación española se centra en nueve de los 61 temas que recoge la clasificación MSC'91, mientras que casi el 90% se centra en 35 de ellos. Como se muestra en la tabla, el tema más productivo en la investigación matemática en cuanto a número absoluto de documentos es el de *Análisis Funcional* (no. 46), seguido del *Ecuaciones en derivadas parciales* (no. 35) y *Análisis numérico* (no. 65).

MSC	MSC 1991	Nº art.	%	% acum
46	Análisis funcional	561	9,0%	9,0%
35	Ecuaciones en derivadas parciales	377	6,1%	15,1%
65	Análisis numérico	377	6,1%	21,1%
62	Estadística	357	5,7%	26,9%
90	Economía, investigación operativa, programación, juegos	317	5,1%	32,0%
58	Análisis global, análisis en variedades	312	5,0%	37,0%
53	Geometría diferencial	274	4,4%	41,4%
34	Ecuaciones diferenciales ordinarias	182	2,9%	44,3%
68	Ciencias de la computación	181	2,9%	47,2%
20	Teoría de grupos y generalizaciones	180	2,9%	50,1%
14	Geometría algebraica	167	2,7%	52,8%
16	Anillos y álgebras asociativos	167	2,7%	55,5%
42	Análisis de Fourier	158	2,5%	58,0%

60	Teoría de la probabilidad y procesos estocásticos	156	2,5%	60,5%
93	Teorías del control y sistema	155	2,5%	63,0%
76	Mecánica de fluidos	141	2,3%	65,3%
17	Anillos y álgebras no asociativos	135	2,2%	67,5%
47	Teoría de operadores	121	1,9%	69,4%
54	Topología general	107	1,7%	71,1%
13	Anillos conmutativos y álgebras	103	1,7%	72,8%
32	Varias variables complejas y espacios analíticos	94	1,5%	74,3%
73	Mecánica de sólidos	94	1,5%	75,8%
70	Mecánica de sistemas y partículas	92	1,5%	77,3%
03	Lógica y fundamentos	88	1,4%	78,7%
05	Combinatoria	88	1,4%	80,1%
30	Funciones de una variable compleja	82	1,3%	81,4%
81	Teoría cuántica	82	1,3%	82,8%
41	Aproximaciones y expansiones	81	1,3%	84,1%
55	Topología algebraica	73	1,2%	85,2%
11	Teoría de números	71	1,1%	86,4%
82	Mecánica estadística, estructura de la materia	68	1,1%	87,5%
33	Funciones especiales	61	1,0%	88,5%
15	Álgebra lineal y multilineal, teoría de matrices	60	1,0%	89,4%
28	Medida e integración	48	0,8%	90,2%
49	Cálculo de variaciones, optimización	46	0,7%	90,9%
94	Información y comunicaciones, circuitos	45	0,7%	91,7%
57	Variedades y complejos celulares	44	0,7%	92,4%
92	Biología y otras ciencias naturales, ciencias del comportamiento	41	0,7%	93,0%
83	Relatividad y teoría gravitatoria	38	0,6%	93,6%
04	Teoría de conjuntos	36	0,6%	94,2%
37*	Sistemas dinámicos y teoría ergódica	34	0,5%	94,8%
91*	Teoría de juegos, economía, ciencias sociales y del comportamiento	32	0,5%	95,3%
18	Teoría de categorías, álgebra homológica	31	0,5%	95,8%
52	Geometría convexa y discreta	28	0,5%	96,2%
12	Teoría de cuerpos y polinomios	25	0,4%	96,6%
26	Funciones reales	23	0,4%	97,0%
78	Óptica, electromagnetismo	23	0,4%	97,4%
01	Historia y biografías	21	0,3%	97,7%
22	Grupos topológicos, grupos de Lie	17	0,3%	98,0%
39	Ecuaciones de diferencias finitas y funcionales	17	0,3%	98,2%
44	Transformaciones integrales, cálculo operacional	17	0,3%	98,5%
74*	Mecánica de sólidos deformables	14	0,2%	98,7%
06	Retículos, estructuras algebraicas ordenadas	13	0,2%	99,0%
31	Teoría del potencial	13	0,2%	99,2%
45	Ecuaciones integrables	13	0,2%	99,4%
80	Termodinámica clásica, transmisión del calor	10	0,2%	99,5%
86	Geofísica	8	0,1%	99,7%
43	Análisis armónico abstracto	6	0,1%	99,8%
19	K-teoría	5	0,1%	99,8%
85	Astrofísica y astronomía	4	0,1%	99,9%
51	Geometría	3	0,0%	100,0%
08	Sistemas matemáticos generales	2	0,0%	100,0%
00	General	1	0,0%	100,0%
	Total	6220		

* Proceden de MSC 2000

Tabla 5.10. Producción matemática por clasificación MSC

Evolución de la producción matemática de España por clasificación MSC y por año.

La tabla 5.11 muestra la evolución de la producción matemática por clasificación MSC y los incrementos experimentados a lo largo de la década por cada tema.

Considerando los códigos MSC con más de 100 artículos en la década, resultan destacables los incrementos en la publicación de temas referentes a Estadística (no. 62), Análisis Numérico (no. 65), Geometría Diferencial (no. 53) y Ciencias de la Computación (no. 68).

MSC	90-91	92-93	94-95	96-97	98-99	Total	Incr
46	80	97	117	119	148	561	85%
35	48	81	75	93	80	377	67%
65	40	53	67	81	136	377	240%
62	30	41	64	96	126	357	320%
90	33	41	64	81	98	317	197%
58	34	65	69	88	56	312	65%
53	27	33	48	75	91	274	237%
34	21	19	23	53	66	182	214%
68	18	30	27	47	59	181	228%
20	29	28	40	37	46	180	59%
14	25	20	28	43	51	167	104%
16	24	29	33	35	46	167	92%
42	16	32	30	31	49	158	206%
60	19	25	28	39	45	156	137%
93	16	23	33	36	47	155	194%
76	11	12	39	33	46	141	318%
17	14	27	33	30	31	135	121%
47	17	21	12	29	42	121	147%
54	14	18	9	29	37	107	164%
13	12	16	27	24	24	103	100%
32	9	14	20	28	23	94	156%
73	10	13	26	21	24	94	140%
70	12	12	21	23	24	92	100%
03	7	20	13	18	30	88	329%
05	7	9	10	30	32	88	357%
30	11	14	21	21	15	82	36%
81	16	13	15	17	21	82	31%
41	4	12	12	30	23	81	475%
55	9	15	11	17	21	73	133%
11	13	13	13	9	23	71	77%
82	15	10	14	10	19	68	27%
33	5	3	9	16	28	61	460%
15	10	9	12	14	15	60	50%
28	3	9	12	13	11	48	267%
49	6	14	8	10	8	46	33%
94	3	12	7	8	15	45	400%
57	5	10	11	8	10	44	100%
92	8	1	5	13	14	41	75%
83	5	10	5	7	11	38	120%
04	5	9	7	12	3	36	-40%
37*				1	33	34	3200%
91*					32	32	
18	3	2	9	4	13	31	333%
52	2	2	3	11	10	28	400%
12	6	4	6	4	5	25	-17%
26	1	1	4	8	9	23	800%
78			4	8	11	23	175%
01	3	6	2	5	5	21	67%

22	1	2	3	4	7	17	600%
39	3	2	3	4	5	17	67%
44	1	3	1	8	4	17	300%
74*					14	14	
06	1	0	0	3	9	13	800%
31		3	2	4	4	13	33%
45	2	2	3	3	3	13	50%
80		4	3	2	1	10	-75%
86		2	1	2	3	8	50%
43	1	0	2	2	1	6	0%
19		1	1	3	0	5	-100%
85	3	0	0	0	1	4	-67%
51			2	0	1	3	-50%
08			1	0	1	2	0%
00		1	0	0	0	1	-100%
Total	718	968	1168	1500	1866	6220	

Notas: Incrementos calculados respecto al primer bienio o, en su defecto, respecto al primer bienio con publicación

* Proceden de MSC 2000

Tabla 5.11. Evolución de la producción matemática por MSC

En el siguiente gráfico se muestra la evolución anual (90-99) de los cinco temas MSC con mayor producción en la década. Es de destacar el crecimiento continuado y de rápida pendiente del código Análisis Numérico (no. 65).

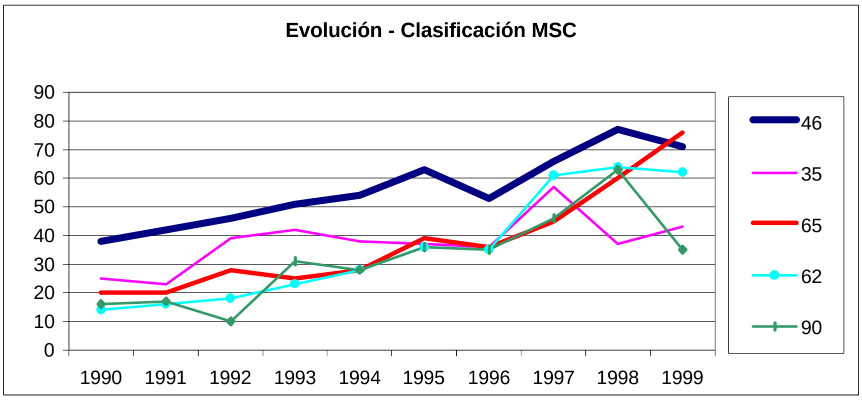


Gráfico 5.7. Evolución de la producción por clasificación MSC

Centros más productivos en los temas MSC con mayor producción

La tabla 5.12 muestra los centros que más han trabajado en cada uno de los cinco códigos MSC con una mayor producción durante la década 1990-1999. La última columna indica el porcentaje del total de documentos en el código que corresponde al centro en cuestión. Así, se tiene que la Universidad Complutense de Madrid es la autora del 35% de todos los artículos españoles de la década en el código “Ecuaciones en derivadas parciales”.

Centro	Nº art.	%
46 - Análisis funcional		
Universidad Complutense de Madrid	84	13,1%
Universidad de Granada	78	12,1%
Universidad Politécnica de Valencia	72	11,2%
Universidad de Sevilla	69	10,7%
Universidad de Valencia	59	9,2%
35 – Ecuaciones en derivadas parciales		
Universidad Complutense de Madrid	146	35,0%
Universidad Autónoma de Madrid	91	21,8%
Universidad del País Vasco	35	8,4%
Universidad de Granada	23	5,5%
Universidad Politécnica de Madrid	18	4,3%
65 - Análisis numérico		
Universidad de Zaragoza	81	19,6%
Universidad de Valladolid	73	17,7%
Universidad Politécnica de Valencia	42	10,2%
Universidad de Málaga	28	6,8%
Universidad de Alicante	23	5,6%
62 – Estadística		
Universidad Complutense de Madrid	80	18,1%
Universidad de Barcelona	39	8,8%
Universidad de Granada	36	8,1%
Universidad Politécnica de Madrid	36	8,1%
Universidad de Cantabria	34	7,7%
90 – Economía, investigación operativa, programación, juegos		
Universidad de Alicante	52	14,4%
Universidad Autónoma de Barcelona	45	12,5%
Universidad de Sevilla	27	7,5%
Universidad Complutense de Madrid	21	5,8%
Universidad de Zaragoza	21	5,8%

Tabla 5.12. Centros más productivos en los temas con mayor producción

Temas MSC más estudiados en los cinco centros con mayor producción

La tabla 5.13 muestra los códigos por MSC que más han investigado los cinco centros españoles con una mayor producción. La última columna indica el porcentaje que representa el código en la producción total del centro. Así, el 20,6% de la producción matemática de la Universidad Complutense se centra en las ecuaciones en derivadas parciales.

Aunque por su menor producción no aparece en la siguiente tabla, cabe señalar que la producción del CSIC, que es en realidad la del Instituto de Matemáticas y Física Fundamental, se centra en dos temas de la clasificación MSC: un 36,6% se clasifica en "Geometría diferencial" (no. 53) y un 34,4% en "Análisis global, análisis en variedades", (no. 58).

Tema MSC	Nº art.	%
Universidad Complutense de Madrid		
Ecuaciones en derivadas parciales	146	20,6%
Análisis funcional	84	11,8%
Estadística	80	11,3%
Geometría algebraica	43	6,1%
Análisis global, análisis en variedades	37	5,2%
Universidad de Granada		
Geometría diferencial	108	19,7%
Análisis funcional	78	14,2%
Estadística	36	6,6%
Ecuaciones diferenciales ordinarias	27	4,9%
Anillos y álgebras asociativos	26	4,7%
Universidad Politécnica de Cataluña		
Ciencias de la computación	81	18,3%
Combinatoria	74	16,7%
Análisis global, análisis en variedades	35	7,9%
Mecánica de fluidos	25	5,7%
Mecánica de sólidos	25	5,7%
Universidad de Barcelona		
Teoría de la probabilidad y procesos estocásticos	57	13,4%
Análisis global, análisis en variedades	49	11,5%
Geometría algebraica	48	11,3%
Estadística	39	9,2%
Anillos conmutativos y álgebras	29	6,8%
Universidad de Zaragoza		
Análisis numérico	81	19,6%
Anillos y álgebras no asociativos	60	14,5%
Análisis global, análisis en variedades	26	6,3%
Aproximaciones y expansiones	25	6,1%
Teoría de grupos y generalizaciones	23	5,6%

Tabla 5.13. Temas con mayor producción en los centros más productivos

5.5. Datos generales por Áreas de Conocimiento

Distribución de la producción matemática por Áreas de conocimiento

La tabla 5.14 recoge la distribución de la producción matemática por las Áreas de Conocimiento en las que se pueden clasificar los artículos del presente estudio. El área con mayor número de publicaciones es la de "Matemática Aplicada" que constituye el 43,8% de la producción total. También es el área que contiene mayor número de profesores adscritos a la misma, por lo que presentamos una relativización de la producción por número de profesores numerarios en el área. Teniendo en cuenta la ratio de nº de artículos por profesor, Geometría y Topología es el área más productiva, siendo seguida a cierta distancia por Álgebra. Una vez más recordamos que la mayoría de los códigos de la MSC están adscritos a varias áreas de conocimiento, por lo que la suma total de artículos supera los 6.220 documentos reales de la base de datos, y la suma de los porcentajes supera 100.

Area de conocimiento	Nº art.	%	Nº profesores	Nº art/prof
Geometría y Topología	1585	25,5%	169	9,38
Algebra	1314	21,1%	205	6,41
Análisis Matemático	1871	30,1%	322	5,81
Matemática Aplicada	2723	43,8%	1334	2,04
Estadística e IO	1010	16,2%	705	1,43
Ciencia de la Computación e IA	338	5,4%	389	0,87
Total real	6220			

Tabla 5.14. Producción matemática por Áreas de Conocimiento

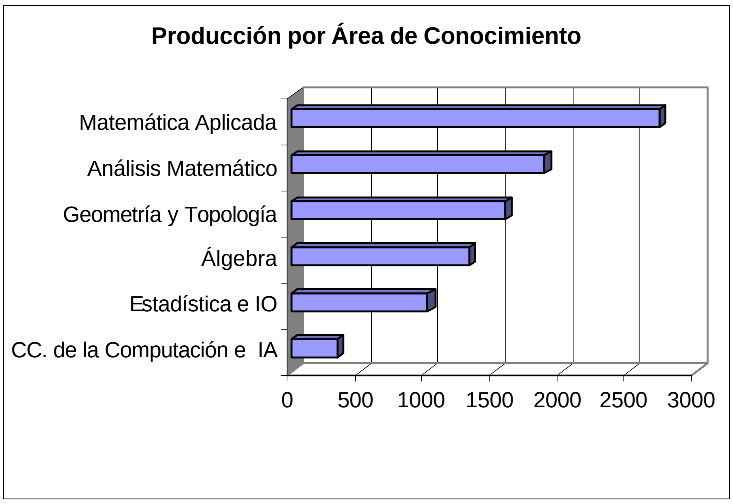


Gráfico 5.8. Producción por Áreas de Conocimiento

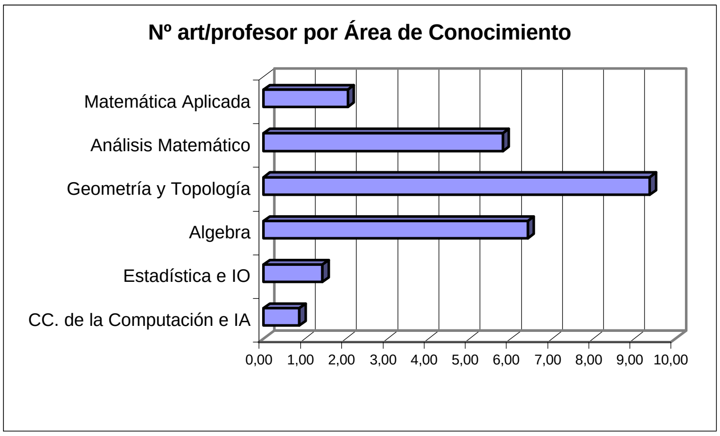


Gráfico 5.9. Ratio nº de artículos por profesor

Evolución de la producción matemática por Áreas de Conocimiento

La tabla 5.15 muestra la evolución y el crecimiento de la producción matemática por las áreas de conocimiento. El gráfico 5.10 muestra la evolución anual de la producción en las distintas áreas de conocimiento.

	90-91	92-93	94-95	96-97	98-99	Total	Incr
Matemática Aplicada	315	453	512	669	774	2723	146%
Análisis Matemático	225	314	358	463	511	1871	127%
Geometría y Topología	198	261	300	402	424	1585	114%
Álgebra	171	213	254	303	373	1314	118%
Estadística e IO	107	133	183	261	326	1010	205%
CC. de la Computación e IA	31	69	50	78	110	338	255%
Total real	718	968	1168	1500	1866	6220	

Nota: Incrementos calculados respecto al primer bienio o, en su defecto, respecto al primer bienio con publicación

Tabla 5.15. Evolución de la producción por Áreas de Conocimiento

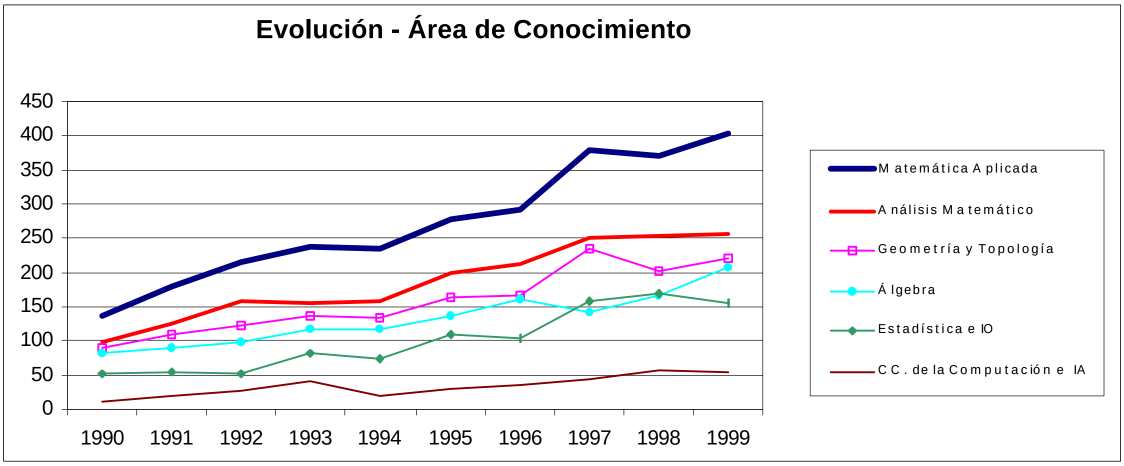


Gráfico 5.10. Evolución de la producción por Áreas de Conocimiento

5.6. Relativización de la producción matemática

Relativización de la producción matemática de las CCAA por número de habitantes

En la tabla 5.16 se relativiza la producción matemática española por el número de habitantes de las comunidades autónomas. Los datos referentes a la población son a fecha del 1-1-1998 y se han obtenido de la página web del Instituto Nacional de Estadística. En la última columna de la tabla se muestra la ratio por CCAA de la producción matemática por población, obteniéndose el número de documentos en la década por cada 10.000 habitantes. La tabla se ha ordenado de forma descendente por este dato.

En la presentación de los resultados generales de la producción matemática por Comunidades Autónomas, se ha visto como Madrid y Cataluña son las que acaparan la mayoría de la producción pero que, sin embargo, son Aragón y Cantabria las Comunidades con una mayor proporción de artículos por profesor. Si tenemos en cuenta la tabla 5.16 se observa que precisamente son estas comunidades las que muestran también una mayor proporción de artículos por cada 10.000 habitantes.

Las comunidades de Madrid y Cataluña, responsables de más del 40% de la producción matemática en España, pasan a ocupar un tercer y cuarto puesto respectivamente si tenemos en cuenta el número de documentos producidos por cada 10.000 habitantes, mientras que las comunidades más productivas en cuanto a capacidad por habitante pasan a ser las de Aragón y Cantabria. Navarra, que aparecía como una de las comunidades menos productivas, pasa a ser la sexta comunidad con mayor producción por habitante. También resulta destacable el caso de Andalucía, que aunque ocupaba el tercer puesto en cuanto a número de documentos publicados, teniendo en cuenta la producción respecto al número de habitantes, resulta ser una comunidad no muy productiva.

CCAA	Nº art.	%	Nº art./10000 hab
Cantabria	205	3,3%	3,89
Aragón	413	6,6%	3,49
Madrid	1391	22,4%	2,73
Cataluña	1212	19,5%	1,97
Valencia	665	10,7%	1,65
Navarra	87	1,4%	1,64
Murcia	174	2,8%	1,56
Andalucía	1092	17,6%	1,51
Canarias	225	3,6%	1,38
Castilla-León	335	5,4%	1,35
País Vasco	278	4,5%	1,32
Galicia	349	5,6%	1,28
La Rioja	26	0,4%	0,99
Asturias	106	1,7%	0,98
Extremadura	99	1,6%	0,93
Baleares	43	0,7%	0,54
Total real	6220		

Tabla 5.16. Producción matemática en las CCAA por nº de habitantes

Profesorado Universitario de Matemáticas por cada 10.000 habitantes y CCAA.

La tabla 5.17 muestra el ratio de profesores por cada 10.000 habitantes y se observa que son Madrid y Cantabria las comunidades con mayor proporción. Se observan diferencias significativas entre unas Comunidades y otras en la tasa de profesores por 10.000 habitantes, que seguramente obedecen a razones históricas.

CCAA	Nº Prof (2000)	Población	Prof/10000 hab
Madrid	583	5.091.336	1,15
Cantabria	56	527.137	1,06
Asturias	101	1.081.834	0,93
Aragón	108	1.183.234	0,91
La Rioja	22	263.644	0,83
Canarias	130	1.630.015	0,80
Andalucía	570	7.236.459	0,79
Galicia	212	2.724.544	0,78
Valencia	295	4.023.441	0,73
Castilla-León	182	2.484.603	0,73
Navarra	36	530.819	0,68
País Vasco	133	2.098.628	0,63
Murcia	70	1.115.068	0,63
Cataluña	383	6.147.610	0,62
Baleares	41	796.483	0,51
Extremadura	36	1.069.419	0,34

Tabla 5.17. Proporción nº de profesores de matemáticas por 10.000 hab.

6. ESTUDIO DE LA CALIDAD EN LA INVESTIGACIÓN

A la hora de valorar la calidad de la investigación, además del análisis cuantitativo realizado, es necesario utilizar indicadores bibliométricos de impacto basados en el número de citas que obtienen los trabajos, con el fin de permitir las comparaciones. Al valorar los resultados que proporcionan lo indicado es bibliométricos hay que tener en cuenta las limitaciones que presentan y que han sido comentadas en la sección "Metodología" del presente estudio.

6.1. Distribución de la producción por cuartiles

La base de datos ISI clasifica las revistas por disciplinas en función de su temática, y dentro de cada disciplina ordena las revistas en función de su factor de impacto. Este orden permite clasificar las revistas por cuartiles. En la tabla 6.1 se muestra la distribución de la producción matemática por los cuartiles en que se clasifican las revistas en las que ha habido algún artículo español, según la última versión del Journal Citation Reports (JCR 1999).

Cuartil	Nº revistas	Nº artículos
1	107	16%
2	105	23%
3	110	36%
4	91	26%

Tabla 6.1. Distribución de la producción española por cuartiles

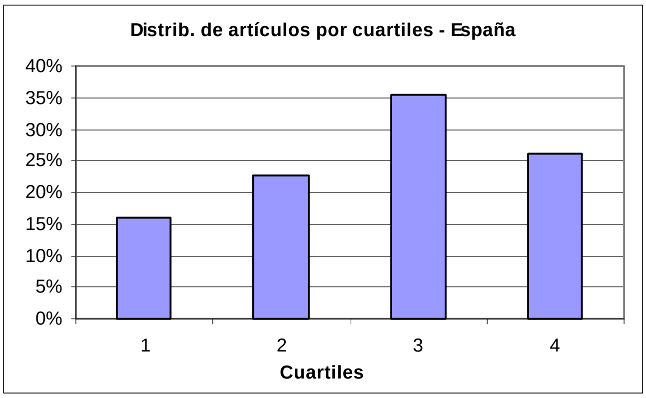


Gráfico 6.1. Distribución por cuartiles de la producción española

Para comparar la distribución por cuartiles de la producción española con la distribución mundial, se ha obtenido la distribución de los artículos que recoge la base MathSci durante la última década, aparecidos dentro de las revistas de los respectivos cuartiles del ISI. La distribución resultante de la producción matemática se muestra en la tabla 6.2.

Se puede observar que la distribución española está desplazada hacia el tercer cuartil de modo mucho más acusado que la distribución mundial, en detrimento del número de trabajos colocados en el primer cuartil. Los porcentajes en el segundo y cuarto cuartil son similares en España y el resto del mundo. Aproximadamente el 39% de los artículos se publican en revistas con un factor de impacto por encima de la media. A nivel mundial este porcentaje alcanza el 44%.

Cuartil	Nº revistas	Nº artículos
1	107	22%
2	105	22%
3	110	29%
4	91	27%

Tabla 6.2. Distribución de la producción mundial por cuartiles

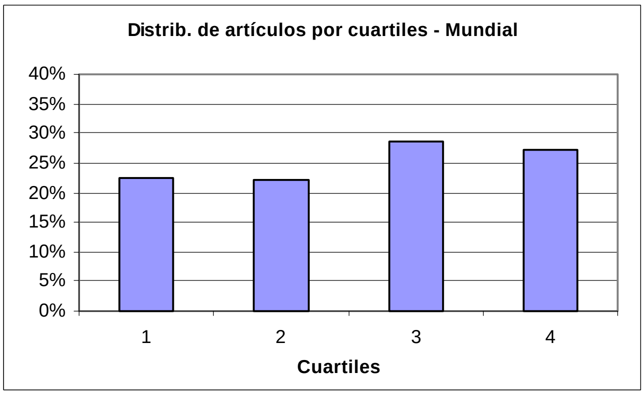


Gráfico 6.2. Distribución por cuartiles de la producción mundial

Evolución de la distribución por cuartiles

La tabla 6.3 muestra la evolución de la distribución por cuartiles de la producción matemática. Se puede observar que la distribución no ha variado significativamente durante la última década, aunque en valores absolutos el número de publicaciones en revistas de calidad sí que ha aumentado, como se ha puesto de manifiesto anteriormente.

	90-91	92-93	94-95	96-97	98-99
Cuartil 1	17%	16%	16%	16%	15%
Cuartil 2	22%	23%	25%	23%	21%
Cuartil 3	34%	37%	35%	34%	37%
Cuartil 4	27%	24%	25%	27%	26%

Tabla 6.3. Evolución de la distribución por cuartiles

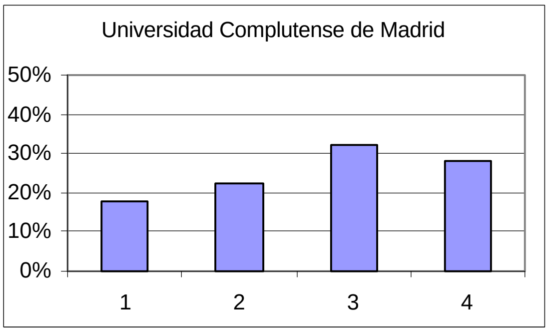
Distribución por cuartiles de la producción de los centros universitarios y del CSIC

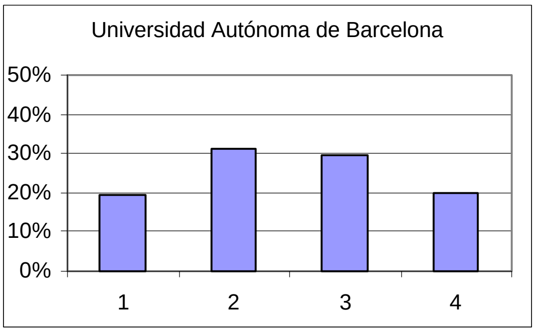
La tabla 6.4 muestra la distribución de la producción de cada centro universitario y el Instituto Miguel Catalán por los cuartiles en los que se encuentran las revistas donde se han publicado. La tabla se ha ordenado de forma descendente por la producción de cada centro. Se resaltan en negrilla los porcentajes de los centros cuya producción en el primer cuartil es superior a la media nacional. Asímismo se muestran los gráficos de los centros de mayor producción.

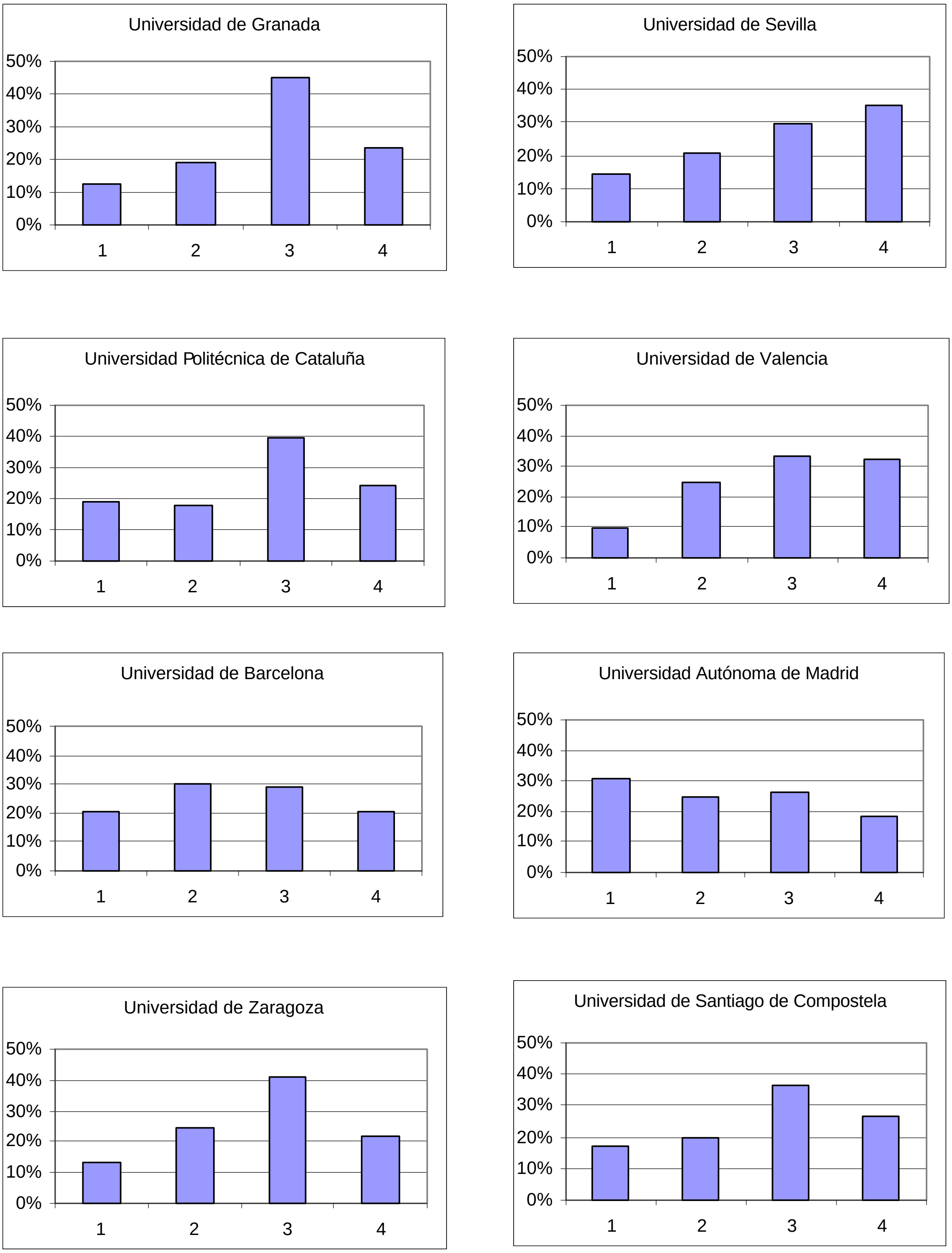
Las universidades con una mayor calidad en su producción (sin olvidar las límitaciones señaladas al concepto de calidad en base al índice de impacto), son la Universidad Autónoma de Madrid, la Universidad de Valladolid y la Universidad de Salamanca.

Centro	Cuartil 1	Cuartil 2	Cuartil 3	Cuartil 4
Universidad Complutense de Madrid	**18%**	22%	32%	28%
Universidad de Granada	13%	19%	45%	24%
Universidad Politécnica de Cataluña	**19%**	18%	40%	24%
Universidad de Barcelona	**21%**	30%	29%	20%
Universidad de Zaragoza	13%	25%	41%	22%
Universidad Autónoma de Barcelona	**19%**	31%	30%	20%
Universidad de Sevilla	14%	21%	30%	35%
Universidad de Valencia	10%	25%	33%	32%
Universidad Autónoma de Madrid	**31%**	25%	26%	18%
Universidad de Santiago de Compostela	**17%**	20%	37%	27%
Universidad del País Vasco	**18%**	24%	36%	21%
Universidad Politécnica de Madrid	12%	24%	36%	28%
Universidad de Valladolid	**29%**	21%	34%	17%
Universidad Politécnica de Valencia	6%	8%	33%	53%
Universidad de La Laguna	12%	14%	47%	27%
Universidad de Cantabria	15%	20%	36%	29%
Universidad de Murcia	7%	24%	44%	26%
Universidad de Málaga	7%	22%	39%	32%
Universidad de Alicante	**21%**	15%	31%	33%
Universidad de Oviedo	11%	28%	36%	25%
Universidad de Extremadura	7%	34%	21%	37%
Universidad Carlos III de Madrid	**17%**	29%	41%	12%
Centro de Física Miguel A. Catalán	10%	30%	39%	22%
UNED	9%	27%	41%	23%
Universidad Pública de Navarra	5%	19%	56%	21%
Universidad de Salamanca	**28%**	30%	25%	18%
Universidad de Vigo	9%	13%	60%	18%
Universidad de Almería	9%	15%	64%	13%
Universidad Jaume I	9%	24%	20%	48%
Universidad de las Islas Baleares	**26%**	23%	42%	9%
Universidad de Córdoba	6%	9%	42%	42%
Universidad de La Rioja	8%	23%	19%	50%
Universidad de La Coruña	12%	16%	40%	32%
Universidad de Alcalá de Henares	0%	33%	33%	33%
Universidad de Cádiz	**22%**	44%	22%	11%
Universidad Pompeu Fabra	**29%**	36%	29%	7%
Universidad de Lleida	11%	0%	44%	44%
Universidad de Jaén	14%	14%	43%	29%
Universidad de Las Palmas de Gran Canaria	20%	0%	20%	60%
Universidad de Navarra	0%	0%	0%	100%
Universidad de Burgos	0%	0%	0%	100%
España	**16%**	**23%**	**36%**	**26%**

Tabla 6.4. Distribución por cuartiles de la producción universitaria







Gráficos 6.3.-6.12. Distribución por cuartiles de la producción universitaria

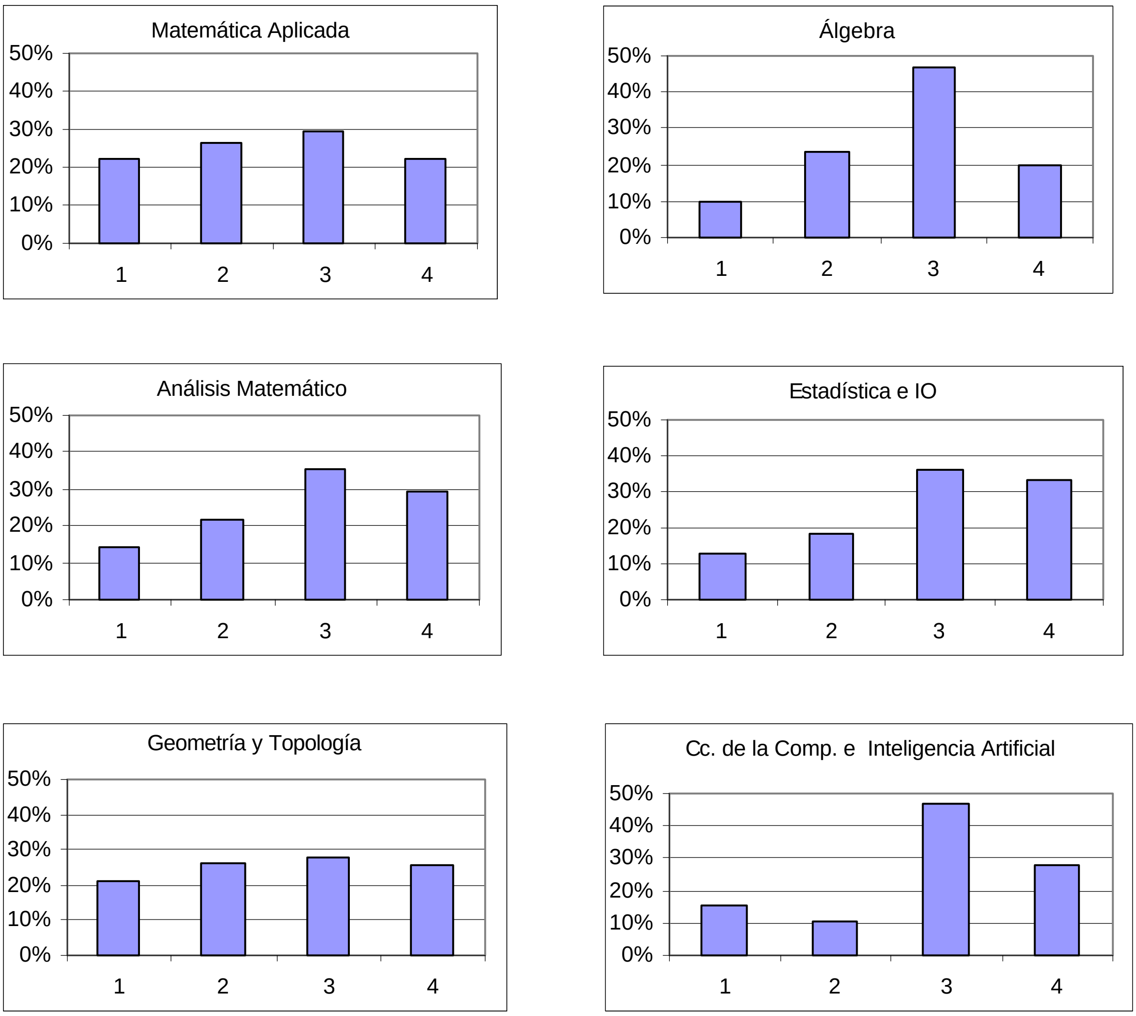
Distribución por cuartiles de la producción por áreas de conocimiento

La tabla 6.5 muestra la distribución de la producción de áreas de conocimiento por los cuartiles en los que se encuentran las revistas donde se han publicado. La tabla se ha ordenado de forma descendente por la producción de cada área. Se resaltan los porcentajes de las áreas cuya producción en el primer cuartil es superior a la media nacional y se muestran los gráficos de la distribución por cuartiles de cada una de ellas.

Las áreas de conocimiento que publican en revistas con índices de impacto superiores, son las de Matemática Aplicada y Geometría y Topología.

Centro	Cuartil 1	Cuartil 2	Cuartil 3	Cuartil 4
Matemática Aplicada	**22%**	26%	30%	22%
Análisis Matemático	14%	22%	35%	29%
Geometría y Topología	**21%**	26%	28%	25%
Álgebra	10%	24%	47%	20%
Estadística e IO	13%	18%	36%	33%
CC. de la Computación e IA	15%	10%	47%	28%
España	**16%**	**23%**	**36%**	**26%**

Tabla 6.5. Distribución por cuartiles de la producción por áreas de conocimiento



Gráficos 6.13.-6.18. Distribución por cuartiles de la producción por área de conocimiento

De nuevo, hay que tomar ciertas cautelas a la hora de interpretar los resultados de estas tablas. Por ejemplo, la tabla 6.6 pone de manifiesto que una buena cantidad de las publicaciones de Álgebra se realizan en las revistas Communications in Algebra, y Journal of Pure and Applied Algebra que, al tratarse de revistas especializadas, y ser el colectivo de algebristas relativamente bajo, se sitúan en el cuartil 3 dentro del epígrafe de Mathematics, pese a tratarse de revistas de calidad dentro de su área.

6.3. Revistas ISI con un mayor número de documentos publicados en ellas, su factor de impacto medio y cuartil

En las siguientes tablas se muestran las distintas categorías ISI a las que pertenecen las cincuenta revistas de la tabla anterior, junto con el factor de impacto medio de los últimos diez años de la revista. Junto a estos datos aparece el cuartil que ocupa la revista dentro de su epígrafe. Hemos elaborado datos de la distribución por cuartiles de los artículos para los epígrafes de Matemáticas, Matemática Aplicada y Estadística. Llama la atención que la distribución por cuartiles de estas dos últimas áreas no coincide con la presentada en la sección anterior, seguramente por la publicación de artículos de ellas en revistas de otros epígrafes, principalmente el de Matemáticas.

Mathematics

Revista	Nº art.	FI	Cuartil
Proceedings of the American Mathematical Society	206	0,280	4
Communications in Algebra	199	0,283	3
Comptes Rendus de l'Academie des Sciences I. Mathematique	186	0,325	3
Journal of Mathematical Analysis and Applications	178	0,325	3
Journal of Algebra	158	0,422	2
Archiv der Mathematik	111	0,238	4
Nonlinear Analysis	108	0,330	3
Journal of Pure and Applied Algebra	94	0,378	3
Studia Mathematica	82	0,314	3
Journal of Differential Equations	76	0,687	1
Transactions of the American Mathematical Society	68	0,545	1
Manuscripta Mathematica	62	0,278	4
Bulletin of the Australian Mathematical Society	58	0,194	4
Mathematical Proceedings of the Cambridge Philosophical Society	54	0,402	2
Journal of Approximation Theory	52	0,392	1
Mathematische Nachrichten	51	0,250	3
Acta Mathematica Hungarica	50	0,141	4
Israel Journal of Mathematics	49	0,352	2
The Journal of the London Mathematical Society	49	0,404	2
Pacific Journal of Mathematics	44	0,371	2
The Rocky Mountain Journal of Mathematics	42	0,174	4
Mathematische Zeitschrift	41	0,432	2
Discrete Mathematics	40	0,224	3
Topology and its Applications	40	0,252	4
Journal of Functional Analysis	39	0,785	1
Proceedings of the Royal Society of Edinburgh A. Mathematics	38	0,410	3
Glasgow Mathematical Journal	34	0,258	3
Mathematische Annalen	34	0,574	1
Universitatis Debreceniensis	32	0,088	4
Geometriae Dedicata	30	0,272	3

Tabla 6.6. “Matemáticas”

Cuartil	Nº artículos
1	16%
2	23%
3	36%
4	26%

Tabla 6.7. Distribución por cuartiles: Matemática

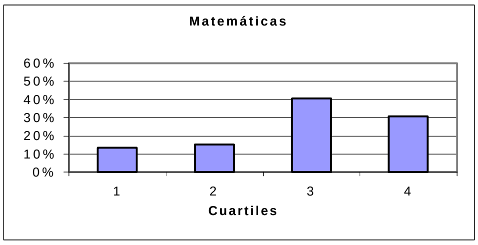


Gráfico 6.19. Distribución por cuartiles

Mathematics, applied

Revista	**Nº art.**	**FI**	**Cuartil**
Proceedings of the American Mathematical Society	206	0,280	4
Journal of Mathematical Analysis and Applications	178	0,325	3
Journal of Computational and Applied Mathematics	139	0,373	3
Nonlinear Analysis	108	0,330	4
Linear Algebra and its Applications	104	0,372	3
Fuzzy Sets and Systems	102	0,489	3
Journal of Pure and Applied Algebra	94	0,378	3
Applied Mathematics and Computation	62	0,241	4
Computers and Mathematics with Applications	51	0,296	4
Applied Mathematics Letters	47	0,338	3
Topology and its Applications	40	0,252	4
Internat. J. of Bifurcation and Chaos in Applied Sci. and Engineering	38	0,794	2
Proceedings of the Royal Society of Edinburgh A. Mathematics	38	0,410	3
Applied Numerical Mathematics	37	0,493	2
Numerical Algorithms	33	0,454	3
SIAM Journal on Mathematical Analysis	30	0,701	1

Tabla 6.8. "Matemática aplicada"

Cuartil	Nº artículos
1	2%
2	6%
3	56%
4	36%

Tabla 6.9. Distribución por cuartiles: Matemática Aplicada

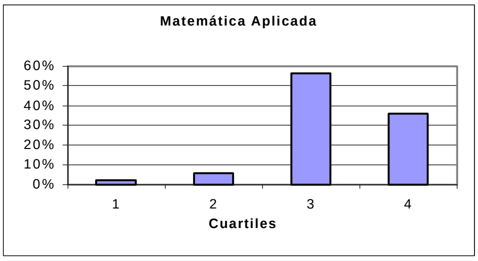


Gráfico 6.20. Distribución por cuartiles

Statistics & Probability

Revista	**Nº art.**	**FI**	**Cuartil**
Fuzzy Sets and Systems	102	0,489	3
Communications in Statistics. Theory and Methods	61	0,158	4
Statistics and Probability Letters	60	0,253	3
Journal of Statistical Planning and Inference	36	0,278	3

Tabla 6.10. "Estadística y probabilidad"

Cuartil	Nº artículos
1	0%
2	0%
3	76%
4	24%

Tabla 6.11 Distribución por cuartiles: Estadística y probablidad

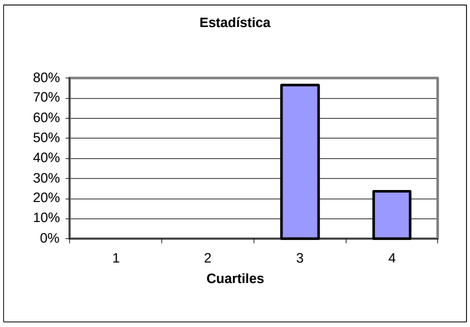


Gráfico 6.21: Distribución por cuartiles.

Astronomy & Astrophysics

Revista	Nº art.	FI	Cuartil
Celestial Mechanics and Dynamical Astronomy	41	0,420	4

Tabla 6.12. "Astronomía y Astrofísica"

Physics

Revista	Nº art.	FI	Cuartil
Journal of Physics A. Mathematical and General	61	1,799	2

Tabla 6.13. "Física"

Physics, mathematical

Revista	Nº art.	FI	Cuartil
Journal of Physics A. Mathematical and General	61	1,799	2
Journal of Mathematical Physics	41	0,947	3

Tabla 6.14. "Física matemática"

Computer Science, Theory & Methods

Revista	Nº art.	FI	Cuartil
Fuzzy Sets and Systems	102	0,489	3
Theoretical Computer Science	30	0,394	3

Tabla 6.15. "Ciencias de la Computación, teoría y métodos"

Computer Science, Interdisciplinary Applications

Revista	Nº art.	FI	Cuartil
Computers and Mathematics with Applications	51	0,296	4
Computer Methods in Applied Mechanics and Engineering	30	0,864	1

Tabla 6.16."Ciencias de la Computación, aplicaciones interdisciplinares"

Computer Science, Information Systems

Revista	Nº art.	FI	Cuartil
Information Processing Letters	32	0,269	4

Tabla 6.17. "Ciencias de la Computación, Sistemas de Información"

Mathematics, miscellaneous

Revista	Nº art.	FI	Cuartil
Mathematical Social Sciences	30	0,328	4

Tabla 6.18. “Matemáticas, miscelánea”

Engineering, mechanical

Revista	Nº art.	FI	Cuartil
Computer Methods in Applied Mechanics and Engineering	30	0,864	1

Tabla 6.19. “Ingeniería mecánica”

Mechanics

Revista	Nº art.	FI	Cuartil
Computer Methods in Applied Mechanics and Engineering	30	0,864	1

Tabla 6.20. “Mecánica”

Multidisciplinary Sciences

Revista	Nº art.	FI	Cuartil
International J. of Bifurcation and Chaos in Applied Scie. and Engineering	38	0,794	2

Tabla 6.21. “Ciencias multidisciplinares”

6.4 Revistas con mejor posición normalizada y número de documentos publicados en ellas

En la tabla 6.22 se muestran las cincuenta revistas con mayor número de documentos ordenadas por su posición normalizada e indicando el número de documentos publicados en cada una de ellas. En el apéndice se incluye una tabla similar de las cincuenta revistas con mejor posición normalizada independiente del número de documentos publicadas en ellas. Como era de esperar esta segunda tabla no contiene casi ninguna revista de las que figuran en 6.24 ya que el número de documentos españoles publicados en ellas es pequeño. Todo ello incide en el comentario ya hecho de que aún queda mucho por avanzar en cuanto al incremento de calidad de la producción española o al menos en cuanto a publicar en las revistas más prestigiosas.

Recordemos que la posición normalizada de las revistas nos permite comparar revistas de distintas disciplinas ISI, algo que el factor de impacto por sí sólo, no nos permite hacer.

Revista	Pos. Norm.	Nº art.
Computer Methods in Applied Mechanics and Engineering	0,91	30
Journal of Functional Analysis	0,88	39
Journal of Differential Equations	0,86	76
Journal of Physics. A. Mathematical and General	0,80	61
Transactions of the American Mathematical Society	0,79	68
Journal of Approximation Theory	0,78	52
Fuzzy Sets and Systems	0,77	102
Mathematische Annalen	0,77	34
Applied Numerical Mathematics	0,74	37
SIAM Journal on Mathematical Analysis	0,73	30
Mathematische Zeitschrift	0,71	41
Int. Journal of Bifurcation and Chaos in Applied Sciences and Engineering	0,70	38
Israel Journal of Mathematics	0,68	49
Mathematical Proceedings of the Cambridge Philosophical Society	0,67	54
Journal of Algebra	0,66	158
Journal of Pure and Applied Algebra	0,66	94
Journal of Mathematical Analysis and Applications	0,59	178
The Journal of the London Mathematical Society. Second Series	0,58	49
Pacific Journal of Mathematics	0,50	44
Proceedings of the Royal Society of Edinburgh. Section A. Mathematics	0,50	38
Glasgow Mathematical Journal	0,47	34
Discrete Mathematics	0,45	40
Proceedings of the American Mathematical Society	0,43	206
Applied Mathematics Letters. An International Journal of Rapid Publication	0,43	47
Numerical Algorithms	0,41	33
Geometriae Dedicata	0,41	30
Journal of Mathematical Physics	0,40	41
Mathematische Nachrichten	0,39	51
Studia Mathematica	0,39	82
Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique	0,37	186
Linear Algebra and its Applications	0,36	104
Communications in Algebra	0,35	199
Nonlinear Analysis. Theory, Methods and Applications	0,34	108
Computers and Mathematics with Applications. An International Journal	0,34	51
Journal of Computational and Applied Mathematics	0,33	139
Topology and its Applications	0,32	40
Theoretical Computer Science	0,32	30
Journal of Statistical Planning and Inference	0,30	36
Statistics and Probability Letters	0,28	60
Information Processing Letters	0,23	32
Manuscripta Mathematica	0,22	62
Archiv der Mathematik. Archives of Mathematics. Archives Mathematiques	0,21	111
Celestial Mechanics and Dynamical Astronomy	0,19	41
Applied Mathematics and Computation	0,18	62
Acta Mathematica Hungarica	0,17	50
Bulletin of the Australian Mathematical Society	0,16	58
The Rocky Mountain Journal of Mathematics	0,12	42
Mathematical Social Sciences	0,10	30
Universitatis Debreceniensis. Publicationes Mathematicae	0,10	32
Communications in Statistics. Theory and Methods	0,09	61

Tabla 6.22. Posición normalizada

6.5. Revistas con un mayor número de documentos publicados en ellas y su disciplina ISI, sin filtrado de áreas fronterizas

Tanto en la introducción del presente informe como en el apartado referente a la metodología, se explica que para obtener la base de datos en la que se ha basado el presente estudio se realizó un filtrado manual para eliminar artículos que se clasifican en áreas fronterizas de las matemáticas, que aunque la AMS las considera de producción matemática, muchos de sus artículos no se considerarían como tal por el colectivo matemático.

Estas áreas de difícil clasificación son: Física, Física Matemática, Física Nuclear, Física de Partículas, las relativas a Informática, Mecánica, Ingeniería Mecánica, y Astronomía y Astrofísica.

No obstante, con el fin de presentar los datos que resultarían sin haber procedido a dicho filtro, la tabla 6.23 muestra las cincuenta revistas donde habría mayor producción, la cantidad de artículos que contendrían y la disciplina ISI donde se clasificarían. Recordemos que la base de datos teniendo en cuenta todos estos artículos pertenecientes a áreas fronterizas con las matemáticas constaría de 7.419 artículos.

Revista	Nº art.	Disciplina
Journal of Physics A. Mathematical and General	263	Physics. Physics, mathematical
Proceedings of the American Mathematical Society	206	Mathematics. Mathematics, applied
Communications in Algebra	199	Mathematics
Comptes Rendus Acad. Sciences I. Mathematique	186	Mathematics
Physics Letters B	179	Physics
Journal of Mathematical Analysis and Applications	178	Mathematics. Mathematics, applied
Journal of Mathematical Physics	175	Physics, mathematical
Nuclear Physics B	160	Physics, nuclear. Physics, particles & fields
Journal of Algebra	158	Mathematics
Journal of Computational and Applied Mathematics	137	Mathematics, applied
Classical and Quantum Gravity	119	Physics
Physical Review D	115	Physics, particles & fields
Archiv der Mathematik	111	Mathematics
Nonlinear Analysis.Theory, Methods and Applications	107	Mathematics. Mathematics, applied
Physics Letters A	106	Physics
Linear Algebra and its Applications	104	Mathematics, applied
Fuzzy Sets and Systems	103	Computer Science, Theory & Methods. Mathematics, Applied. Statistics & Probability
Journal of Pure and Applied Algebra	93	Mathematics. Mathematics, applied
Studia Mathematica	81	Mathematics
Journal of Differential Equations	76	Mathematics
Transactions of the American Mathematical Society	68	Mathematics
Applied Mathematics and Computation	62	Mathematics, applied
Manuscripta Mathematica	62	Mathematics
Communications in Statistics. Theory and Methods	61	Statistics & probability
Statistics and Probability Letters	60	Statistics & probability
Bulletin of the Australian Mathematical Society	58	Mathematics
Computers and Mathematics with Applications	54	Computer Science, Interdisciplinary Applications. Mathematics, applied
Math. Proceed. of the Cambridge Philosoph. Society	54	Mathematics
International J. of Modern Physics A. Particles and Fields. Gravitation. Cosmology. Nuclear Physics	52	Physics, nuclear. Physics, particles & fields
Journal of Approximation Theory	52	Mathematics

Mathematische Nachrichten	51	Mathematics
Acta Mathematica Hungarica	50	Mathematics
The Journal of the London Mathematical Society	49	Mathematics
Israel Journal of Mathematics	47	Mathematics
Modern Physics Letters A. Particles and Fields, Gravitation, Cosmology, Nuclear Physics	47	Physics, mathematical. Physics, nuclear. Physics, particles & fields
Pacific Journal of Mathematics	44	Mathematics
International Journal of Bifurcation and Chaos in Applied Sciences and Engineering	42	Mathematics, applied. Multidisciplinary Sciences
Journal of Geometry and Physics	42	Mathematics, applied. Physics, mathematical
The Rocky Mountain Journal of Mathematics	42	Mathematics
Celestial Mechanics and Dynamical Astronomy	41	Astronomy & Astrophysics
Mathematische Zeitschrift	41	Mathematics
Discrete Mathematics	40	Mathematics
Topology and its Applications	40	Mathematics. Mathematics, applied
Applied Mathematics Letters	39	Mathematics, applied
Journal of Functional Analysis	39	Mathematics
Applied Numerical Mathematics	37	Mathematics, applied
Proceedings of the Royal Society of Edinburgh A. Mathematics	37	Mathematics. Mathematics, applied
Journal of Statistical Planning and Inference	36	Statistics & probability
Computer Methods in Applied Mechanics and Engineering	35	Computer Science, interdisciplinary applications. Engineering, mechanical. Mechanics
General Relativity and Gravitation	35	Physics

Tabla 6.23. Revistas con mayor producción y disciplina ISI

Como se observa, aparecen una gran cantidad de artículos en el área de Física. Precisamente, la constatación de este fenómeno y de que la gran mayoría de sus autores pertenecían a Departamentos de Física y no Matemáticas, fue lo que nos motivó a realizar el filtrado manual que se ha explicado en la introducción.

7. COLABORACIÓN EN LA INVESTIGACIÓN MATEMÁTICA

De nuevo, los datos que se ofrecen en este capítulo están basados en la base de 6.220 artículos de la base ISI española.

Colaboraciones entre autores – Índice de autoría

Índice de coautoría

El índice de coautoría es el número medio de autores que participan en un documento. Dicho índice, así como el número medio de autores españoles que participan en el documento, se muestran en la tabla 7.1. Observamos que como media, de los 2,16 autores que firman un artículo, 1,70 son españoles.

Nº medio de autores por artículo	2,16
Nº medio de autores españoles por artículo	1,70

Tabla 7.1. Índice de coautoría

Evolución del índice de coautoría

En la tabla 7.2 se muestra la evolución del índice de coautoría en la producción española durante la última década. Podemos apreciar un ligero pero continuado aumento de la colaboración en la investigación matemática. El número medio de autores por artículo ha crecido un 18,2% en los últimos diez años, pasando de 1,91 a 2,25 autores por documento.
Entre autores españoles exclusivamente, el aumento es algo menor, pero también se observa una tendencia creciente, habiendo pasado de 1,59 autores españoles por artículo en 1990 a 1,8 en 1999. Este incremento sostenido en la colaboración seguramente se debe a una mayor incorporación de nuestros investigadores a los foros internacionales y a la generalización del uso de Internet. Podemos afirmar que se está cambiando el modo de hacer matemáticas pasando de una individualización a un trabajo cada más de equipo.

	1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**Incr**
Nº medio autores por artículo	1,91	2,02	2,04	2,06	2,06	2,21	2,17	2,21	2,27	2,25	18,2%
Nº medio autores españoles /art	1,59	1,59	1,55	1,60	1,61	1,71	1,69	1,73	1,81	1,79	12,7%

Tabla 7.2. Evolución del índice de coautoría

Documentos con un único autor y su evolución

De los 6.220 artículos con los que se realiza el presente estudio, 1.506 de ellos han sido firmados por un único autor. En la tabla 7.3 aparece la evolución de este tipo de artículos.

Aunque el número absoluto de documentos realizados en solitario ha crecido en términos absolutos, pasando de 115 documentos en 1990 a 196 en 1999, teniendo en cuenta el aumento de la producción, el porcentaje de documentos realizados por un único autor ha disminuido a lo largo de la década, pasando de significar el 35% de la producción en 1990 a ser sólo el 20% en 1999.

	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	Incr
Nº art. con un único autor	115	110	128	144	138	132	152	204	187	196	70,4%

Tabla 7.3. Evolución en la producción con un único autor

Colaboraciones entre instituciones – Nº medio por artículo

Nº medio de instituciones por artículo

En la tabla 7.4 se muestra el número medio de instituciones que participan en un documento. De la media de 1,55 instituciones por artículo, 1,19 son españolas.

Nº medio de instituciones por artículo	1,55
Nº medio de instituciones españoles por artículo	1,19

Tabla 7.4. Nº medio de instituciones por artículo

Evolución del número medio de instituciones por artículo

En la tabla 7.5 se muestra la evolución del número medio de instituciones que participan en un documento durante la última década.
Vuelve a observarse un continuado aumento de la colaboración en la investigación matemática. El número medio de instituciones por artículo ha crecido un 15,7% en los últimos diez años, pasando de ser 1,36 en 1990 a 1,58 en 1999.
Igualmente, la colaboración entre instituciones españolas ha crecido durante la década.

	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	Incr
Nº medio instituciones / artículo	1,36	1,52	1,50	1,53	1,54	1,54	1,54	1,61	1,58	1,58	15,7%
Nº medio instituciones esp /art	1,11	1,15	1,13	1,18	1,18	1,18	1,18	1,22	1,22	1,22	10,3%

Tabla 7.5. Evolución del nº medio de instituciones por artículo

Tasas de colaboración entre instituciones

Tasas de colaboración nacional e internacional en la producción matemática de España

La tabla 7.6 recoge los datos referentes a la colaboración entre las instituciones. Podemos observar que la mayoría de la investigación matemática, exactamente un 55,9%, se realiza sin colaboración entre distintas instituciones.
No debemos olvidar que los documentos firmados por varios autores con la misma dirección no son considerados como colaboración institucional. Así, si un artículo está escrito por dos profesores del mismo Departamento, no se considera como colaboración.

	Nº art.	%
Sin colaboración	3480	55,9%
Colaboración nacional	1064	17,1%
Colaboración nacional intramuros	*235*	*3,8%*
Colaboración nacional extramuros	*875*	*14,1%*
Colaboración internacional	1862	29,9%
Total real	6220	

Tabla 7.6. Tasas de colaboración

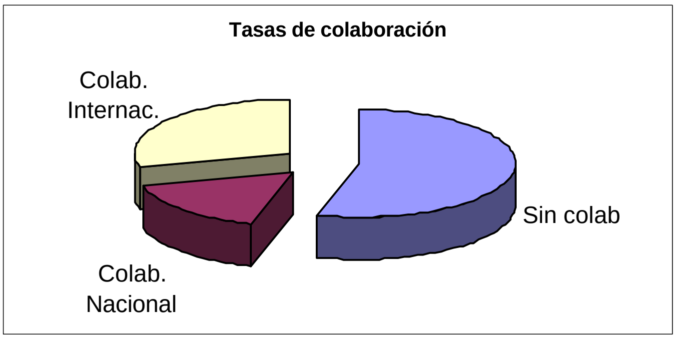


Gráfico 7.1. Tipos de colaboración

Evolución anual de la colaboración matemática

La tabla 7.7 muestra la evolución anual de la colaboración en la última década y los incrementos que han experimentado los distintos tipos de colaboración.

Se observa una tendencia a aumentar la colaboración, habiéndose incrementado las tasas de colaboración tanto nacional como internacional muy por encima de la tasa de los trabajos sin colaboración.

	90-91	92-93	94-95	96-97	98-99	Total	Incr
Sin colab.	443	567	651	814	1005	3480	127%
Colab. nacional	91	136	195	269	373	1064	310%
Col. intramuros	*17*	*25*	*43*	*63*	*87*	*235*	*412%*
Col. extramuros	*75*	*118*	*159*	*222*	*301*	*875*	*301%*
Colab. intern.	196	292	358	456	560	1862	186%
Total real	718	968	1168	1500	1866	6220	

Nota: Incrementos calculados respecto al primer bienio o, en su defecto, respecto al primer bienio con publicación

Tabla 7.7. Evolución de la colaboración matemática

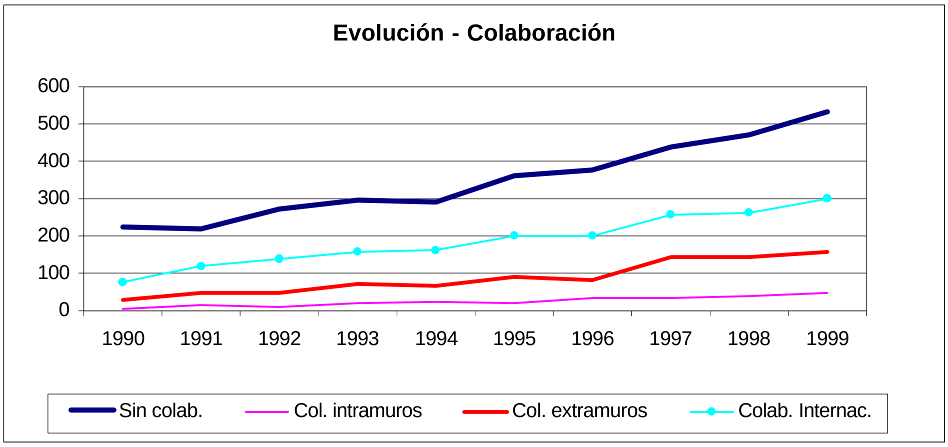


Gráfico 7.2. Evolución de la colaboración matemática

Colaboración entre Comunidades Autónomas

En la tabla 7.8 se muestra la colaboración entre las distintas comunidades autónomas. Cada fila corresponde a una Comunidad y contiene los porcentajes (redondeados) que la

colaboración con la CCAA de la columna correspondiente supone respecto del total de la colaboración de la comunidad de la fila. Por consiguiente, la tabla no es simétrica sino que hay que leerla por filas.
En general, la tabla indica que el patrón geográfico juega un papel importante en la colaboración. Más que por facilidad para la comunicación esto se debe seguramente a que los equipos investigadores de una Comunidad se han formado a partir de un equipo “madre” en una Comunidad vecina. Por ejemplo, se puede observar que las mayores proporciones de colaboración se dan entre Baleares y Cataluña, La Rioja y Aragón, y Navarra y Aragón. La comunidad de Madrid es la colabora con mayor número de CCAA.

	And	Arag	Ast	Bale	Can	Cant	CL	Cat	Val	Ext	Gal	Rioj	Mad	Mur	Nav	PV
Andalucía		6	1		1		2	5	10	6	9		39	16		6
Aragón	6		9		6	1	2	3	5		2	6	16		31	13
Asturias	3	27				27	3	19			14		8			
Baleares								78	22							
Canarias	2	11				8	6	2	2			2	62			8
Cantabria		1	14		7		29		7	4			26		7	3
Cast-León	3	3	2		7	33		3	15	5			25			5
Cataluña	6	3	7	7	1		2		3		10		54	2	1	6
Valencia	16	9		3	1	7	13	4		3	3		13	6	17	4
Extremad.	35					15	15		10				15	10		
Galicia	13	3	7					13	3				40	16	1	4
La Rioja		64			9								9		18	
Madrid	16	6	1		14	6	5	20	3	1	11	0		2	0	14
Murcia	39							4	9	4	26		11			7
Navarra		60				9		2	21	0	2	3	2			2
País Vasco	8	16			6	2	3	7	3	0	3		45	3	1	

Tabla 7.8. Colaboración entre comunidades autónomas (porcentajes)

<u>Colaboración internacional</u>

Producción matemática española en colaboración internacional por países colaboradores

Los patrones de colaboración internacional en grandes áreas geográficas se muestran en la tabla 7.9 Los porcentajes se refieren al total de colaboración internacional.

La comparación de los ejes de colaboración permite observar que los investigadores españoles colaboran sobre todo con la Unión Europea, un 16,1% del total de documentos se ha escrito en colaboración con la UE, y un 9% con EE.UU. y Canadá. Con los países europeos no pertenecientes a la UE es con quien menos se colabora en la investigación matemática.

	Nº art.	**%**
Unión Europea	1003	53,9%
Resto de Europa	207	11,1%
EE.UU. y Canadá	562	30,2%
Latinoamérica	184	9,9%
Otros países	262	14,1%
Total real	1862	

Tabla 7.9. Colaboración internacional

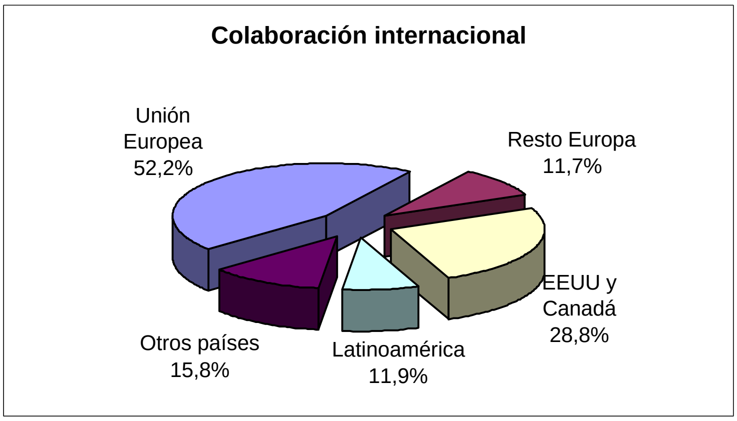


Gráfico 7.3. Colaboración internacional

En las tablas 7.10, 7.11, 7.12, 7.13 y 7.14 se estudian las colaboraciones internacionales. Dentro de la colaboración con la Unión Europea, el país con el que más se colabora es Francia (un 27,8% de la colaboración con la UE), seguido de Gran Bretaña (14,3%) y de Italia (13,9%). Es destacable la colaboración con Estados Unidos, que alcanza el 8% del total de documentos. Entre los países latinoamericanos, Brasil es el país con el que más colaboraciones realiza España (58 documentos), seguido por Argentina (42 documentos), y México (38 documentos). La colaboración con los países del sudeste asiático es escasa.

Países de la Unión Europea	Nº art.
Francia	279
Gran Bretaña	143
Italia	139
Alemania	135
Bélgica	115
Holanda	61
Portugal	31
Suecia	27
Finlandia	25
Austria	20
Irlanda	13
Grecia	9
Dinamarca	6
Total	1003

Tabla 7.10. Colaboración con la UE

EEUU y Canadá	Nº art.
EEUU	507
Canadá	55
Total	562

Tabla 7.11. Colaboración con EEUU y Canadá

Resto de Europa	Nº art.
Polonia	61
Rumania	33
Noruega	30
Croacia	19
Bulgaria	16
República Checa	16
Hungría	10
Ucrania	7
Eslovaquia	4
Suiza	4
Uzbekistan	3
Bielorusia	2
Estonia	1
Yugoslavia	1
Total	207

Tabla 7.12. Colaboración con resto de Europa

Latinoamérica	Nº art.
Brasil	58
Argentina	42
México	38
Venezuela	15
Chile	13
Uruguay	10
Cuba	5
Costa Rica	3
Total	184

Tabla 7.13. Colaboración con Latinoamérica

Otros países	Nº art.
Rusia	73
Israel	36
República Popular China	33
Australia	23
Japón	17
Nueva Zelanda	12
Vietnam	11
Turquía	10
Marruecos	8
India	7
República de Corea	7
Sudáfrica	7
Georgia	4
Armenia	2
Líbano	2
Singapur	2
Taiwan	2
USSR	2
Zimbawue	2
Egipto	1
Irak	1
Total	262

Tabla 7.14. Colaboración con resto del mundo

Patrón de colaboración por centro de investigación

La tabla 7.15 estudia los patrones de colaboración en las universidades y en el Centro de Física Miguel A. Catalán (que en realidad corresponde a los matemáticos del Instituto de Matemáticas y Física Fundamental), ordenados de mayor a menor producción de artículos de matemáticas. Se marcan en negrita porcentajes mayores que el porcentaje medio de la categoría.

La Universidades de Málaga, Extremadura y Sevilla son las universidades con una mayor tasa de "no colaboración", ambas muy por encima de la tasa nacional, seguidas por la Universidad de La Rioja y Granada. La Universidad de Burgos no resulta representativa por tener un único documento. A su vez, las universidades con menor tasa de "no colaboración" son las de Alcalá de Henares y La Coruña. La Universidad de Navarra no resulta representativa por tener un único documento.

Destacan por sus altas tasas de colaboración intramuros (colaboración interdepartamental) la Universidad Politécnica de Cataluña, la Universidad de Valladolid y la Universidad de Cantabria. Por otra parte, las universidades de Las Palmas, Córdoba, Lleida y Alcalá de Henares, son las que presentan una mayor tasa de colaboración extramuros (interfacultativa). Casi todas estas universidades son de escasa producción. Entre las de mayor producción, tienen mayor tasa de colaboración extramuros la Universidad Politécnica de Madrid y la Universidad de Cantabria.

En la colaboración internacional, destacan por sus altas tasas la Universidad Autónoma de Madrid, la Universidad Autónoma de Barcelona, la Universidad Carlos III de Madrid y la Universidad de las Islas Baleares. Las universidades de Pompeu Fabra y de Lleida tienen unas altas tasas de colaboración internacional a pesar de su escasa producción. La Universidad Complutense de Madrid que es la universidad con una mayor producción, presenta tasas por debajo de la media nacional tanto en "no colaboración" como en los dos tipos de colaboración nacional, pero sí que presenta un alto nivel de colaboración internacional.

Centro	Sin colab.	Intramuros	Extramuros	Internacional
Universidad Complutense de Madrid	43,0%	2,5%	9,9%	**36,1%**
Universidad de Granada	**63,9%**	**4,9%**	6,6%	19,3%
Universidad Politécnica de Cataluña	51,4%	**7,0%**	10,2%	27,1%
Universidad de Zaragoza	50,4%	**5,1%**	**17,2%**	25,4%
Universidad de Barcelona	45,4%	0,5%	10,4%	35,3%
Universidad Autónoma de Barcelona	40,7%	0,5%	**15,0%**	**39,9%**
Universidad de Sevilla	**66,3%**	5,7%	6,6%	18,8%
Universidad de Valencia	52,7%	1,0%	13,4%	26,2%
Universidad Autónoma de Madrid	43,8%	0,0%	7,4%	**44,1%**
Universidad de Santiago de Compostela	50,3%	**5,2%**	11,5%	27,3%
Universidad del País Vasco	50,0%	3,2%	**15,1%**	23,4%
Universidad Politécnica de Madrid	31,0%	**4,7%**	**26,6%**	28,5%
Universidad Politécnica de Valencia	55,5%	**5,5%**	11,8%	23,6%
Universidad de Valladolid	**56,1%**	**5,9%**	6,7%	25,3%
Universidad de La Laguna	46,8%	**4,1%**	12,7%	**30,0%**
Universidad de Cantabria	37,1%	**5,9%**	**22,0%**	**30,7%**
Universidad de Murcia	52,9%	1,1%	9,8%	27,0%
Universidad de Málaga	**69,3%**	**3,9%**	6,5%	14,4%
Universidad de Alicante	53,3%	3,7%	6,5%	29,0%
Universidad de Extremadura	**66,7%**	2,0%	13,1%	16,2%
Universidad de Oviedo	52,8%	0,0%	**15,1%**	22,6%

Universidad Carlos III de Madrid	31,2%	1,1%	**20,4%**	**39,8%**
Universidad de Salamanca	43,2%	3,7%	13,6%	**35,8%**
UNED	39,1%	0,0%	**18,5%**	26,1%
Universidad de Vigo	31,3%	**4,5%**	**38,8%**	23,9%
Universidad Pública de Navarra	33,7%	3,5%	12,8%	18,6%
Universidad de Almería	38,3%	0,0%	**25,5%**	**36,2%**
Universidad de las Islas Baleares	44,2%	0,0%	14,0%	**39,5%**
Universidad Jaume I	45,7%	0,0%	**17,4%**	23,9%
Universidad de Córdoba	36,4%	3,0%	**36,4%**	15,2%
Universidad de La Rioja	**65,4%**	0,0%	**19,2%**	3,8%
Universidad de La Coruña	20,0%	0,0%	**24,0%**	**32,0%**
Universidad de Alcalá de Henares	16,7%	0,0%	**33,3%**	27,8%
Universidad Pompeu Fabra	28,6%	0,0%	**21,4%**	**50,0%**
Universidad de Cádiz	44,4%	0,0%	**16,7%**	5,6%
Universidad de Lleida	33,3%	0,0%	**33,3%**	**44,4%**
Universidad de Jaén	28,6%	0,0%	**42,9%**	0,0%
Universidad de Las Palmas de Gran Canaria	40,0%	0,0%	**60,0%**	0,0%
Universidad de Burgos	100,0%	0,0%	0,0%	0,0%
Universidad de Navarra	0,0%	0,0%	0,0%	100,0%
Centro de Física Miguel A. Catalán	18,3%		**43,0%**	32,3%
España	**55,9%**	**3,8%**	**14,1%**	**29,9%**

Tabla 7.15. Patrón de colaboración por centro de investigación

Patrón de colaboración por clasificación MSC

Patrón de colaboración en los 20 temas MSC con mayor producción

La tabla 7.16 estudia los patrones de colaboración de los veinte temas de la clasificación MSC con una mayor producción durante la última década. Resaltamos en negrita los porcentajes mayores que la media.

Destacan las altas tasas de colaboración internacional que muestran los temas “16: Anillos y álgebras asociativos” y “35: Ecuaciones en derivadas parciales”.
Por el contrario los temas “13: Anillos conmutativos y álgebras”, “65: Análisis numérico” y “93: Teorías del control y sistema” son los que presentan una mayor tasa de “no colaboración”, todas por encima del 60%.
“93: Teorías del control y sistema” también presenta una alta colaboración intramuros, al igual que lo hace la “76: Mecánica de fluidos”.
Por último, “53: Geometría diferencial”, “58: Análisis global, análisis en variedades” y “54: Topología general” son los que más se trabajan en colaboración extramuros.

MSC	Tema	Sin colab.	Intramuros	Extramuros	Internacional
46	Análisis funcional	**57,0%**	1,4%	13,3%	28,3%
65	Análisis numérico	**66,1%**	3,3%	9,3%	21,3%
35	Ecuaciones en derivadas parciales	49,9%	0,8%	9,4%	**39,9%**
62	Estadística	**56,2%**	2,6%	**19,3%**	21,9%
58	Análisis global, análisis en variedades	37,8%	3,6%	**24,8%**	**33,8%**
90	Economía, investigación operativa, programación, juegos	54,0%	**4,3%**	12,8%	29,0%
53	Geometría diferencial	52,9%	2,4%	**25,3%**	19,4%
68	Ciencias de la computación	46,6%	3,1%	12,4%	**37,8%**
34	Ecuaciones diferenciales ordinarias	**59,8%**	**4,8%**	**14,8%**	20,6%
20	Teoría de grupos y generalizaciones	**60,9%**	0,5%	**16,3%**	22,3%
16	Anillos y álgebras asociativos	45,9%	2,9%	9,4%	**41,8%**

14	Geometría algebraica	**58,6%**	3,0%	6,5%	**32,0%**
42	Análisis de Fourier	44,4%	**4,3%**	**19,1%**	**32,1%**
93	Teorías del control y sistema	**62,1%**	**9,3%**	6,2%	22,4%
60	Teoría de la probabilidad y procesos estocásticos	48,1%	2,5%	12,0%	**37,3%**
76	Mecánica de fluidos	50,7%	**7,6%**	4,9%	**36,8%**
17	Anillos y álgebras no asociativos	**56,8%**	1,4%	11,5%	**30,2%**
47	Teoría de operadores	**60,3%**	1,7%	**14,9%**	23,1%
54	Topología general	42,9%	**4,5%**	**24,1%**	28,6%
13	Anillos conmutativos y álgebras	**68,9%**	**4,9%**	2,9%	23,3%
	España	**55,9%**	**3,8%**	**14,1%**	**29,9%**

Tabla 7.16. Patrón de colaboración en los temas con mayor producción

8. CONCLUSIONES

- Tanto en el mundo como en los ámbitos europeo y español, la década de los 90 se caracteriza por un aumento de la producción matemática recogida en la base de datos MathSci. La producción española crece a un ritmo mayor que la del resto del mundo, habiendo pasado de ser el 1,7% de la mundial en el año 1990 al 3,2% en el año 1999. Esto también ocurre si comparamos en el seno de la UE, donde la producción española durante la última década ha pasado de suponer el 8,9% en 1990 al 13,0% en 1999 (cf. pag. 20).

- Los códigos "Trasformaciones integrales, cálculo operacional" (no. 44), "Análisis funcional" (no. 46), Teoría de conjuntos (no. 04), "Geometría diferencial" (no. 53) y "Análisis de Fourier" (no. 42), tienen en España un porcentaje de producción respecto a la producción total española muy por encima de la mundial (cf. pag. 21).

- También en la producción ISI la producción española ha crecido a un ritmo mucho más rápido que la producción mundial. La aportación española en esta base de datos ha pasado de representar el 1,7% de la producción mundial en 1990 (con 330 artículos) al 3,9% en 1999 (con 983 artículos), y ha continuado creciendo hasta situarse en el 4,18% según los últimos datos del ISI del 2001. Simplificando, cabe decir que la producción española se ha incrementado en un 300%, mientras que la producción mundial lo ha hecho en menos de la mitad (pags. 24 y ss).

- Comparando con otras disciplinas científicas, las Matemáticas ocupan en España el tercer lugar en cuanto a lo que supone su aportación dentro de la producción mundial (el 4,18%), por detrás de Astrofísica y Ciencias Agrarias. Sin embargo la media de citas por artículo está un 16% por debajo de la media mundial (pag. 27).

- Las Comunidades Autónomas con mayores cifras absolutas de producción matemática son Madrid, Cataluña y Andalucía superando todas ellas el millar de documentos ISI en la década y sumando entre las tres el 60% de la producción total española. Esto pone de manifiesto la gran concentración de la investigación existente en Madrid y Cataluña. Sin embargo, relativizando la producción por el número de profesores, las Comunidades con mejor ratio de artículos por profesor resultan ser Aragón, Cantabria y Cataluña, por este orden, (pag. 29).

- La media de artículos ISI por profesor está en 2,22, y contando la totalidad de documentos MathSci resulta una productividad de 3,78 artículos por profesor en la década. Haciendo una pequeña prospección de estos datos podemos aventurar que a lo sumo 2/5 del total de 3.124 profesores universitarios de Matemáticas están activos en lo que respecta a publicar asiduamente (cf. pag. 29).

- Por sectores institucionales la Universidad es el sector más productivo participando en el 98,6% de los documentos, mientras que el CSIC lo hace en el 2,3%. Por su parte el sector privado está totalmente ausente de la producción matemática española, lo que pone de manifiesto el poco a nulo interés de la empresa privada en la investigación y la inexistencia de matemáticos en labores de I+D en el ámbito empresarial (pag. 32).

- Por universidades, la Universidad Complutense de Madrid es la que realiza una mayor aportación a la investigación matemática (el 11,4% de la producción total), seguida a una cierta distancia por las Universidades de Granada y Politécnica de Cataluña (8,8% y 7,1% respectivamente). Las Universidades de Burgos, Navarra, Las Palmas, Jaén y Lleida son las Universidades de menor producción matemática, con un porcentaje sobre el total menor del 0,15%. Relativizando la producción por el número de profesores en cada Universidad, las universidades con mayor ratio de documentos por profesor son la Universidad de Barcelona, la Universidad Autónoma de Madrid y la Autónoma de Barcelona (pag. 34).

- En el CSIC, el 80% de toda la producción matemática está concentrada en el Centro de Física Miguel A. Catalán (CFMAC), integrado sólo por tres investigadores. Llama la atención la carencia de un instituto propio de Matemáticas en el CSIC, situación sin paralelo en los paises de la UE (cf. pag. 36).

- Más del 50% de la investigación española se centra en nueve de los códigos de la MSC, mientras que casi el 90% se centra en 35 de ellos. Los tres códigos más productivos en en cuanto a número absoluto de documentos son Análisis Funcional (no. 46), Ecuaciones en derivadas parciales (no. 35) y Análisis numérico (no. 65). Los incrementos mayores en producción a lo largo de la década se han dado en Estadística (no. 62), Análisis Numérico (no. 65), Geometría Diferencial (no. 53) y Ciencias de la Computación (no. 68) (cf. pag. 38).

- Asignando los códigos MSC a Áreas de Conocimiento “Matemática Aplicada” resulta ser el área más productiva, con el 43,8% de la producción total. También es el área que con mayor número de profesores adscritos a la misma, por lo que teniendo en cuenta la ratio de nº de artículos por profesor, Geometría y Topología resulta ser el área más productiva, seguida a cierta distancia por Álgebra (pag. 43).

- Analizando la distribución de la investigación española por cuartiles dentro de la clasificación del ISI por factor de impacto se observa que está desplazada hacia el tercer cuartil de modo mucho más acusado que la distribución mundial, en detrimento del número de trabajos colocados en el primer cuartil. Los porcentajes en el segundo y cuarto cuartil son similares en España y el resto del mundo. Aproximadamente el 39% de los artículos se publican en revistas con un factor de impacto por encima de la media. A nivel mundial este porcentaje alcanza el 44% (pags. 46 y ss.). Además la distribución por cuartiles no ha variado sensiblemente a lo largo de la década (cf. pags. 46 y ss.). Sería conveniente, pues, orientar las publicaciones hacia revistas más valoradas internacionalmente, aunque ello suponga someterse a procesos de valoración más rigurosos.

- Este mismo escoramiento hacia el tercer cuartil se aprecia en la mayoría de los centros, aunque en diferente medida. Destacan por la calidad de su investigación (porcentaje de trabajos en el primer cuartil) la Universidad Autónoma de Madrid, las Universidades de Valladolid, Salamanca y Barcelona (cf. pags. 48 y 49).

- Por Áreas de Conocimiento, Matemática Aplicada y Geometría y Topología son las que tienen un tanto por ciento mayor de publicaciones en los cuartiles superiores, mientras que Álgebra y Cc. de la Computación e Inteligencia Artificial están muy escoradas hacia el tercer cuartil (cf. pags. 50 y ss.).

- En lo que respecta a los patrones de colaboración podemos decir que cada vez es mayor la proporción de trabajos firmados por más de un autor. Parece razonable suponer que la comunicación electrónica ha influido sensiblemente en este fenómeno que está transformando las formas de colaboración en la escritura de trabajos de Matemáticas. No obstante todavía se aprecia una fuerte incidencia del patrón geográfico en la colaboración entre las distintas CCAA. Por lo que respecta a colaboración internacional destaca la colaboración con Estados Unidos y con la Unión Europea, siendo dentro de ésta Francia el país con mayor índice de cooperación (cf. pags. 58 y ss.).

9. APÉNDICE

Clasificación MSC 2000

00 General
01 History
03 Mathematical logic and foundations
04 Set theory
05 Combinatorics
06 Order, lattices, ordered algebraic structures
08 General mathematical systems
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12 Field theory and polynomials
13 Commutative rings and algebras
14 Algebraic geometry
15 Linear and multilinear algebra; matrix theory
16 Associative rings and algebras
17 Nonassociative rings and algebras
18 Category theory, homological algebra
19 K-theory
20 Group theory and generalizations
22 Topological groups, Lie algebras
26 Real functions
28 Measure and integration
30 Functions of a complex variable
31 Potential theory
32 Several complex variables and analytic spaces
33 Special functions
34 Ordinary differential equations
35 Partial differential equations
37 * *Dynamical systems and ergodic theory*
39 Finite differences and functional equations
40 Sequences, series, summability
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43 Abstract harmonic analysis
44 Integral transforms, operational calculus
45 Integral equations
46 Functional analysis
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65 Numerical analysis
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73 Mechanics of solids
74 * *Mechanics of deformable solids*
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92 Biology and behavioral sciences
93 Systems theory, control
94 Information and communication, circuits

* Proceden de MSC 2000

Revistas con un mayor número de documentos publicados en ellas y su disciplina ISI

En la tabla aparecen las cincuenta revistas donde más han publicado los autores españoles y el número de documentos publicados en cada una de ellas. Junto a estos datos aparece la disciplina o disciplinas ISI en la que se clasifica la revista.

Revista	Nº art.	Disciplina
Proceedings of the American Mathematical Society	**206**	Mathematics. Mathematics, applied
Communications in Algebra	**199**	Mathematics
C. R. Academie des Sciences I. Mathematique	**186**	Mathematics
Journal of Mathematical Analysis and Applications	**178**	Mathematics. Mathematics, applied
Journal of Algebra	**158**	Mathematics
Journal of Computational and Applied Mathematics	**139**	Mathematics, applied
Archiv der Mathematik	**111**	Mathematics
Nonlinear Analysis	**108**	Mathematics. Mathematics, applied
Linear Algebra and its Applications	**104**	Mathematics, applied
Fuzzy Sets and Systems	**102**	Computer Science, Theory & Methods. Mathematics, applied. Statistics & Probability
Journal of Pure and Applied Algebra	**94**	Mathematics. Mathematics, applied
Studia Mathematica	**82**	Mathematics
Journal of Differential Equations	**76**	Mathematics
Transactions of the American Mathematical Society	**68**	Mathematics
Applied Mathematics and Computation	**62**	Mathematics, applied
Manuscripta Mathematica	**62**	Mathematics
Communications in Statistics. Theory and Methods	**61**	Statistics & Probability
Journal of Physics. A. Mathematical and General	**61**	Physics. Physics, mathematical
Statistics and Probability Letters	**60**	Statistics & Probability
Bulletin of the Australian Mathematical Society	**58**	Mathematics
Math. Proceed.. Cambridge Philosophical Society	**54**	Mathematics
Journal of Approximation Theory	**52**	Mathematics
Computers and Mathematics with Applications	**51**	Computer Science, Interdisciplinary Applications. Mathematics, applied
Mathematische Nachrichten	**51**	Mathematics
Acta Mathematica Hungarica	**50**	Mathematics
Israel Journal of Mathematics	**49**	Mathematics
The Journal of the London Mathematical Society.	**49**	Mathematics
Applied Mathematics Letters	**47**	Mathematics, applied
Pacific Journal of Mathematics	**44**	Mathematics
The Rocky Mountain Journal of Mathematics	**42**	Mathematics
Celestial Mechanics and Dynamical Astronomy	**41**	Astronomy & Astrophysics
Journal of Mathematical Physics	**41**	Physics, mathematical
Mathematische Zeitschrift	**41**	Mathematics
Discrete Mathematics	**40**	Mathematics
Topology and its Applications	**40**	Mathematics. Mathematics, applied
Journal of Functional Analysis	**39**	Mathematics
Inter. J. of Bifurcat. and Chaos in App. Sci. and Engin	**38**	Mathematics, applied. Multidisciplinary Sciences
Proc. Royal Soc. of Edinburgh. Sect. A. Mathematics	**38**	Mathematics. Mathematics, applied
Applied Numerical Mathematics	**37**	Mathematics, applied
Journal of Statistical Planning and Inference	**36**	Statistics & Probability
Glasgow Mathematical Journal	**34**	Mathematics
Mathematische Annalen	**34**	Mathematics
Numerical Algorithms	**33**	Mathematics, applied
Information Processing Letters	**32**	Computer Science, Information Systems
Universitatis Debreceniensis	**32**	Mathematics
Computer Methods in Applied Mechanics and Engineering.	**30**	Computer Science, Interdisciplinary Applications. Engineering, Mechanical. Mechanics
Geometriae Dedicata	**30**	Mathematics
Mathematical Social Sciences	**30**	Mathematics, miscellaneous
SIAM Journal on Mathematical Analysis	**30**	Mathematics, applied
Theoretical Computer Science	**30**	Computer Science, Theory & Methods

Revistas con mejor posición normalizada y nº de documentos publicados en ellas

En la tabla se muestran las cincuenta revistas con mejor posición normalizada, indicando los documentos publicados en cada una de ellas. La posición normalizada de las revistas nos permite comparar revistas de distintas disciplinas ISI, algo que el factor de impacto no nos permite hacer.

Revista	Pos. Norm.	Nº art.
Annals of Mathematics. Second Series	0,99	3
Chaos. An Interdisciplinary Journal of Nonlinear Science	0,99	4
Memoirs of the American Mathematical Society	0,99	2
SIAM Journal on Optimization	0,99	4
IEEE Transactions on Image Processing	0,99	2
Journal of the Royal Statistical Society. Series B. Methodological	0,98	6
Operations Research	0,98	2
Acta Mathematica	0,98	3
Communications on Pure and Applied Mathematics	0,97	7
Journal of the American Mathematical Society	0,97	3
Econometrica. Journal of the Econometric Society	0,96	6
Mathematical Programming	0,96	7
American Mathematical Society. Bulletin. New Series	0,96	1
Biometrics. Journal of the International Biometric Society	0,96	1
Inventiones Mathematicae	0,95	8
Journal of the ACM	0,95	2
IEEE. Transactions on Information Theory	0,95	10
Constructive Approximation	0,94	11
International Journal for Numerical Methods in Engineering	0,94	19
Journal of Nonlinear Science	0,94	5
Mathematics of Operations Research	0,94	4
Numerical Linear Algebra with Applications	0,94	4
Journal of the American Statistical Association	0,94	14
SIAM Journal on Control and Optimization	0,94	15
Journal of Algebraic Combinatorics. An International Journal	0,93	1
Biometrika	0,93	7
Geometric and Functional Analysis	0,92	1
Artificial Intelligence	0,92	4
Advances in Mathematics	0,92	5
Computer Methods in Applied Mechanics and Engineering	0,91	30
Naval Research Logistics. An International Journal	0,91	6
The Annals of Statistics	0,91	15
SIAM Journal on Scientific Computing	0,91	12
Duke Mathematical Journal	0,90	19
Applied and Computational Harmonic Analysis	0,90	1
Journal de Mathematiques Pures et Appliquees. Neuvieme Serie	0,90	20
IEEE. Transactions on Software Engineering	0,90	1
Computer Physics Communications	0,89	6
Computational Geometry. Theory and Applications	0,89	4
SIAM Journal on Numerical Analysis	0,88	28
Annales Scientifiques de l'Ecole Normale Superieure. Quatrieme Serie	0,88	7
Journal of Global Optimization	0,88	1
Inverse Problems	0,88	11
Archive for Rational Mechanics and Analysis	0,88	20
Journal of Functional Analysis	0,88	39
SIAM Review	0,87	8
Journal of Differential Geometry	0,87	6
Journal of Computational Physics	0,87	19
Journal of Differential Equations	0,86	76
Commentarii Mathematici Helvetici	0,86	14

The Importance of Mathematics in the development of Science and Technology

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Do not worry too much about what is Mathematics
Before you try your luck with them. Then you'll see.

Abstract

Mathematicians often say that the essence of Mathematics lies in the beauty of numbers, figures and relations, and there is truth in that. But the driving force of mathematical innovation in the last centuries has been the desire to understand how Nature works. This aspect often goes unmentioned.

Together with the experimental method, Mathematics forms the conceptual scheme on which modern science is based and which supports technology, with close interactions among them. Upon these bases the Industrial Society was born some centuries ago, and the new Information Society is built in the present along the same lines.

In the article we give a brief outline of this scientific connection and how it came to work and the heroes that made it what it is, a view to the future, and a short comment on Mathematics in Spain.

1 Introduction. Essence and role of Mathematics

Mathematics is an autonomous intellectual discipline, one of the clearest exponents of the creative power of the human mind. On the other hand, it plays a fundamental role in modern Science, has a strong influence on it and it has been influenced by it in an essential way. Here are, briefly presented, two conceptions that symbolize different ways of seeing the great edifice that is present-day Mathematics. These options are reflected in the denominations of Pure and Applied Mathematics. But then, are there two different Mathematics? and, if this true, can they healthily co-exist and interact, or do they actually exist separated from, even hostile to each other? In the present article we will see that, today as in the past, both views of Mathematics are faces of the same coin, looking at times so different, at times so similar.

A first dimension of Mathematics is in fact the **pure** aspect, Mathematics as an art in its own right, a game that is played in our minds. Indeed, Mathematics is an art that expresses beauty in the form of axioms, theorems and logical or numerical relations; it attracts the researcher precisely because of its logical perfection, by being one of the most compelling examples of the human capacity for reasoning and analysis, by imposing order and harmony where formerly we only saw disorder and chaos. This is the dimension which lies closest to the researcher and, as every pure form of art, it has a fascination that explains why professionals devote an enormous and quite exclusive part of their lives to it. It is natural for professional mathematicians to tend to see their science from the point of view of the art in itself, with its concepts, conjectures, results and methods of proof, with its time-honored areas: arithmetic, algebra, geometry and analysis, and the new sprouts: statistics, calculus of probabilities, mathematical logic, computation,... and above all, with its perfect logical deductions. Great scholars, from Pythagoras and Plato to Gauss, have even seen in Mathematics a world of order, more perfect that the everyday physical world. In fact, few professional mathematicians have missed the feeling that the true Mathematics inhabits somewhere beyond, in an ideal world, waiting to be discovered by the artist. Some could go very far in these ideal directions: thus, Carl G. J. Jacobi sustained that Mathematics exists only “for the honour of the human mind”. Hence, the popular conception, at the same time romantic and misleading, of the mathematician as a distracted *savant* with little or no practical mind.

Is this the whole picture of Mathematics? Indeed, Mathematics is much more, there is whole new way of looking at them, and doing them: next to the experimental method, it is the basis upon which modern Science has been built and, as a consequence, the modern technological development rests. It permeates today all aspects of contemporary society from engineering to information, management business and finance, not forgetting the movement of the social disciplines toward the status of sciences, which amounts, in other words and with the proper nuances, to the use in these disciplines of the mathematical and experimental methods in combination.

Now, the practical importance of Mathematics in Science is indisputable, and it is not under discussion to a certain level, since the overwhelming majority of scientists are well aware of the *instrumental value* of some Mathematics. Thus, a quantitatively very important part of the Mathematics that is taught at universities all over the world is devoted to the education of engineers, physicists, chemists, computer scientists, economists and professionals of several other disciplines. However, the "applied" role of Mathematics goes far beyond this description, is more *essential.* In fact:

(i) Mathematics has played a fundamental role in the formulation of modern Science since the very beginning; a scientific theory is a theory that has an adequate mathematical model;

(ii) the Mathematics that can be applied today covers all the fields of the mathematical science and not only some special topics; it concerns Mathematics of all levels of difficulty and not only simple results and arguments;

(iii) the sciences continue to require today new results from ongoing research and present multiple new directions of inquiry to the researchers, but the rhythm of the contemporary society makes the time lapse substantially shorter and the request more urgent;

(iv) the capabilities of scientific computation have made *numerical simulation* an indispensable tool in the design and control of industrial processes.

In this article we will deal with this aspect whereby *Mathematics is the language* in which the pages of Science are written. thanks to it there has been a development of the combination Science-Technology that has changed the life of the citizen of technologically advanced societies in the last four centuries in a more radical way than the Neolithic revolution had done in the ninety previous centuries, and the change has been more dramatic in the last decades than in whole centuries before. Indeed, the daily practice of the physical sciences and engineering hides huge amounts of higher mathematics. Moreover, the very concepts on which their theories are based are essentially *mathematical concepts.* In the last decades we have seen the trend towards mathematization reach other disciplines, like Economics, particularly the financial market, branches of Chemistry, Biology and Medicine, and even the social sciences. It is true that the mathematical machinery, imposing or not, is most often carefully concealed from the public eye.

In the hands of the scientist, *Mathematics should permit to assimilate the data and to understand the phenomena.* In the hands of the engineer, it is the tool that makes possible to build a numerical or qualitative *model* whose analysis allows to *make decisions and design artifacts in an efficient and reliable way.* This activity is what, lacking a better name, we call **Applied Mathematics**. It covers the classical areas like Mathematical Physics and Mathematical Methods for Engineering, but it has today broader contours with the advent of scientific computation and numerical simulation. Modeling, computational simulation and data analysis are essential tools in modern science and industry. Applied Mathematics is just the **Mathematics of Reality**, i.e., the real world, whatever this sentence means to each individual reader.

Let us point out that there are other complementary visions of Mathematics: its cultural aspect, its importance in teaching and education as a vehicle for rational thought, its importance in understanding the daily world ("the Mathematics for the common man"), its aspect as a challenging intellectual game. It is at the same time the science of the exact and the calculation of the probable. It is the science of abstract and symbolic reasoning, and it is also today synonymous to computational virtuosity, of capacity to effectively process information, such an important quality in the present world. It tells us about the pure scientist who works with a piece of paper, and also about the world of modeling, computation and control of industrial processes. The layman thinks that Mathematics is tied to the quest of infinite precision. In practice, much of the art of contemporary mathematics is based on estimating. All of these aspects are part of the multiple legacy of Mathematics[1].

We turn next our attention toward the past and present of Applied Mathematics. The reader may find it convenient in a first reading to skip the information contained in the footnotes. Besides, a number of famous and important formulas and equations will appear scattered through the pages. They are not meant to be studied as part of this text! The purpose is rather to remind the initiated reader of their beauty and relevance, and at the same time to make the point that there is no *royal way* to Mathematics, namely that a real understanding of the topics outlined here implies serious study.

2 Galileo's and Newton's heirs

Two great historical figures fixed the *key role* of Mathematics in the moments in which modern Science was being born. *Galileo formulated it, Newton demonstrated it.* We ought to add that back in History Pythagoras of Samos (569bC-475bC) sustained that *All is number* and found the wonderful connections between Music and Arithmetic, while Archimedes of Syracuse joined Geometry and Mechanics in the IIIrd century b.C (d. 212 b.C.). And one century before Galileo, the universal genius of *Leonardo da Vinci guessed the role* of Mathematics in Science. A pleiad of great mathematicians, the heroes of our story, followed them[2]. The mathematicians who are busy with the

[1] We have written about these subjects in [41].

[2] In the story that follows the names of Galileo and Newton are accompanied by other eminent mathematicians, some of which will be assigned a prominent role in the narrative. Such a selection has been useful to set the main hits and to get to know the heroes of our private adventure, but is no doubt unfair from a strictly historical point of view with personalities like Fermat, Leibniz or Gauss, and we want to make it clear at this point. We hope to be excused because of the brevity of

application of their art stand truly upon the shoulders of giants[3].

Let us proceed in parts: it is true that from the oldest times Mathematics has been related, even motivated, by practical problems. Arithmetic originates from the activities of counting and adding, Geometry stems from measuring lines, surfaces and bodies. But it is also true that Mathematics as a logico-deductive science, just as it was elaborated and bequeathed to us by the Greeks from Pythagoras to Euclides, had a net intellectual, we could say ideal, basis that it has always conserved since then and that is a fundamental part of pure Mathematics, that is to say, of Mathematics in itself. This intellectual process lives in its own world and does not owe anything of its merit or beauty to the possible utility or practical application, not more than a poem or a painting do. An easy and frequently made syllogism would lead from here to conclude that the authentic Mathematics lives essentially alien to the adventure of science and technology. We contend that this syllogism is false by a great deal, even if it has been sustained by many mathematicians, and we will make our case clear in what follows by using opinions of famous scientists, but mainly by presenting a record of factual evidence. Indeed, *History shows us that the symbiosis with Science and Technology has been fundamental and fruitful and that Mathematics owes a great deal of its present being and of its main topics to its adventure companions, and conversely the latter to the former.*

As is well known, modern Science appeared in Europe at the end of the Renaissance. It is not based upon Mathematics alone. The fundamental pillar of the building in germ was aptly formulated by the English philosopher and politician Francis Bacon circa 1620 and consists of the *experimental method*[4]. Nature becomes the preferential object of philosophical investigation, we should learn to read and to understand it, and eventually to control it; observation is the means for comprehension and experiment is the test of our predictions. The sciences were formed around this method, first Physics, then Biology, Geology, Chemistry and so on.

Mathematics is, since the very beginning, the other pillar of the sciences. It was Galileo GALILEI (1564-1642) who pointed out in the clearest form that course for the budding sciences at the beginning of the XVII century. His is the famous quotation taken from his letter "Il saggiatore"[5] that we reproduce in detail: *"Philosophy is*

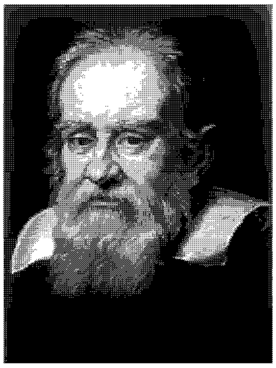
the text (the famous *narrow margin* referred to by Fermat) and also because the purpose we have in mind is not the history of science.

[3]Newton's opinion on his predecessors in a letter to R. Hooke, 1675: "If I have seen farther than others, it is by standing on the shoulders of giants". I have endeavoured to include in the text and notes some of the most celebrated phrases of mathematicians and scientists about Mathematics and its application.

[4]The inductive method is presented in his work *Novum Organum* or *New Instrument*, 1620.

[5]1623, cf. Opere, VI, p. 232; "The Assayer", translated into English by S. Drake, Doubleday Anchor Books, New York, 1957.

written in that great book that stands constantly open to our gaze, the Universe, but it cannot be understood unless one first learns to comprehend the language in which it is written and its characters. It is written in the language of Mathematics, and its characters are triangles, circles and other geometrical figures,..."[6]



GALILEO GALILEI

Galileo was of course a committed defender of the experimental method, to which he contributed his famous astronomical and mechanical observations [7]. The attitude of Galileo had precedents, the most remarkable being as we said Pythagoras and Archimedes in the Ancient Times and Leonardo da Vinci (1452-1519)[8] a century before, but his formulation was determined and put to practice, and it happened in a suitable historical context; it eroded the bases of Aristotelism and Scholastics dominant until then in the intellectual world. It bore fruit in a short time and the scientists see themselves reflected in it.

Indeed, philosophies are a small thing if they remain words and polemics, if they are not carried out. The glory of the XVIIth century resides in a series of great philosophers-scientists (called at that time *natural philosophers*), who, without forgetting metaphysics, threw themselves determinedly to the pursuit of the knowledge of Nature and of mathematical invention: René Descartes studied the principles of reasoning, as well as mechanics and the universe; he tied geometry to algebra and wrote "The Discourse of the Method"[9]; Blaise Pascal wrote his "Pensées" but also investigated the principles of fluids (like pressure), geometry, calculus and probabilities. And so did Pierre de Fermat, Edmond

[6]The famous words are not usually printed in the original Italian: *"La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi è impossibile a intendere umanamente parola, senza questi è un aggirarsi vanamente per un oscuro labirinto.*

[7]He wrote down his ideas on Physics, Mathematics and Engineering in the book *Discourses and mathematical proofs concerning the two new sciences*, written in Florence before 1633 but only published abroad in 1638 after the problems with the Church. The two new sciences are mechanics and the science of motion. In 1995 the space probe *Galileo* reached Jupiter and with it the 4 planets discovered by him in 1610.

[8]The interests of Leonardo, a truly universal genius, cover painting and sculpture, engineering and architecture, Physics and Mathematics. Scientist and visionary, he drew the plans of a flying object (forerunner of the helicopter) and coined the term turbulence. Here is a relevant quotation from Leonardo: "No certainty exists where it is not possible to apply the mathematics or in what cannot be related to mathematics".

[9]*Le Discours de la Méthode*, Leiden, 1637, a capital work in the history of science. His work *Les Météores* is considered to be the first attempt to put the study of weather on a scientific basis.

Halley, Christiaan Huygens and Gottfried W. Leibniz, a most renowned mathematician, logician and philosopher.

We are ready to meet one of the crucial characters and moments in the history of science. Indeed, the century reaches its culmination with the figure of Isaac NEWTON (1642-1727), who shows the incontestable success of Galileo's proposal as applied to mechanics. He attacks the basic problems debated during the century and

ISAAC NEWTON

(i) concludes that the movement of solid bodies follows a simple mathematical law that relates the second derivative of space to an invisible *but real* entity, the force. In mathematical words, $\mathbf{F} = m\mathbf{a}$;

(ii) upon applying this theory to the heavenly bodies, he concludes that they move along their orbits in agreement with the law of universal attraction. In formulas, $F = Gmm'/r^2$.

In order to mathematically support the movements resulting from these laws he discovers what we know as infinitesimal calculus and solves differential equations. Moreover, the very formulation of his laws is not possible without the new concepts taken from Differential and Integral Calculus, that carries the names of Newton and Leibniz, and was invented by combining the intuitions of mechanics and geometry[10].

In 1687, when his monumental work, the *Principia*, is published[11], Mechanics is solidly founded upon the same bases it still has. Mathematics is not only an indispensable tool, *it is the language in which Science is conceived and expressed*, this is the reason of the book's title. From that moment on, the description of the dynamics and evolution of mechanical systems are an essential part of Mathematics. An enormous period of development follows during which Mathematics tries to fulfill this new fundamental role.

Newton is generally considered the most influential scientist in the history of mankind, cf. [36]. Let us provide some additional data in order to better understand the greatness of his legacy. If to his credit we may list the foundations of Mechanics and Astronomy, of Differential and Integral Calculus and Differential Equations, he also studied the nature of light, laid the foundations to Optics and contributed remarkable technical advances, like the refraction telescope. On top of this, he studied the fluids that are today called Newtonian, explained the operation of tides, computed the velocity of sound (and was also interested in Theology, Alchemy and Astrology,

[10] In placing Newton in proper perspective we have to combine his mathematical formation with the astronomical knowledge he inherited from Tycho Brahe, Johannes Kepler and Galileo.

[11] *Philosophiae Naturalis Principia Mathematica*, i.e., "Mathematical Principles of Science".

a quite common feature of the times)[12]. His prestige among his contemporaries was enormous and the most brilliant philosophers of the XVIIIth century (Hume, Kant, Voltaire[13]) studied his work and thought about expanding his fabulous success to all fields of philosophy, a task that turned out to be of a higher difficulty. Indeed, we are still busy with it.

The immensity of the task of understanding Nature did not escape a penetrating person like Newton, with all his success. One of his most celebrated opinions runs as follows: "I do not know what I will look like to others; to myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me".

3 The century of reason and lights

During the following three centuries, a part of that ocean has been filled with truth, science and mathematics. Science and Technology, the basis of the Industrial Revolution, have advanced with theories, reasoning and experiments. As a consequence, the society of the XXth century has changed more radically with respect to the XVIIth century than anything that had happened in several thousand years before, since the onset of the great agricultural civilizations. The comfort of house, transportation and communications, and the health of the present-day citizen rest upon technical bases completely unknown to the people of the XVIIth century.

Starting with G.W. Leibniz, a great philosopher and Newton's rival in the famous and a bit sad "dispute of the Calculus", a series of brilliant mathematicians (we would say physicist-mathematicians), like the Bernoulli family, Euler, D'Alembert,... exploited the potential of the new Calculus and formulated mathematically all types of mechanical problems: shooting problems, problems concerning the fall of bodies, the motion of fluids, mechanical vibrations, minimization,...

Infinitesimal methods are likewise powerful in their application to geometry, a discipline that lives in close symbiosis with mechanics. Scholars study the Calculus of Variations, a name for the calculus of minimum values of so-called "functionals", that will bloom in the XXth century as a fundamental topic of Functional Analysis,

[12]He was quite confident in his powers. Here is a quotation from Principia: "From the same principles, I now demonstrate the frame of the System of the World".

[13]It is worth remembering that the Principia were translated into French by the friend of the latter, the Marquise de Châtelet, with his collaboration, 1756. She is described in Encyclopaedia Britannica as "Gabrielle-Émilie Le Tonnelier de Breteuil, Marquise du Ch., French mathematician and physicist who was the mistress of Voltaire", and only in the text of the article her many accomplishments are described.

by then not even foreseen. Jean Le Rond D'Alembert[14] studied the vibration of a string and wrote the wave equation, that led him to decompose a function into a sum of elementary waves, a task also undertaken by Leonhard EULER (1707-1783) who carried out the decomposition into a possibly infinite sum of sinusoidal functions. Euler is perhaps the most prolific mathematician in history, he made fundamental contributions to Geometry, Analysis and Number Theory, but also to the different branches of Mechanics, Elasticity, Hydrodynamics, Acoustics, and even Music. His Latin is not difficult and his textbooks can be read today with profit and pleasure (preferably after translation!). He lived a great part of his life in St Peterburg, so he is credited with the foundation of Russian Mathematics, together with Daniel Bernoulli.

LEONHARD EULER

The problem of infinite sums will worry mathematicians in the near future, but not in these moments of discovery and euphoria, and even less L. Euler whose intuition seems to know no limits.

Some of the glories and griefs of Mathematics as the language of Mechanics can be observed in the study of fluids. A systematic theory escaped even the genius of Newton. Indeed, the most difficult aspect of this theory consisted precisely in finding the exact mathematical hypothesis that permit to build a mathematical model, i.e., to mathematize it *just as it really is* [15]. Toward the year 1738 Johann and Daniel Bernoulli establish the theoretical science of Hydrodynamics on the idealized basis of the so-called *perfect fluids.* The study is continued by Euler, who writes the famous equations (1755)

$$\rho\,(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = 0, \qquad \nabla\cdot\mathbf{u} = 0,$$

(in today's notation) whose analytical solution turns out to be intractable at the time[16]. Moreover, D' Alembert exposes the limitations of the idealization implicit in the concept of perfect fluid by showing that a solid obstacle submitted to a "perfect wind" would suffer no net *drag* and no net *lifting force.* Indeed, this happens because theoretical mechanics does not deal with Nature, that escapes in its pure essence our curiosity, but it rather deals with the mathematical model that we are able to form about it. Experimental agreement allows us to confirm that a theory is good as a

[14] a well-known representative of the French *Illustration*, who combined a brilliant mathematical career with the publication of the famous Encyclopedia, jointly with D. Diderot.

[15] we recall here Newton's saying about his mechanics: *hypotheses non fingo*, I do not invent the hypothesis or axioms.

[16] and they keep some of their mystery today: the existence of classical solutions given smooth initial data in 3 space dimensions is still an open problem.

model of the physical world, but never that it is perfect[17].

In spite of the relative failure with the fluids, a feeling of optimism invades the minds of the best mathematicians - mechanicians at the end of the XVIIIth century, like Joseph Louis Lagrange[18] or Pierre Simon LAPLACE. The latter publishes his monumental book "Mécanique céleste" (1788). He is also the author of the "Théorie Analytique des Probabilités", 1812, a most important reference in the development of probability theory[19]. Based on his mechanical studies he thought that the universe functions like a clock (determinism) and declared that the most important mathematical problems were already posed and solved, or about to be solved in a short time. Fortunately, History would prove the great man wrong on these issues. Does this bring to our minds recent heated debates about the end of Physics or History?

PIERRE S. LAPLACE

4 The XIXth century, the great century of Science

The contribution of the XIXth century to Mathematics, both pure and applied, is surprising by its novelty, by its richness and multiplicity of topics, and by its very unexpectedness. Let us begin our review with the Mathematics that came from Physics.

• ELECTRICITY AND MAGNETISM: From Michael Faraday to J.C. Maxwell, experiments and partial laws cover a road that counts the names of Gauss, Ampère, Biot, Savart, Lenz, ... till we arrive at the system of partial differential equations that relates the electric and magnetic fields (1863), the work of James Clerk MAXWELL[20] Maxwell's equations are one of the major achievements of Mathematics in the 19th century. Thanks to J.C. Maxwell the new branch of science, whose existence was unsuspected a century before, reached the level of mathematical perfection which Newton accorded to Mechanics. As a consequence, the wave equation is the tool that allows us to describe the propagation of electro-magnetic phenomena in the form of waves characterized by three parameters: first, the amplitude A;

[17] we will return to this subject when speaking of Einstein.

[18] Author of a *Mécanique analytique*, where the general equations of motion, Lagrange equations, are described

[19] Engineers and applied scientists are used to the Laplace Transform.

[20] publication in final form in *Treatise on Electricity and Magnetism*, 1873.

JAMES C. MAXWELL

second, the speed c that depends on the medium (and is therefore constant in the vacuum); third, the frequency ω of oscillation, that is a variable quantity. In short,

$$u_{tt} = c^2 u_{xx} \quad \Rightarrow \quad u = A\cos(kx - \omega t),$$

where $k = \omega/c$ is called the wave number. Do we need this formula to proceed? The answer is yes, since soon afterwards, and as reflection of the generality of the parameter ω in the mathematical model, Heinrich R. Hertz predicts and discovers electro-magnetic waves outside of the visible range (radio waves, 1888), and Guglielmo Marconi discovers wireless telegraphy, that is to say, the radio (1895), introducing us to the world of communications, which is the soul of the XXth century. On the other hand, an incompatibility appears with Newton's mechanics, about which we will speak in a moment. Let this be said about the consequences of the mathematical formulation on the evolution of science[21].

• THE REAL FLUIDS, from Claude Louis Navier to George Gabriel Stokes, 1821 to 1856 and later. The Navier-Stokes equations describe real fluids and they govern the behavior of atmospheric phenomena (climate, Meteorology, Hydrology, the future Aeronautics). The correct formulation of the equations describing the movement of real fluids took therefore some 180 years, after the attempts by Newton. A brilliant series of mathematicians figure among the modelers, like S. Poisson and J. C. Saint Venant, as well as the medical doctor J.L.M. Poiseuille, who investigated the blood flow. Lord Kelvin and H. Helmholtz set the bases for the mathematical study of vortices and turbulent fluids, already mentioned by Leonardo, but the full mathematical understanding of the latter is *still an open problem.*

In order not to extend our text excessively we will only mention two further physical theories of great mathematical significance:

• THERMODYNAMICS, which studies the exchange of heat, acquires solid mathematical foundations with James Joule, Saadi Carnot, J.R. Mayer, ... It has strong influence on the calculus with partial derivatives and the concept of exact differential. This theory includes the famous Second Law of Thermodynamics (law of entropy growth in the universe), a fundamental law in science. While its mathematical statement is simple, its practical interpretation has deep implications and puzzles

[21]Maxwell is considered the major theoretical physicist of the XIXth century, Einstein sustained that Maxwell's work represented the most significant revolution in the study of physics since Newton. The theory of wave propagation is one of the classical branches of applied mathematics nowadays in its multiple variants. An excellent mathematician, Maxwell was an advocate of the probabilistic approach to Science, which he applied to the study of gases, and is credited with saying that "the true Logic for this world is the Calculus of Probabilities"

generation after generation of scientists[22].

• Finally, let us mention STATISTICAL MECHANICS, associated to the names of L. Boltzmann and W. Gibbs[23], who carved a branch of Mathematical Physics on the basis of the calculus of probabilities, a discipline that had remained very much at the margin of this scientific adventure[24]. Indeed, the mathematical idealization of chance had been elaborated in the fabulous XVIIth century (ca. 1650) by B. Pascal, P. Fermat and C. Huygens to understand games of chance, and advanced later by Buffon, Bernoulli, De Moivre and Laplace among others. Suddenly, the concept of probability acquires a life of its own in Physics when attempting to model the behavior of huge quantities of particles[25]. This is why the need arises: particles obey of course Newton's mechanical law, but given that Avogadro's number[26] is so huge, approx. 6×10^{23}, it is absolutely impossible to follow individual particle trajectories. Statistical mechanics proposes an average behavior with surprising effectiveness: the prediction of the ideal relationship between temperature, energy and pressure for a perfect gas is immediate and turns out to be quite accurate!

BERNHARD RIEMANN

We change the scene to portray another of our heroes, an "exemplary life", Bernhard RIEMANN (1826-1866), one of those surprising figures whose work contains the best of pure and applied mathematics. The great German mathematician, who died quite young, is well known as a giant of pure mathematics. He bequeathed to us the hypothesis about the zeros of the "Zeta function" (*Riemann's Hypothesis*) whose proof is considered to be the most famous open problem of Mathematics upon entering the XXI century, after the recent solution of Fermat's conjecture. The Riemann hypothesis asserts that all interesting solutions of the equation $\zeta(s) = 0$ lie on a straight line in the complex plane, precisely at $Re(s) = 1/2$. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every integer solution would shed light on many mysteries, from the distribution of prime numbers to theoretical Physics. Riemann was a scholar with a geometrical mind who thought of complex

[22]with unsuspected consequences: entropy is nowadays a central concept in Information Theory after the work of C. Shannon, *The mathematical theory of communication*, Bell. Syst. Techn. Journal **27**, pp. 379-423, 623-658 (1948).

[23]not to forget Maxwell, cf. the Maxwell-Boltzmann distribution.

[24]Boltzmann's tomb in Vienna has as sole ornament the entropy formula of statistical mechanics $S = k \log W$.

[25]This was not a trivial step. Boltzmann relied on his belief in atoms, a view strongly opposed at the time by famous scientists like E. Mach. The bitter controversy seriously affected his health.

[26]that measures the number of molecules of a gas per unit volume (22.4 l) under normal temperature and pressure conditions.

analysis in terms of conformal transformations and had the vision of general spaces of several dimensions defined in terms of their local geometry[27]. Today we call them *Riemannian geometries* and they are the foundation upon which theoretical physics is built. Now, the same Riemann studied the propagation of compressible gases and arrived at the conclusion that the mathematical model[28], understood in the sense of classical solutions, is contradictory (because it predicts characteristic lines that intersect each other, so that on them the physical variables - density, pressure and speed - would take on several values simultaneously). However, he ventured that the theory was correct if *the point of view were radically changed*; as solutions of the differential equations we must admit functions that are not differentiable, not even continuous. Such boldness, so typical of the best Mathematics of the XIXth and XXth centuries, reminds us again of Newton: Riemann was not "inventing" a theory. The theory of *shock waves* is today a fundamental topic in gas dynamics with its application to Aeronautics, and is therefore one of the most active areas of mathematical research in partial differential equations, ... and engineering.

Inner Evolution. But, even after mentioning Riemann, the present vision would be totally inaccurate if it did not take more explicitly into account the internal evolution of Mathematics, that had by then attained a high level of maturity. We will comment only briefly on this issue since it is better known by the mathematical public. The following are some of the star topics. Many of them appeared unexpectedly, but they were meant to have a brilliant future. Let us mention non-Euclidean geometries by J.C.F. Gauss[29], J. Bolyai and N.I. Lobachevski, the rigorous foundation of Infinitesimal Calculus by Augustin L. Cauchy, the theory of functions by Karl Weierstrass, mathematical logic by George Boole and followers, set theory by Georg Cantor, where we mention only a relevant name next to each chapter.

There are research fields in which Mathematics clearly takes the relay from Physics in the task of extracting the substance contained in a concept. This happens with the problem of representing a function as a sum of simple functions, solved by Brook Taylor and Colin McLaurin for sums of powers and posed by Daniel Bernoulli (1753) and Leonhard Euler for trigonometric sums as they appear in the wave and heat equations. Thanks to the insistence of Joseph Fourier (1822)[30] mathematicians enlisted in the adventure of giving a clear rigorous sense to general infinite sums of trigonometric

[27]his famous article *On the hypotheses which lie at the foundations of Geometry*, in German *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, 1854, published in 1868.

[28]a nonlinear system of partial differential equations of hyperbolic type.

[29]the "Prince of Mathematicians".

[30]article of 1807, memory presented to the Paris Academy of Sciences and published in 1822.

functions,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\omega x) + b_n \sin(\omega x)\}.$$

This is the origin of a major area of the theory of functions, known as Fourier Analysis. The task was fraught with baffling difficulties and great successes. Thus, when Paul du Bois Raymond constructed (1873) a continuous and periodic real function whose Fourier series does not converge at all points it seemed that something was quite wrong with the mathematics of wave analysis. On close inspection three options lay open to the researcher: (i) modify the notion of function, (ii) modify the definition of convergence, (iii) replace the basis of sine and cosine functions by better-suited candidates. It is to the credit of mathematicians that *all three courses* have been pursued with amazing success. The fundamental theorem about summation of Fourier series is due to Lennart Carleson, 1966[31], and needs *almost everywhere convergence*, L^2 spaces and the impressive analysis machinery developed in the XXth century[32].

Social Context. It may be interesting to say some words on the social evolution of Science in the XIX century. This is the century in which the bourgeois, industrial and democratic revolutions take root in Europe, bringing along the extension of scientific and industry-related studies in universities and in other specialized centers centers, like the technical schools. That development enlarged the body of professors and researchers at an exponential rate. Progress was so impressive that at the end of the century we find again a frank optimism in the mathematical opinion, if we for instance let ourselves be led by the history written by the German geometer Felix Klein[33]. Another characteristic of this period is the deep separation taking place between mathematicians and physicists and engineers, a consequence of the enormous growth of their respective fields of study. Such a separation will have serious consequences on the evolution of Mathematics in the XXth century, and even on the very concept of Mathematics.

[31] *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), pp. 135–157.

[32] Here are two quotations from Fourier that will help kindle the debate on Pure versus Applied Mathematics: The first is "The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory". Now the second one: "The profound study of nature is the most fertile source of mathematical discoveries".

[33] *Lectures on the development of mathematics in the 19th century.* Here is a significant quote from Klein: "The great mathematicians like Archimedes, Newton or Gauss always united theory and applications in equal measure".

5 An agitated turn of the century

In any case, the turn of the century is spectacular in Physics as in Mathematics. Two extraordinary figures appear in the mathematical arena, Henri POINCARÉ (1854-1912) and David HILBERT (1862-1943). They make a deep imprint in the Mathematics of the XXth century. But a great part of the retrospective brilliance is due to the fact that the turn of century was *a time of crisis*, since the evidence of phenomena that did not fit into the "great explanation" at hand kept mounting.

HENRI POINCARÉ

DAVID HILBERT

• The experiment of Michelson-Morley (1887) showed that the speed of light is really constant, as predicted by the wave theory based on Maxwell's equations. The mechanical model of the world of Euclides-Newton sees a first huge crack.

• The movement of particles suspended in gases reveals a highly irregular movement, the Brownian movement (Robert Brown, 1827). This is a blow for Euclides' geometry based on points, straight lines and smooth curves (or at least piece-wise smooth).

• The surprises of the theory of functions lead to the Theory of Sets (Georg Cantor) that together with Logic (George Boole, Gottlieb Frege, Giuseppe Peano) form the basis in the attempt to provide rigorous foundations to Mathematics once for all. Mathematics proposes to Science the concepts of *consistent* and *complete* theory. Disputes and different schools arises: logicism (Alfred N. Whitehead and Bertrand Russell[34]), intuitionism (Luitzen Brouwer), formalism (D. Hilbert). Then paradoxes appeared (Russell, Burali-Forti, Richard) and that sowed a notable chaos in weak and not so weak spirits.

• No efficient analytical or computational tools are available to tackle the complexities of the equations governing continuous media, like fluids. Consequently, the practical Mathematics of engineering plunges into a series of approximations and rules

[34]their famous book *Principia Mathematatica* dates from 1910.

that divorce them from the theory.

• Even the classical questions of the general integration of the equations of movement for three or more (heavenly) bodies turns out to be impossible[35]. Big problems, big remedies: H. Poincaré proposes the qualitative methods and opens the doors to algebraic geometry and topology (called then Analysis Situs, 1895). But, at the time he discovers with his theoretical methods the tremendous complexity hidden in the mathematical model (i.e., the dynamical systems). The hidden monsters are called homoclinical orbits and they will infest with *chaos* the whole body of celestial mechanics when Poincaré is finally well understood (this took several decades)[36].

• Let us add some optimistic notes. Thus, the theory of integration of functions is crowned in the works of E. Borel and H. Lebesgue. Now Calculus possesses a concept of integral where the process of taking limits is natural. Functional Analysis is born (Hilbert spaces) and the famous Dirichlet Problem has a solution (in a sense seen then as quite unusual). The price to pay is the construction of a sophisticated mathematical theory that students of science and engineering must absorb, or at least learn to live together with, paraphrasing J. von Neumann.

• Main discoveries of a mathematical nature occur in other sciences and will bear fruit in the next century. The Russian scientist Dmitri I. Mendeleev found order in the chaos of chemical elements and proposed the Periodic Table in 1869, the basis of today's physico-mathematical treatment of Chemistry. On the other hand, the Austrian monk, botanist and plant experimenter Gregor J. Mendel formulated the rational laws of inheritance, thus laying the mathematical foundation of the science of Genetics[37].

6 The XXth century, a century of wonders

At this height, we expect to have impressed upon the reader a feeling of the deep symbiosis of Mathematics with Physics, of their surprising and in many cases unexpected interactions. By this time this symbiosis includes advanced technological applications, a prelude of what the new century will be. The explosion of Mathematics and Science in the XXth century makes it advisable to reduce our text to some of the most important items. A main feature that stands out is the progressive math-

[35] as exposed by H. Poincaré in his book *Méthodes nouvelles of the mécanique céleste,* Paris, 1899.

[36] In order to measure the stature of our hero the following quotation could be useful: "in his courses at the Faculté des Sciences de Paris since 1881, and later at the Sorbonne since 1886, Poincaré changed subject every years, touching upon Optics, Electricity, Astronomy, the equilibrium of fluids, Thermodynamics, Light and Probability".

[37] *Versuche über Pflanzenhybriden* (Experiments with Plant Hybrids), published 1886.

ematization of other sciences, which makes them appear as new horizons for Applied Mathematics.

New Mathematics that came from Physics

• THE THEORY OF RELATIVITY. Albert EINSTEIN, the Man of the Century according to *Time magazine* (year 2000), proposed the two versions of relativity in 1905[38] (special relativity) and in 1916 (general relativity). It will be small surprise to the reader if we say that in both cases it is a matter of an in-depth reflection upon the Mathematics that lie at the basis to Physics. Special relativity has as precursors Lorentz, Poincaré and Minkowski, who studied the invariance group that corresponds to the new geometry of space-time. General relativity uses the geometrical concepts that Riemann elaborated more than a century earlier as a pure *Gedankenexperiment*, i.e., a thought exercise upon the "hypotheses which lie at the foundations of Geometry", and that were developed by the Italian differential geometry school of Ricci, Levi-Civita and Bianchi. Relativity was destined to be a great ball-game for differential geometry in the XXth century. We go from Einstein's equations to the Big Bang and to black holes (Oppenheimer and Snyder, 1939; Penrose and Hawking). All can be seen as a piece of pure mathematics building a model for a branch of Physics. It is befitting however not to forget the other face of Relativity: since the first experimental confirmation by Sir Arthur Eddington in 1919, an incessant number of experiments have served to confirm (or rather, with Einstein's modesty, not to refute) the theory of Relativity. Indeed, hypotheses are not invented in real science[39].

ALBERT EINSTEIN

Let us pause to take a look at some of the main formulas. In September 1905 Einstein published a short paper in which he proved the fundamental formula $E = m\,c^2$ about the mathematical equivalence of mass and energy, which has become a classic in the popular culture of the XXth century. On the other hand, the transformation laws of Special Relativity that replace the Galilean transformation

[38] 1905 was the *annus mirabilis* for Einstein. In three separate papers he explained the photoelectric effect, Brownian motion and the theory of relativity. It is unlikely that such a feat will be repeated.

[39] Here is a significant opinion of Einstein on the role of mathematics: "Mathematics deals exclusively with the relation of concepts to each other without consideration of their relation to experience. Physics too deals with mathematical concepts; however, these concepts attain physical content only by the clear determination of their relation to the objects of experience", in *The theory of Relativity*, 1950. Einstein's opinions are all the more interesting since, contrary to other outstanding figures in the history of Physics, like Newton or Maxwell, he was not himself an outstanding mathematician, at least technically. He left however an impressive legacy to Mathematics through his theories.

laws at high relative velocities, known as Lorentz transformation laws, are:

$$x = \gamma x' + \gamma v t', \qquad t = \gamma t' + \frac{v}{c^2}\gamma x',$$

where the constant γ is called the time dilation factor. It depends on the relative velocity v and is given by the expression: $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Consequently, the addition of velocities follows the surprising rule

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}},$$

very much against what we were used to believe (i.e., $u = u'+v$). All in all , Einstein's most recognized formula is of course $E = m\,c^2$, which forms with Planck's quantum formula $E = h\,\nu$ the new vision of energy at the beginning of the century. Precisely, quanta are our next subject.

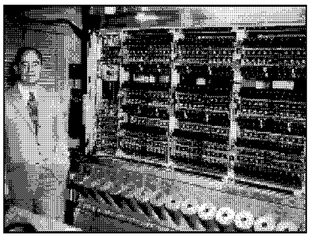
• Quantum Mechanics. The second magical tour[40] takes us from Max Planck's Hypothesis of the Quanta, 1900, to the Schrödinger Equation (Erwin Schr., 1926) passing by Niels Bohr, Louis de Broglie, Max Born, Werner Heisenberg and Paul Dirac. The door to the atomic world is coded in the marvelous equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\,\psi,$$

where $\hbar$ is the reduced Planck constant, $\hbar = h/2\pi$, $i = \sqrt{-1}$, Δ is the Laplacian operator and $V = V(x,y,z,t)$ is the potential. All this may really seem like a piece of Kabbala, and at first the experts discussed heatedly about the meaning to be given to the variable $\psi(x,y,z,t)$ called "wave function". Such is the power of Mathematics, these great physicists had found a piece of the Mathematical Code of the Universe but did not know how to interpret the cipher. In 1928 the probabilistic interpretation was proposed by Max Born, where $|\psi|^2$ is the probability density of finding a particle at the location (x,y,z) at the instant t, and this is widely accepted, not without resistance, following Einstein in that[41]. Because Quantum Mechanics is a fundamental challenge to the previously admitted way of looking at the world, to traditional determinism and causality. We may say that Determinism is based on the assumption that "the exact knowledge of the present allows the future to be calculated". Is it not that the dream of the exact sciences, and does not Quantum Mechanics subvert that belief? Pondering on the issue, W. Heisenberg found in 1927 the following answer: "not the conclusion [of the deterministic assumption], but the initial hypothesis is false".

[40] quotation in homage to "The Magical Mystery Tour", Lennon and McCartney, 1967.
[41] his famous comment: "God does not play dice".

Leaving the world of interpretations aside, we must report that this theory, based on the highest level of mathematical abstraction, will be confirmed by a century of experiments. Its magical part has a stellar moment when Paul A.M. Dirac, using the relativist formulation, proposes the existence of a particle, called today positron (1932), because "the equations admit the sign change with respect to the solution describing the electron",... and the positron was duly discovered[42] by experimental physicists shortly afterwards (Anderson and Blacket, 1932-33). Dirac predicted the existence of the antiproton that was confirmed by Segrè in 1955, and also of the magnetic monopole, but this time existence went without confirmation up to the present day. Dirac's predictions are a remarkable example, in no way unique, where mathematical modeling goes ahead of the experimental evidence[43]. Does this remind us of Hertz?



J. V. NEUMANN

The mathematical harvest is not scarce: the theory of self-adjoint operators in Hilbert spaces with the corresponding spectral theory were developed by John VON NEUMANN (Janos v.N., 1903-1957), one of most versatile geniuses of the century[44], with the purpose of giving sense to the operators that appear in the Schrödinger equation, Laplacians and the rest. He is based on the work of S. Banach and the Italian experts in the Calculus of Variations, but Quantum Mechanics has its whims: it needs some sophisticated mathematical objects, so-called "unbounded linear operators in Hilbert spaces". We are therefore at the edge or beyond the syllabus of undergraduate Mathematics. This is interesting information for those who claim *that all useful mathematics is necessarily easy*[45]. Together with the Calculus of Variations, Quan-

[42]should we said found? or recognized?

[43]On the other hand, science based solely on mathematical arguments or analogies can be wrong science. Thus, there is strong mathematical tendency to assert that in the realm of particles certain mathematical symmetries are "laws" of nature. A telling counterexample is provided by the law of conservation of parity that specifies that elementary particles and their mirror images *must* behave identically; in 1956-57 three sino-americans T. D. Lee, C. H. Yang and C. S. Wu first conjectured and then proved that there are subatomic processes that violate that law.

[44]J. von Neumann, *Mathematische Grundlage der Quantenmechanik,* "Mathematical Foundations of Quantum Mechanics", Springer, 1932. Von Neumann's trajectory travels through the most diverse areas of Mathematics, pure and applied: in his youth he modified the ZF set theory, he creates the v.N. algebras in operator theory, he is the father of Game Theory ("Theory of games and economic behaviour", J. von Neumann and O. Morgenstern, 1944) and we will see him later at the Institute for Advanced Studies in Princeton as one of the fathers of the first modern computer. After the war he was busy with hydrodynamics, numerical methods (Monte Carlo, stability for finite difference schemes), the theory of automata, and so on.

[45]I refer specifically to the opinions of the famous English mathematician G.H. Hardy in his book *A Mathematician's apology,* [15], that reflects very different points of view from the ones maintained

tum Mechanics has been a continuous source of problems for Functional Analysis, a branch of Mathematics that takes on its own flight.

Mathematics that came from Engineering

• AERONAUTICS. After the impressive advances of Mathematical Physics in the XIXth century, and in particular of fluid mechanics, it could seem that the old problem of flight, that had already occupied Leonardo da Vinci, had to be solved for good. And the experiments with balloons had been conducted with success a century before[46]. Moreover, the theory of complex variables and of potential and vortex flows had obtained remarkable progress. But with all this progress, real *propelled flight* was not understood nor practiced, and a discouraged W. Thomson Lord Kelvin recognized towards the end of the century that the dream of propelled flight was maybe impossible[47]. Then, and after a number of partial successes in different countries, the experimental method was vindicated by the brothers Wilbur and Orville Wright, manufacturers of bicycles and accomplished experimenters with no academic training. They were able to fly a propelled artifact in the inhospitable beaches of Kitty Hawk, North Carolina, in the cold morning of December 17, 1903. An Engineering discipline is born, Aeronautics.

The reaction of the scientific community was immediate and up to the challenge. During the period 1905-10 the main mathematical ingredients missing in the theoretical model were understood (L. Prandtl, M. Kutta, N. E. Zhukovski, S.A. Chaplygin). They deal with the concepts of sustentation, circulation, boundary layer, separation, laminar and turbulent regime. In 30 years the new scientific discipline carries us beyond the sound barrier. And with this discipline new branches of applied mathematics see the light, such as the theory of singular perturbations, the theory of supersonic and transonic flows and the mathematical theory of combustion[48].

We restrain here from listing other branches of Engineering that have had a similarly active interaction with Mathematics, but see Section 8.

Great news coming from Mathematics

Mathematics have lived throughout the XX century quite focused on the internal development of the ideas received from the fabulous previous century. Fortunately,

in this article, cf. specially his section 26. It is a well-known book, of great interest, but time does not seem to have proven the author right. It is to be considered that in 1940 the practical relevance of sophisticated theories like Quantum Mechanics could very well not be clear, as it is today.

[46]Brothers Montgolfier, 1783.

[47]"heavier-than-air flying machines are impossible", he said in 1895.

[48]More toward theoretical mathematics we have the mathematical theories of front propagation and that of singularity formation, like blow-up for nonlinear differential equations. Let us add that, though the engineering practice of aeronautics rests on firm theoretical foundations, the deep mathematics involved are far from being well understood and research is quite active

the always difficult and generally failed attempt to predict the main lines of the future has had an exceptional counterexample in the famous proposal by D. Hilbert at the II International Congress of Mathematicians, celebrated in Paris in 1900. Hilbert summarized in 23 problems the main challenges faced by Mathematics, going from the most theoretical aspects of pure mathematics to the problems of mathematical physics[49], cf. reference [17]. Those 23 problems have been of great importance in the course of the century, but other lines have come to compete for the limelight, and how! Let us point out three important developments among many others.

• THE CALCULUS OF PROBABILITIES. It may look like an answer to the needs presented by Quantum Mechanics, but in reality it happened independently. In the 30's Andrei N. Kolmogorov put in Moscow the foundations of axiomatic probability[50] upon set theory, and abstract measure theory is born. The names of P. Levy in France and N. Wiener in the USA are usually associated with this discovery. We should not forget the precedents: Boltzmann studied Brownian motion, L. Bachelier wrote his thesis in 1900 in an (unsuccessful at the time) attempt to model financial markets, and Einstein obtained the Nobel Prize in 1921, not for the theory which made him famous, but for his studies on the photo-electrical effect and... on Brownian motion. Markov chains had been studied since 1900 by A.A. Markov.

Nowadays, the theory of Stochastic Processes is a main area of this booming branch of Mathematics, and the Itô Calculus is an essential tool of continuous stochastical analysis to be compared to the classical infinitesimal calculus of Newton and Leibnitz. All this development was completely unknown, even unsuspected, to older ages and it takes upon itself the task of informing us about uncertain and random events and their probable outcome or evolution. As usual in our narrative, it is not just an academic pursuit, it has very important applications in scientific, industrial and financial processes.

• DETERMINISTIC CHAOS. The study of chaos generated by differential equations, already announced by Poincaré, whose Mathematics had matured thanks to the efforts of different mathematicians, especially G. Birkhoff, had to wait for the work of a physicist devoted to the study of weather to acquire a dramatic impulse. In effect, this merit is attributed to Edward Lorenz, from MIT [51]. Interested in the study of convective processes in the atmosphere, he proposed a very simplified model consisting of three ordinary differential equations and I will not resist the temptation

[49]Though it must be said that the latter were relatively under-represented, and Hilbert worked on the subject in subsequent years.

[50]His book *Grundbegriffe der Wahrscheinlichkeitsrechnung*, "Foundations of the Calculus of Probabilities", was published in 1933.

[51]his famous publication is *Deterministic non-periodic flow*, J. Atmos. Sci **20** (1963), 130–141.

of reproducing it for you

$$\begin{cases} x' = -10x + 10y, \\ y' = 28x - y + xz, \\ z' = \frac{8}{3}z + xy. \end{cases}$$

For this particular choice of parameters he found to his surprise that the numerical trajectories produced by the computer did not converge to a fixed point or a periodic solution. The 12 page paper dates from 1963. Deterministic chaos was born, along with strange attractors and a whole branch of Mathematics, at the beginning quite experimental, then theoretical, a great novelty made possible by the advent of the computer. Authors like S. Smale, D. Ruelle and M. Feigenbaum become world-famous[52]. Objects like the *fractal sets* of B. Mandelbrot[53], already announced in the work of G. Julia in the 1920's, enter the scene. The study of fractal, chaotic and turbulent processes is one of the border-lines of present mathematical thought, the relation of deterministic chaos to natural chaotic and turbulent phenomena still being largely unknown.

• New concepts of solution in differential equations. Toward the 1930's it was clear for many researchers that the concept of classical solution was not sufficient to build a theory of differential equations for use in mathematical physics which would satisfy the requirements of the applied science. In effect, it is natural in this discipline to work with *problems*, i.e., with sets of equations and additional data, and to require them to be *well posed*. Following J. Hadamard, this means that such problems should have a solution, that this one has be to unique if sufficient data are given, and finally that the solution should depend continuously on the data. Now, it may happen in the real science that classical solutions do not to exist and this fact can even be proved in a rigorous way, and even then the problem could be reasonable from the physical point of view. Or it may simply happen that the concept of solution whose existence turns out to be natural and simple to show is not the classical concept.

Faced with this challenge mathematicians have developed a diverse set of notions of *generalized solutions* with physical meaning. A remarkable example arises in Dirichlet's problem of energy minimization already mentioned[54]. Another basic example arises with Riemann's problem of gas dynamics. Yet another similar prob-

[52]cf. Ian Stewart, *Does God play dice? The New Mathematics of Chaos*, Penguin, London, 1989.

[53]cf. B. Mandelbrot, *The fractal geometry of Nature,* 2nd ed., San Francisco, 1982.

[54]It deals with minimizing the energy integral $\int_\Omega |\nabla u|^2\,dx$ among all the admissible functions $u = u(x)$ defined in a domain of the space, Ω, and which take assigned values on the border of Ω; ∇u denotes the gradient of u. The crucial question in order to envisage the correct solution, is to decide what is understood under the label *admissible* function. The answer motivates Hilbert spaces.

lem is tackled by J. Leray (1933)[55] in the study of the solutions of the Navier-Stokes equations for real (viscous) fluid in tri-dimensional space. Thanks to the work of functional analysts (S.L. Sobolev, L. Schwartz, ...) the concepts of *weak solution* and *solution in the sense of distributions* are developed to suit those needs. Summarizing a great deal, the main idea is not to ask the solutions to possess all the derivatives implicit in the equation, but rather to comply with a family of tests. With the experts in conservation laws (P. Lax, O. A.Oleinik, S.N. Kruzhkov) we arrive at the concept of *entropy solutions*, needed for gas dynamics where weak solutions are insufficient. Entropy solutions of gas dynamics equations "solve" the differential equations but may not even be continuous (and thus we recover the legacy of Riemann, Rankine and Hugoniot and their shock waves).

In our days new concepts of solution appear to suit new needs, such as the *viscosity solutions* of M.G. Crandall, L.C. Evans and P.L. Lions. L. Caffarelli extends the concept to the problems of phase transition or free boundary, in which the discontinuity is a fundamental part of the mathematical setting. And the saga continues with so-called mild solutions, semigroup solutions, renormalized solutions,...

One of the most striking aspects of these new concepts is their compatibility with the *numerical solutions* produced by the discrete methods of numerical calculus. We find thereby a surprising alliance of the abstract and the numerical concepts against "the unflexibility of the classical concepts".

7 Engineering and Mathematics in the last revolution of the century. Computers and computational mathematics

The practical realization of the old dream of building a calculating machine takes shape in form of the modern computer that originates from two sources, Technology and Mathematics. Both combine towards a fabulous invention in the year 1946[56]. From one side, we have the old project of the calculating machine, already thought of by B. Pascal[57] and G. Leibniz in the XVIIth century[58], which owes so much to Ch. Babbage at the beginning of the XIXth century, and finally is to be realized in the XXth century in an efficient form thanks to the progress of electronics: first, the

[55]Jean Leray published three papers on the subject in 1933-34. The last is *Essai sur les mouvements plans d'un liquide visqueux emplisssant l'espace*, Acta Math, **63**, 1934.

[56]with this date I refer to the ENIAC computer.

[57]his *machine à calculer*, the *Pascaline*, is famous.

[58]Leibniz thought in the direction of algebra and symbolic logic. Recent investigations indicate that the first of such calculating machines is due to a German, Schickard, 1623.

vacuum tube and then a line of impressive technical progress that leads to semiconductors, miniaturization and the *chip*[59].

But the computer is not born as a passive calculating machine, it is born with a program. This is the legacy of mathematical logic, from G. Boole with his algebra to the program of formalization of Mathematics by D. Hilbert, that leads to Kurt Gödel's incompleteness proof in 1931[60], one of the absolute Mathematical Hits in the XXth century. Which in turn provokes the interest of a mathematical genius,

A. TURING

Alan TURING (1912-1954), who translates the program of formalization to the language of machines (*On Computable Numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, 1937), and invents, together with Alonzo Church, Computability theory. All this happened years before a physical computer was to see the light. There follows a historical moment: the war effort, deciphering the German code Enigma,... Enters von Neumann with the idea of the stored program and ENIAC is built in 1946[61]. The modern computer appears as an effective calculating machine with four characteristics: general purpose, electronic, digital and programmable; *the two latter are directly related to mathematics.* The first commercial computer, UNIVAC, was released in 1951. In the short period of a bit more than 50 years we have seen the evolution from huge machines, that could handle kilobytes to megabytes, to the personal computer with capacity of several gigas and to the World Wide Web. Duality in the computer world continues in the form of the famous couple Hardware and Software[62].

THE COMPUTATIONAL WORLD, A NEW WORLD FOR MATHEMATICS. The computer world is changing little by little the daily life of the citizen: banking transactions, electronic mail, ticket reservations,... Its effect upon Mathematics, less known by the general public, is even more dramatic. On the one hand, new branches appear like theoretical Computational Mathematics, or the theory of automata and formal lan-

[59]the integrated circuit was invented by R. Noyce and J. Kilby in 1958.

[60]the incompleteness of formal systems, was published in *Ueber formal unentscheidbare Saetze der Principia Mathematica und verwandter Systeme*, "On formally undecidable propositions of Principia Mathematica and other related systems".

[61]ENIAC stands for Electronic Numerical Integrator and Computer, built by J.W. Mauchly and J.P. Eckert at the Univ. of Pennsylvania; today the pioneering work of J.V. Atanasoff is recognized. Mention should be made of the English Colossus, 1942, and the German Z1 to Z4 machines, cf. ref. [24]. All of these machines had a military purpose.

[62]Personal computers appear in 1977 and, against the predictions of the gurus, have taken up the scene, thanks no doubt to the impressive progress of hardware: a chip may contain at the end of the century up to 10^9 transistors

guages. But all branches of Mathematics, pure and applied, are affected by the sudden ability to actually calculate what before could only be imagined, and this works like an infection on the everyday practice of mathematics: mathematicians, scientists and engineers calculate orbits of satellites or trajectories of dynamical systems, numerical distributions or time series of real processes, weather maps or mathematical studies of singularities, temperature distributions in a furnace or statistical properties of the zeros of Riemann's Zeta function, ...

Among the most remarkable novelties, Mathematics has an important role in industrial and other applied processes in which laboratory experiments are combined with the new tools derived from Mathematics: there appears the combination of **mathematical modeling - mathematical and numerical analysis - simulation - visualization - control,** that forms a usual tool in the most diverse fields: communications, weather prediction, astrophysics, mining, industrial engineering, the car industry, the oil industry, environmental problems, economy and finance, communications, and quite recently biology and medicine, as we will see with some detail in Section 8. This area of mathematics has the task of approximating in an effective way the solutions of mathematically sophisticated models. Interest in its development and application gives rise to big institutes and computation centers all over the world. New disciplines arise, like CFD, i.e., Computational Fluid Dynamics, or CB, Computational Biology.

The new concepts: numerical model, computer simulation, numerical experiment or exploration, dynamical visualization,... have become daily practice in scientific and industrial media. The development of methods of numerical formulation of the continuous models of physics, like differential and integral equations, is a fundamental branch of computational mathematics (viz, the methods of finite differences, finite elements[63], finite volumes,...). The study of the properties and convergence of these methods constitutes Numerical Analysis, that has a deep connection to Algebra. On the other hand, the computation capacity gives new life to the branches of discrete mathematics, as graph theory, with its important applications (for example, to the telephone networks and in general to the world of communications).

In summary, a view has emerged where **Computational Science** is now the third leg of the scientific method together with Theory and Experiment, and this view is nowadays strongly practiced in Physics, Chemistry and Engineering.

[63]Finite elements are a wonderful example of the development of a mathematical-numerical tool by the parallel but separate efforts of mathematicians and engineers, see an interesting historical account in [2]. The phenomenon is not isolated, cf. the recent history of wavelets. These examples should lead us to think a bit more about the benefits of communication.

8 Trends at the beginning of the XXIth century. Mathematics in the Sciences, Industry, Management and Business

We have seen the recent evolution of pure and applied mathematics towards theoretical consistency and universality of interests. In consonance with this, the panorama of current interests and future trends in the world of Mathematics offers an impressive variety. Using a somewhat rhetorical language, we may say that Mathematics is today *ubiquitous*, it is everywhere, and *relevant,* it matters. Mathematical modeling plays a bigger role than ever in science, engineering, business and the social sciences.

We will mention next some of the main applied topics as they appear in the literature, in conferences, in programs of major research institutes. We have also benefitted from different sources like [11, 12, 13, 26, 31] and others. We point out in italics related mathematical aspects for the reader's convenience.

• Celestial Mechanics. Problems of aerospatial science. *Stability and chaos in dynamical systems. Strange attractors.* Mechanics of solids and fluids in zero gravity.

• Theory of fluids. Application to meteorology and climatology. Ocean engineering. Complex Environmental Problems, global warming and other geo-social issues. *Global circulation models, balance models; stochastic climate modeling; hierarchies of intermediate complexity models, like the geostrophic model.* Glaciology. Acoustics and application to the sound industry. Industrial fluids, lubrication. Turbulence. *Predictability and chaos. Stability, bifurcation. Free boundary problems.* Cross-areas, like fluid-structure interaction.

• Aeronautics. Hydrodynamical problems, supersonic and transonic flight. Airfoil design. Problems of combustion (flame propagation, detonation). *Shock waves and hyperbolic equations. Boundary Layers and asymptotic developments. Traveling Waves.*

• Modern Physics. The Mathematics of the atomic world and of elementary particles. The standard model, quantum electro-dynamics, quantum chromo-dynamics. *Group theory, renormalization and gauge theories, supersymmetry, Yang-Mills equations, instantons, dilatons, branes,... exotic geometries and topologies in higher dimensions.*

• Astrophysics. General relativity, stellar models. Mathematics of plasma physics, magnetohydrodynamics. *Kinetic equations (Boltzmann, Landau, Fokker-Planck, Vlasov, ...).*

• Geosciences. Problems of resources and mining. Environmental Problems: climate research. Pollution transport in air, water, and soil. Computational hydrology.

The equations of oil extraction, of groundwater filtration, of contaminant dispersal: nonlinear systems of PDEs and free boundary problems. Mathematics of seismic phenomena, *wave propagation, inverse problems.*

• Materials Science. Modeling and simulation of composite materials, magnetic material, polymers, glass, and paper. Crack propagation and further failure mechanisms. *Linear and nonlinear Elasticity. Calculus of variations. Homogenization theory.* Phase transitions, crystal growth, superconductivity and hysteresis.

• Nanotechnology. Integrated optics, optical networks. Quantum electronics and optics. Nanoscale techniques in medicine, porous materials. *Coupling of quantum states, mesoscopic and continuous models. Semiclassical Boltzmann theory, Wigner equation.*

• Industrial Engineering. Steel industry, blast furnaces. Prototypes for the car industry (fluids, aerodynamics, materials and fracture theory).

• Communications. Telecommunication and optical networks: analysis, simulation, optimization, transmission rate optimization, network design. Antennae, radar and sonar. *Electromagnetic field theory.* Microwave ovens couple Maxwell equations with Fourier heat theory.

• Discrete Mathematics. *Graph theory, combinatorics.* Applications to management, scheduling, routing,...

• Computer Science. *Mathematical logic, algorithmics, computational complexity, parallelization.* Finite automata, formal languages, *algebra.* Machine learning, data mining, artificial intelligence, natural language processing.

The design of the quantum computer would open a new world to computation.

• Control. Optimal control, robust control, nonlinear control. Predictive control. Fuzzy control systems. Neural networks, Fault Detection and Diagnosis in Industrial Processes. Modeling and Control of Economic Systems. Constraint Based Scheduling. Communication and Control of Distributed Hybrid Systems.

• Automation and Robotics. *Algebraic geometry and computation.* Computer Vision and Virtual Reality. Biological and Computational Learning.

• Information theory. Coding of messages, error-correcting codes. Surprising applications of *number theory and algebra.* Image Processing and Compression. *Wavelets, fractals, nonlinear PDE theories.* Speech and image recognition.

• Statistics in Science, Industry, Government and Business. Estimation and hypothesis testing, design of experiments. Reliabilitiy, survival analysis. Stochastic processes. Time series. Epidemiology. Quality control. Analysis of Variance. Multivariate analysis. Survey sampling, polls.

• Optimization Theory and Mathematical Programming. Integer Programming:

Facets, Subadditivity, and Duality. Nonlinear programming, convex programming. Iterative Methods. Industrial Design Optimization. *Numerical methods, partial differential equations, calculus of variations, combinatorics, linear algebra.*

• Problems of optimal transportation. Problems of traffic (with continuous and discrete modeling). Network planning. Traffic in the *Web.*

• Economy. Financial mathematics (option pricing, derivative trading, risk management,...) unites *stochastic differential equations, partial differential equations and free boundary problems.* Models for the global economy.

• Chemistry. Quantum Chemistry: *simulation of atomic and molecular structures through fundamental equations.* Reaction dynamics, combustion. *Mathematics of nucleation, growth of crystals and chemotaxis. Front propagation, traveling waves, chemical oscillators. Chaos.* Drug design.

Life Sciences and Medicine:

• Biology: Mathematical Ecology, Epidemiology, Biometrics, Bio-informatics. Mathematics of Genetics, Computational phylogenetics. Nucleic Acid structure and function. Molecular evolution. Proteomics. Regulatory and developmental pathway inference. *DNA computation.* Sequence alignment, fuzzy reasoning. Mathematical modeling in biopolymerization.

• Medicine: interaction fluid-structure as a model for the blood flow. Modeling and simulation of the function of other organs: brain, lungs and liver. *Self-organization and fractal geometries.* Computational assistence of surgery. Pharmacokinetics, tumor growth modeling. Computational neuroscience. The Mathematics of infectious diseases and epidemic spreading. Artificial organs, immune system modeling.

• Medical imaging methods. Tomography: computerized tomography, 3D image reconstruction. *Fourier and Radon Transforms, inverse problems.*

• Though Computational Mathematics (as different from Computer Science) permeates all fields of application, it deserves a mention in itself: numerical methods and codes; efficient algorithms; approximation, (a priori and a posteriori) error estimates, adaptive methods and adaptive models, multigrid and domain decomposition, multiscale analysis, numerics of random processes, ...

• On the other hand, Mathematical Modeling in its different variants (deterministic, continuous, discrete, ...) leads to the problems of Model Validation and the techniques of obtaining and elaborating data on which validation is based (see Statistics above), as well as the quite important (and debated) concept of hierarchy of models, a progressive way of approaching "reality" that is nowadays recognized and embedded into the toolkit of the applied scientist (the old idealists with their eternal truth will revolve in their graves; or will they not?).

We shall stop here and take a much needed break with some comments. The list is loosely organized by affinity of topics; however, the close interconnection of the branches of applied mathematics forces us to indulge in repetitions, or otherwise to place a subject under one of various possible headings. On the other hand, we are leaving without proper comment a number of fields of application: the theory of complex systems, self-similarity in the natural world, pattern formation and recognition, the global positioning systems (GPS), mathematics of electoral systems, Architecture, the textile or the food industry. And there is the strong trend for Mathematics to play an important role in the visual Arts, as it already does in the Entertainment Industry combined with the formidable progress of computer technology. And how could I forget talking to you about Knot theory, G. Dantzig's Simplex Method or the Kalman Filter? In conclusion, this long list is incomplete, mostly because of the limited knowledge of the author, but I hope that it will impress upon the reader the enormous variety of interests of today's applied mathematics.

I would like to add a final personal reflection on the trends I see underlining all the above diversity. The mathematics that are to come will be much more **stochastic** and **algorithmic** than they used to be in the XXth century, and **mathematical modeling** will come to be considered an essential part of the mathematical education and activity, alongside with computation and simulation. But whatever happens, it looks to me that a clear and complete **proof**, and elegant if possible, will always be the heart of the matter, as it has been since good old Euclid, and future mathematicians will still get excitement from **problems and conjectures,** and as Galileo did, from **looking at the world** (or the stars). And they will build, perched on the shoulders of former giants, these delicate, intricate and elusive objects called **theories,** some of them destined to oblivion, some to eternity, or to the daily wear-and-tear. Who marvels anymore at the surprising existence of electromagnetic waves filling the air, now that they have even become a form of pollution? But so much for philosophy at this moment.

9 From Hilbert's 23 problems in 1900 to the Clay Millenium Problems in 2000

We have already pointed out the deep impact that the list of problems proposed by D. Hilbert in 1900 had upon his contemporaries and successors. One hundred years later different initiatives try to follow the example of the great man, see e.g. the books by Arnold-Atiyah-Lax-Mazur, and by Engquist-Schmid[64]. On Wednesday

[64]For more information see the article by A. Jackson cited in the final references. See also vol. 3, no. 1 (2000) of Gaceta de la Real Sociedad Matemática Española, article by J. L. Fernńdez and M.

March 24th, 2000 at the Collège of France in Paris the official announcement was made of the collection of seven mathematical problems that constitute the Millennium Prize Problems, sponsored by the Mathematics Clay Institute. Remembering Hilbert, it tries to reflect seven of the most important open problems of the mathematical science at the beginning of the new century[65]. These problems cover quite different areas of pure and applied mathematics. Here is the list

1. P versus NP (Computation theory)
2. The Hodge Conjecture (Algebraic geometry)
3. The Poincaré Conjecture (Geometry and topology)
4. The Riemann Hypothesis (Number theory)
5. Yang-Mills Existence and Mass Gap (Theoretical Physics)
6. Navier-Stokes Existence and Smoothness (Fluid Mechanics and PDEs)
7. The Birch and Swinnerton-Dyer Conjecture (Algebraic arithmetic geometry)

Let me add my personal mixed feelings about the list that seems destined to be famous and influential. Fortunately, it includes important open problems that cover varied topics of pure and applied mathematics. However, it does not do full justice to the vision of mathematics as the language and tool of science and engineering.

10 Examples of new courses

After two sections devoted to enumeration, it is time to take a closer look at some of the novelties of present-day mathematics. Among the many options, the following three examples are taken from Finance, Communications and Fundamental Physics.

The Mathematics of financial uncertainty and risk

A remarkable example of the practical applications of Mathematics developed in the last decades is the so-called financial mathematics. The new financial instruments of *derivatives* are based on, and at the same time motivate this new branch of applied mathematics, which combines stochastic processes, partial differential equations and free boundary problems. The most famous result is the *Black-Scholes model*[66] for

de León.

[65]the solution of each problem would mean for the author a prize of 1 million of dollars. All the information about the prize and the problems can be obtained from the website `http: //www.claymath.org/prize_problems`.

[66]F. Black, M.Scholes, *The pricing of options and corporate liabilities*, 1973. Merton and Scholes received the Nobel Prize for Economy in 1997. A first version of the model had been proposed by L. Bachelier in 1900! it took seven decades for the more realistic model and the application to occur.

the option market, which reduces the pricing of an option to the solution of a heat equation (backwards in time). I would like to record this reduction in the (would I say famous?) sequence of formulas

$$\frac{dS_t}{S_t} = \mu dt + \sigma\, dB_t \quad \Rightarrow \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + b\, S\, \frac{\partial P}{\partial S} - r\, P = 0,$$

which passes from a stochastic integral, representing random uncertainty, to a deterministic PDE, that allows for price valuation. This is a surprising example of *concept and technique transfer*, made possible by the common mathematical code (and by the fact that F. Black graduated in Quantum Physics). Since there is an inherent instability in those markets, and they have enormous consequences on the economy, both public and private, it is very important to try to apply mathematical methods to find the mathematical clue to the mechanisms that govern the evolution of such phenomena, and to replace guess-work by mathematics in the financial practice. This is a real-world challenge for the new century.

From Fourier analysis to wavelets

We have discussed a while ago the problem faced by Fourier analysis when Du Bois Raymond proposed his example of non-convergent Fourier series, and we want to recall here that the third option out of the problem consisted in changing the basis of functions used in the representation. This is what A. Haar did in 1909[67], thus solving the difficulty in principle, and we can say that this is the remote origin of wavelets, an idea that took a whole century to come of age. Prior to World War II investigation on the issue seems to have followed an exclusively mathematical interest with no application in mind whatsoever. But after the war engineers and applied scientists landed on the idea led by applications, notably in the information theory of Claude Shannon. Eventually the two strands merged and wavelet analysis has become an important intersection of the frontiers of mathematics, scientific computing and signal processing[68].

The mathematical models of Theoretical Physics

The two great scientific revolutions in XXth century Physics, i.e., Relativity and Quantum Mechanics, have impressed on the discipline a strong connection with pure mathematics and the enormous challenge of building a theory to unite both models into a consistent whole. Experimenters and theoreticians have taken up the quest for the "ultimate theory" which would explain all, from the constitution of the atom to the farthest recesses of the Universe. A final theory is still pending (and might be for a time) but great achievements have been obtained. Here are some milestones,

[67] "Zur Theorie der orthogonalen Funktionensysteme", *Math. Annalen* **69** (1910), pp. 331-371
[68] Most of the data are taken from the book [19], cf. also [16]

all of them deep mathematics. Quantum Electrodynamics (QED) was developed to describe electromagnetic interaction in the framework of Quantum Mechanics, and deals with charges, photons and uses the beautiful Feynman diagrams. Next, Quantum Chromodynamics[69] does a similar job to describe the strong forces among *quarks*, the particles postulated by M. Gellmann and G. Zweig in 1964 as the building blocks of neutrons and protons. Out the four fundamental forces of Nature (gravitational, electromagnetic, weak and strong) the two intermediate have been given a unified theory in 1967 by S. Weinger, Sh. Glashow and Abdus Salam. *Symmetry*, *gauge* and *renormalization group* are keywords in this highly mathematical world. Maxwell's, Schrödinger's and Dirac's equations cede the place to Yang-Mills equations. The work crystallized in the early 70'sin the Standard Model of elementary particles, which explains atomic reality in terms of three generations of quarks and *leptons*. These particles interact through the $SU(2) \times U(1)$ theory for the electroweak force and the $SU(3)_{color}$ theory for the strong force. Mathematics is therefore at the core of the model, in the form of Lie groups, differential geometry (more specifically, connections on a fibre bundle) and partial differential equations.

Grand Unified Gauge Theories attempt to combine both group theories into one. In String Theory the old basic idea of point particles is replaced by the idea of elementary vibrating strings. At the end of the century Superstring Theory proposes a mathematical model for the unification of all forces, hence of all physics. It lacks however sufficient experimental verification; without it a theory is just a theory. And the quest continues. These ideas have motivated quite important mathematical developments associated to names like Atiyah, Donaldson and Witten.

Physicists believe that the combination models-and-experiments will allow us to understand a strange world in which matter, space and time are not what we use to think, where empty space is full of activity and even there could exist many additional space dimensions curled up in ridiculously small distances (a typical distance would be 10^{-35}m, so that we do not see them, *voilà l'astuce*; but we see the mathematics, and in due time will see the consequences, so they say).

11 Facts and opinions

In the words on John Milnor, "pure mathematicians tend to judge any work in the mathematical sciences on the basis of its mathematical depth, the extent to which it introduced new mathematical ideas and methods, or it solves long standing problems". To which I would add that new ideas and methods are tested by their productivity, and mention elegance of proof and insight. He continues thus: "However,

[69]The name refers to the picturesque denomination for the conserved charge, called "color".

when mathematics is applied to other branches of human knowledge, a quite different question must be asked first: to what extent does it increase our understanding of the real world"[70]. Not long ago there was a movement towards separation in Mathematics that seemed to move farther and farther away from each other the cultivators of both genres, pure and applied. And we should not forget the prejudice of many pure scientists against a type of applied mathematics more intent on profit than on scientific standards, and, on the other hand, the prejudice of many applied scientists towards the very artificial worlds of certain pure mathematics. Fortunately, we are witnessing a series of simultaneous events - namely, the explosion of the vitality of theoretical and computational mathematics, the successes of Mathematics in the formulation and solution of the key problems of contemporary Physics, Economy and Engineering, and the unsuspected variety of applications of all branches of Mathematics. These events are deeply modifying the vision of both fields, that tend to merge in one, in the best tradition of the past, as expressed in the words of the XIXth century Russian mathematician P.L. Chebyshev: "Bringing together theory and practice leads to the most favorable results; not only does practice benefit, but the sciences themselves develop under the influence of practice, which reveals *new subjects* for investigation, as well as *new aspects* of familiar subjects"[71].

It is a great mystery for professionals that pure and applied mathematics work like the faces of the same coin. That they are not exactly the same is very well reflected in the words of Albert Einstein: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality"[72]. But the ideal and the practical meet with striking results. The amazement before the practical power of Mathematics is most vividly expressed by E. Wigner in a famous statement, where he wondered about "the unreasonable effectiveness of mathematics in the natural sciences"[73].

A word has to be said about the changes in the way mathematics is being done, specially when it is applied. The emergence of the *computer era* has given mathematics new wings, *we can compute!* Efficient and fast computation has become available and cheap at the beginning of the XXIth century, and society needs more. Theorems will always be theorems and a logical derivation is the key to understanding, but the way to discovery will never be the same, and numerical performance is now central to most of mathematics (all of the applied mathematics). The effects on teaching will be no less drastic, but they are still being developed.

Another key feature of modern applied mathematics is *mathematical model-*

[70]See Notices Amer. Math. Soc., 1998.
[71]Taken from [20]. My emphasis.
[72]From *Geometry and Science*, 1921. Included in *Sidelights of Relativity*, Dover, 1983.
[73]Conference in New York, 1959. Published in *Comm. Pure Applied Math.* **13** (1960), 1-14

ing, the art of devising sensible *representations* of different phenomena of the real world in mathematical terms, based on rational *assumptions* that simplify reality to make it computable. J.L. Lions, the late French mathematician who contributed so much to the present relevance of Mathematics in the industrial world, said in 1991: "Ce que j'aime dans les mathématiques appliquées c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir." And he added: "De toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure." We must bear in mind that a model is only a model and reflects reality in the conflicting way that Einstein describes. But it is everything we have, unless we consider a better model (or even a hierarchy of them). This is the glory and the danger of modeling, a crucial aspect of today's applied mathematics. The current discussion about the predictions of mathematical climate models on global warming on the Earth show how important the issue is and how difficult is to manipulate partial evidence based on partial models and backed up by huge datasets of difficult interpretation. Which brings us back to the merit of the giant modelers of the past like Newton, Maxwell, Einstein and the Quantum people.

As we have seen, a large part of the best Mathematics have originated to explain the working of the physical world. But it must be observed that very often the important consequences of Mathematics have not manifested immediately. The formulation of physical processes in mathematical key in the sense of Galileo requires a maturing process, and this process has its own rules and rhythm, that go from a few years to a one or a few centuries. It would be a blessing if both the administration and education authorities were conscious of this fact in their decision-making.

On a more speculative level, the well-known mathematician and science writer Ian Stewart asserts that it is possible that Mathematics are efficient "because they represent the underlying language in the human brain". In that case we revert Galileo's bet in the sense that we maybe understand the world in mathematical terms because that is precisely the coding system in our mind. But this is a different debate.

Let me summarize some opinions I sustain for the sake of the everlasting debate:

- Only good Mathematics can be good Applied Mathematics. Applied Mathematics as an art which is different and separated from Mathematics as such, simply does not exist[74]. By putting Mathematics to use, application changes them.

- Mathematics is really applied only if it answers an important problem of science, technology, economy, or more generally, society. And we have seen how varied these problems can be.

- Though we can come to judge with a certain degree of reliability what is impor-

[74] I take this forceful idea from A. Rényi, [32], who attributes it in the fiction to Archimedes.

tant today, the task of predicting what mathematics will be important in the long term (so-called strategic planning) exceeds the usual capacity of sensible persons, unless we simply answer in general terms like "good Mathematics will matter" or "the Mathematics of real-world problems will matter". Educated guesses and opinions on specific matters are human and may be useful as personal orientation, but when it comes to decisions and priorities prudence ought to be the rule.

- There is an interesting issue of approach to the job: it has been observed that, when faced with a mathematical riddle, the so-called applied mathematician enjoys constructing and comparing suitable models, and wants this precise riddle *explained* whatever the temporary cost to the perfect logic, while his pure counterpart takes delight in logical proof; only *proof* will rule his day.

So, are pure and applied mathematics the same after all? Or more carefully formulated, are they essentially the same? It is up to the reader to judge. You already know my opinion, but let me add in a relaxed tone a quotation from Yogi Berra[75]: "In theory, there is no difference between theory and practice; in practice, there is".[76]

12 Appendix on Mathematics in Spain

Spain played in a given moment of the late Middle Ages a significant role in the transmission of Arab culture to the West and there even existed a king in Seville[77] who wrote poetry and promoted Mathematics. Al Andalus, the Arab Spain, had solid interests in the sciences, in particular medicine and astronomy, with fine scholars like Azarquiel (or Al-Zarkali, active in Toledo) who composed astronomical tables. The Indian number system based on position was already in use in Al Andalus in the IX century.[78] After its takeover by the Christians (1085 a.C.), Toledo, the city of three cultures -Christian, Arab and Jewish- was for centuries a main center of learning with its School of Translators, which brought into Latin the works of Greek and Arab authors[79]. In another direction, the Majorcan Ramón Llull devised in his *Ars Magna* a whole art of algorithmic reasoning in which we can see the early precedents of the Boole algebra and the computer logic (Llull, who lived in the XIIIth century, is at the

[75] famous American baseball player, well-known for his funny quips.

[76] Here is a joke on the different views of mathematics: engineers say that the equations approximate reality, while physicists think that reality approximates the equations; on their side, mathematicians are astonished at the idea of a connection between 'their' equations and reality.

[77] Alfonso X, called the Wise.

[78] The first real Andalusi school of mathematics seems to have been that Maslama of Magerit, i.e., Madrid, in the Xth century.

[79] The Monastery of Ripoll in Catalonia also had a world-famous library.

same time one of the oldest classics of the Catalan language). A century later nautical maps called *portulanos* from Majorca were the top of the art, and the names of Soler and Cresques are well-known. The latter, a Jew, participated in the organization of the Portuguese nautical school, which was at the origin of the discovery of the way to the Indies around Africa, and, indirectly, also of America.

But the medieval and early Reinassance hopes failed later in Spain, so that mathematics (and other sciences) have had a very humble history for centuries. While the Spanish literature and art stand at the peak of worldwide creation since the XVII th century and up to our time, it is apparent that no Spanish names appear in the famed textbooks of mathematical learning. There are in such texts numerous concepts and results named after authors belonging to the nations with a great scientific tradition: French, English, German, Italian, in more recent times Russian and American,.., as there are also frequent examples of other countries which due to their size and circumstances did not play such a prominent role in history but count in science. During these centuries of glorious development, let us say from Galileo to Einstein, Spanish names are not mentioned. Could history have been different? King Philip II realized the need for science and created a Mathematical Academy in Madrid (1582) under the direction of Juan de Herrera, the architect of El Escorial, but the institution did not take root and closed a few years later, while similar initiatives abroad gave birth to the Royal Society in England, the Académie de Sciences in France, and so on. There have been a number of brilliant isolated men, worthy of mention, like Pedro Ciruelo, Omerique, Jorge Juan and Echegaray, but a school never took root until very recently. For centuries Spanish students and professors were forbidden to travel and learn in foreign countries, a quite strong safety rule that prevented at the same time heterodoxy, science and progress.

This is not the place for a detailed study of History, for which we may refer to the specialists[80], so let me proceed by pointing out how we have recently come to a quite favorable present. Spain appeared to abandon its deep mathematical lethargy in the first half of the last century and the figure of Julio Rey Pastor can serve as a reference to a remarkable effort in making our country up-to-date, an effort based on a couple of main ideas: in the first place, by study abroad in the great foreign centers, and then by the import of the problems and topics that occupy the worldwide community. This method had a striking success in the development of North-American mathematics, and was having good results in our country in the first decades of the XXth century. However, our ill-fated history, and mainly in that respect the civil war, destroyed the effort, or in the best cases forced the scholars to exile, which then gave abundant fruit on Latin-American land, as is the case of well-known mathematicians like Luis Santaló and the blooming of Argentinian

[80]like Juan Vernet, whose work [43] is used above.

mathematics. With a few very honourable exceptions, mathematical activity after the war and up to the 60s returned to the slumber of the past. Little by little began the awakening of Spain to normal mathematical life, specially in the 70s. After a decade of enormous effort of a generation that learnt from the original sources, taught from the most reliable textbooks in the classrooms, organized research seminars and traveled or sent their young students abroad, regular publication in recognized journals and participation in international events increased. In the 80's there came the decade of original creation, reflected in the number and level of the publications in good journals[81]. The signs of good times became many and unequivocal, and we may conclude that Spain is no longer different ("Spain is different" is a famous touristic motto dating from Franco's time, which had an obvious negative reading when applied to the troubles of Spain to become a modern country, and was therefore felt critically by the democratic opposition). The official indicators allow us to put figures to this evidence of change. From them we may deduce two facts which have initially surprised many:

(a) That Spanish mathematics have passed from a very modest place in 1980 (0.3% of the world production by the ISI Data Base) to a very honourable position at the moment, immediately after USA, Germany, England, France, Russia, Italy, Japan and Canada, with a production in relevant journals which has been multiplied by a factor of more than 10 and represents in 2001 a worldwide figure of more than 4.18% (ISI).

(b) That in the comparative outlook of Spanish science Mathematics figures among the best placed specialties.

Another consequence of the creative state of Spanish mathematics is the presence of numerous and valuable textbooks and research monographs in prestigious collections. Let it be said that Spain, which has reached a solid position in research, also counts on a tradition in mathematical education, with a very relevant role in ICMI.

Finally, the trend towards the computational and applied aspects of mathematics, with the emphasis on mathematics as the modeling tool per excellence, is now strongly felt in a community formerly very exclusively tied to pure mathematical thinking. Opening the windows to the wide world outside is an enormous challenge for the health of our mathematics and the welfare of future generations, and all efforts are welcome. Let in the fresh air!

[81] At this moment it is befitting to recall Galileo's words "Science knows only one commandment: contribute to science", according to B. Brecht in his *Life of Galileo*

13 Conclusion

This is the end of our journey. Remembering Galileo, I would like to conclude as follows: the book of Nature is open before our eyes for us to admire in its infinite, changing and surprising beauty. Mathematics, as the language of Science, is here to help us understand it: besides, it may allow us to use it and exploit it, and this aspect is loaded with promises and dangers, as all human endeavours. I am confident that the mathematicians of today will make their contribution to understanding and improving the Information Society whose birth we have had the fortune to witness. In the era of computers and information *reality is in the number*, as Pythagoras would have liked. Or at least a big chunk of it.

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The Appendix reflects ideas of the author on the present of Spanish Mathematics taken with minor additions from reference [42], first section. More on the same subject in [41]. Interesting sources in Spanish are the *Boletines de SEMA*; the *Gaceta de la*

RSME[82] (cf. vol. 3, 1 (2000)) and *Revista Española de Física*, vol 14, no. 5, issues devoted to the state of Mathematics on the occasion of the World Mathematical Year celebration. See also [1, 8, 18, 26, 30, 39]. Finally, the list of references below reflects readings of the author while compiling this text and is not meant as a selection of the best reading on the subject.

References

[1] V. Arnold, M. Atiyah, P. Lax, B. Mazur, "Mathematics: Frontiers and Perspectives", AMS Publications, 2000.

[2] I. Babuska, *Courant Element: Before and After*, in "Finite Element Methods", edited by Krizek, Neittaanmäki and Sternberg, M. Dekker Inc, New York, 1994.

[3] G. I. Barenblatt, *George Keith Batchelor and Daniel George Crighton, Applied Mathematicians*, Notices American Math. Soc., vol 48, no 8 (2001), 800–806.

[4] J. L. Casti, "Five Golden Rules", John Wiley, New York, 1996. "Five More Golden Rules", John Wiley, New York, 2000.

[5] G. Chaitin, "The limits of mathematics", Springer, Singapore, 1998.

[6] B. Cipra, "What is happening in the Mathemaitical Sciences", vols. 1–4, Amer. Math. Soc, Providence, RI.

[7] COMAP, "Las Matemáticas en la vida cotidiana". Addison Wesley - Universidad Autónoma de Madrid. 1998 (Spanish). English: S. Garfunkel et al., "Introduction to Contemporary Mathematics", W.H. Freeman & Co, New York, 1988.

[8] B. Engquist (Editor), W. Schmid (Editor), "Mathematics Unlimited - 2001 and Beyond", Springer Verlag, Berlin, 2001.

[9] R.P. Feynman, "Feynman Lectures On Physics (3 Volume Set)", Reading, Mass., Addison-Wesley Pub. Co. [1963-65] ,

[10] R.P. Feynman, "Six Easy Pieces: Essentials of Physics Explained by Its Most Brilliant Teacher", Helix Books, 1995 (Addison-Wesley Longman, 1996), and "Six Not-So-Easy Pieces: Einstein's Relativity, Symmetry, and Space-Time", Addison-Wesley Pub., Reading, Mass. 1997.

[82] SEMA is Sociedad Española de Matemática Aplicada, and RSME stands for Real Sociedad Matemática Española

[11] I. Fonseca et al., "The Impact of Mathematical Research on Industry and vice versa", Round Table at 3rd European Congress of Mathematics, Barcelona, july 2000.

[12] A. Friedman et al., "Mathematics in Industrial Problems", (a 10 volume collection), IMA Volumes in Mathematics and its Applications, Springer-Verlag (1988-1998).

[13] A. Friedman, J. Lavery, "How to Start an Industrial Mathematics Program in the University", SIAM Report, Philadelphia 1993

[14] J. Gleick, "Chaos: Making a New Science", Penguin Books, Nueva York, 1987.

[15] G. H. Hardy, "A Mathematician's apology", Cambridge, 1940.

[16] E. Hernández, G. Weiss, "A first course on wavelets. With a foreword by Yves Meyer", Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996.

[17] "Mathematical Developments arising from Hilbert Problems", Proceedings of Symposia in Pure Mathematics, XXVIII, Amer. Math. Soc, Providence, 1976.

[18] A. Jackson, *Mathematical challenges of the XXI century*, Notices Amer. Math. Soc., vol. **47**, no 10 (2000), pp. 1271-1273.

[19] S. Jaffard, Y. Meyer, R.D. Ryan, "Wavelets, tools for Science and Technology", SIAM, 2001.

[20] A.I. Khinchin, "Mathematical Foundations of Information Theory", Dover, 1957 (Papers appeared in 1953, 1956 in Uspekhi Mat. Nauk in Russian).

[21] M. Kline, "Mathematical Thought from Ancient to Modern Times", Oxford Univ. Press, 1972.

[22] M. Kline, "Mathematics. The loss of certainty", Oxford Univ. Press, Oxford, 1980.

[23] J. P. Maury, "Galileo, el mensajero de los astros", Claves, Ed. B.S.A., Barcelona, 200 (Spanish).

[24] N. Metropolis, J. Howlett, G. C. Rota, eds., "A History of Computing in the Twentieth Century", Academic Press, 1980.

[25] P. A. Meyers, "Encyclopedia of Modern Physics", Academic Press, San Diego, 1990.

[26] Mitteilungen der Deutschen Math. Vereinigung *(Notices of the German Math. Union)*, 2-1998, contains a report on the Future of Mathematics by D. Mumford, A. Friedman, L. Lovász, Yu. Manin, G.C. Rota, R.B. Jensen and R. Penrose.

[27] K. Moriyasu, "An elementary primer in Gauge Theory", World Scientific, Singapore, 1983.

[28] J. Muñoz Santonja, "Newton, el umbral de la ciencia moderna", col. La Matemática en sus personajes, vol. 3, Nivola ed., Madrid, 1999 (Spanish).

[29] I. Newton, "Philosophiae Naturalis Principia Mathematica", Pepys, London, 1687. In Spanish: "Principios matemáticos de la Filosofía Natural", Alianza Ed., Madrid, 1987.

[30] Obra Colectiva, "Fotografiando las Mathematics", Carroggio SA de Eds, Barcelona 2000 ("Photographing Mathematics", in Spanish).

[31] J.T. Oden et al., "Research directions in computational mechanics", Report of the US National Commitee on Theoretical and Applied Mechanics, Washington DC, 2000.

[32] A. Rényi, "Dialogues on Mathematics", Holden-Day, San Francisco,1967.

[33] Revista Española de Física, vol 14, no. 5, año 2000. Special number "La Física y las Matemáticas".

[34] J. M. Sánchez Ron, "El siglo de la ciencia", Taurus, Madrid, 2000 (Spanish).

[35] M.M. Schiffer, L. Bowden, "The role of Mathematics in Science", The Math. Assoc. of America, New Math. Library vol. 30, 1984.

[36] J. Simmons, "The scientific 100", Citadel Press, Kensington Publ. Corp, Nueva York, 1996.

[37] S. Singh, "The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography", Doubleday & Company, 1999.

[38] D. Stauffer, H.E. Stanley, "From Newton to Mandelbrot, A Primer in Theoretical Physics", Springer, Berlin, 1991.

[39] I. Stewart, "The Problems of Mathematics", Oxford Univ. Press, 1992.

[40] M. Tanur, F. Mosteller et al., "Statistics: A Guide to the Unknown" Brooks/Cole, 1989.

[41] J. L. Vázquez, *Las Matemáticas y los objetivos del año 2000. Un llamamiento a los matemáticos españoles*, Gaceta de la Real Soc. Matemática Española, vol. 3, 1 (2000), (Spanish) pp. 9-22. Cf. also `http://dulcinea.uc3m.es/ceamm`.

[42] J. L. Vázquez, *Mathematical Events in Spain in the Year 2000*, Intelligencer, Springer-Verlag, July 2000, pp. 12-14.

[43] Juan Vernet Ginés, "Historia de la Ciencia Española", Editorial Alta Fulla, 1998 (Spanish).

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Matemáticas, Ciencia y Tecnología: una relación profunda y duradera

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No te preocupes demasiado por lo que son las Matemáticas antes de probar tu suerte. Ya lo irás viendo.

RESUMEN

Los matemáticos suelen decir que la esencia de las Matemáticas reside en la belleza de los números, figuras y relaciones, y hay una gran verdad en ello. Pero la fuerza motriz de la innovación matemática en los siglos pasados ha sido el deseo de entender cómo funciona la Naturaleza. Este aspecto fundamental es pocas veces mencionado.

La Matemática forma junto con el método experimental el esquema conceptual en que está basada la Ciencia moderna y en el que se apoya la Tecnología, existiendo estrechas interacciones entre ellas. Sobre estas bases nació la Sociedad Industrial hace varios siglos, y la nueva Sociedad de la Información se construye en el presente siguiendo las mismas pautas.

En el artículo damos un esbozo de esta relación con la ciencia y la tecnología, de cómo se puso en marcha y de los héroes que la han hecho realidad, seguido de una ojeada al futuro, en que la relación se extiende prácticamente a toda la sociedad. Se añade un corto comentario sobre la Matemática en España.

1 Introducción. Esencia y papel de las matemáticas

La Matemática es una disciplina intelectual autónoma, uno de los exponentes más claros del poder creativo de la mente humana. Por otra parte juega un papel fundamental en la Ciencia moderna, tiene una marcada influencia sobre ella y a su vez se ve influenciada por la ciencia de una manera esencial. Estas son, brevemente presentadas, las dos concepciones que simbolizan las maneras diferentes de ver el gran edificio que es la Matemática actual. Estas opciones se reflejan en las populares denominaciones de Matemática Pura y Aplicada. Pero entonces, ¿es que existen dos matemáticas diferentes? y, si esto es verdad, ¿pueden coexistir pacíficamente e interactuar recíprocamente, o es que viven de hecho separadas e incluso hostiles una a la otra? En el presente artículo intentaremos mostrar que, hoy como ayer, ambas visiones de la matemática son las caras de una misma moneda, que nos parecen a veces tan diferentes, a veces tan semejantes.

Un arte puro. Una primera dimensión de las matemáticas es en efecto el aspecto puro, la matemática como un arte por derecho propio, un juego que se juega en nuestras mentes. La Matemática es un arte que expresa la belleza en forma de axiomas, teoremas y relaciones lógicas o numéricas y atrae al investigador precisamente por su perfección lógica, siendo uno de los ejemplos más claros y convincentes de la capacidad humana para el razonamiento y el análisis. Ella impone orden y armonía donde sólo veíamos desorden y caos.

Ésta es la dimensión más próxima al investigador y, como toda forma pura de arte, tiene una fascinación que explica por qué los profesionales consagramos una parte enorme y bastante exclusiva de nuestras vidas a ella. Resulta natural que los matemáticos profesionales tiendan a ver su ciencia desde este punto de vista del arte en sí mismo, con sus conceptos, conjeturas, resultados y métodos de prueba, con sus áreas venerables: la aritmética, el álgebra, la geometría y el análisis, y los nuevos retoños: la estadística, el cálculo de probabilidades, la lógica matemática, la computación,... Y estimen sobre todo sus *perfectas deducciones lógicas*. Grandes sabios han profundizado en esta dirección: PITÁGORAS ve en los números la clave de la realidad y PLATÓN ve en el mundo de las ideas un mundo de orden más perfecto que el mundo físico cotidiano. De hecho, pocos matemáticos profesionales han sido totalmente ajenos al sentimiento de que la verdadera Matemática habita más allá, en un mundo ideal, esperando a ser descubierta por el artista. En sus fabulosos 13 libros de *Los Elementos,* EUCLIDES de Alejandría (325-265 a.C.) estableció a la vez la teoría y las reglas de un juego que sigue sus pautas hoy como hace 22 siglos. Pocos artistas a lo largo de la Historia podrán decir lo mismo sobre la repercusión y perennidad de su obra: las demostraciones de Euclides son aún hoy día "las demostraciones" en los temas por él tratados. Tal es su influencia intelectual que en el siglo XX los matemáticos asociados bajo el nombre de guerra de Nicolás BOURBAKI osaron repetir la histórica gesta con unos actuales *Elements de Mathématique*[1]. La matemática es pues un arte autónomo que halla la verdad dentro de sí misma. Recordemos a Carl

[1] Con un éxito innegable a pesar de una cierta división de público y crítica.

G. J. JACOBI que sostuvo que la matemática sólo existe *"para honor del espíritu humano"*. Claro que de ahí también se deriva una cierta concepción popular, con su halo romántico pero hoy día un tanto descaminada, que ve al matemático como un sabio irremediablemente distraído, con poca o ninguna mente práctica.

Otra visión, otro papel. ¿Refleja lo anterior el cuadro completo de la Matemática? En absoluto, *la Matemática es eso y mucho más*, hay un modo totalmente distinto de verla y de hacerla que queremos presentar. Junto con el método experimental, las matemáticas son la base sobre la que se asienta la Ciencia moderna y, como consecuencia, en ellas se apoya el desarrollo tecnológico de nuestras sociedades. Penetra hoy todos los aspectos de la sociedad contemporánea desde la ingeniería a la información, el mundo de la empresa, la salud, la administración y las finanzas, sin olvidar el movimiento de las disciplinas sociales hacia el estatus de ciencias, que significa en otros términos y con los matices apropiados, el uso combinado en estas disciplinas de los métodos matemáticos y experimentales. La importancia práctica de las matemáticas en las ciencias es indiscutible, y no está de hecho en discusión pues la mayoría aplastante de los científicos es bien consciente del *valor instrumental* de unas buenas dosis de matemáticas en la ciencia. Así, una parte cuantitativamente importante de las matemáticas que son enseñadas en las universidades de todo el mundo se consagra a la educación de ingenieros, físicos, químicos, biólogos, informáticos, economistas y profesionales de otras varias disciplinas.

Sin embargo, creemos que tal aprecio no hace justicia al papel que las matemáticas juegan en la sociedad. Sostenemos que el *papel de la Matemática que es aplicada* en diversos contextos sociales va más allá de esta descripción, es más *esencial*. De hecho:

(i) las matemáticas han jugado un papel fundamental en la formulación de la ciencia moderna desde sus comienzos; una teoría científica es una teoría que dispone de un modelo matemático adecuado;

(ii) las matemáticas que se pueden aplicar hoy día abarcan todos los campos de la ciencia matemática y no sólo ciertos temas especiales; se trata de matemáticas de todos los niveles de dificultad y no sólo de resultados y argumentos sencillos;

(iii) las ciencias exigen hoy como ayer nuevos resultados de la investigación y plantean nuevas direcciones de estudio a los investigadores. Pero el ritmo de la sociedad contemporánea hace los plazos sustancialmente más cortos y la exigencia más urgente;

(iv) las capacidades del cálculo científico han hecho de la *simulación numérica* una herramienta indispensable en la comprensión, diseño y control de los procesos industriales.

(v) cuando se habla de la utilidad de las matemáticas para las ciencias se incluye implícitamente en este nombre la técnica y la ingeniería. Pero hoy día los contornos son mucho más amplios y difusos; éste es un aspecto de gran importancia en el presente y el futuro de las matemáticas.

En este artículo trataremos de este aspecto en que *la Matemática es el idioma* en que están escritas las páginas de la ciencia; gracias a ella ha habido un desarrollo del combinado ciencia-tecnología que ha cambiado la vida del ciudadano de las sociedades tecnológicamente avanzadas en los últimos cuatro siglos *de una manera más radical que la Revolución Neolítica había hecho en los noventa siglos precedentes*, y el cambio ha sido más dramático en las últimas décadas que en siglos enteros anteriores.

Es un hecho bien conocido por los expertos que la práctica diaria de las ciencias físicas y la ingeniería utiliza cantidades enormes de matemática del más alto nivel. Es más, los mismos conceptos con que se formulan sus teorías son esencialmente *los conceptos matemáticos*. En las últimas décadas hemos presenciado como la tendencia hacia la matematización alcanza a otras disciplinas, como la Economía, particularmente el mercado financiero, ramas de la Química, la Biología y la Medicina, e incluso las ciencias sociales.

Es un hecho comprobado que la maquinaria matemática, sea imponente o no lo sea, *se oculta muy a menudo cuidadosamente* al público en los manuales o en los escritos de divulgación, como si no existiese, pues se supone que no será bien vista del lector (o que éste no la comprenderá). Pero los nuevos tiempos traen cambios saludables: gracias a la simpatía del público por las proezas del cálculo y la informática, las matemáticas subyacentes van saliendo a la luz.

Repercusión de la Matemática. En manos del científico, *la Matemática debe permitir asimilar los datos y entender los fenómenos*. En manos del ingeniero, es la herramienta que hace posible construir un *modelo* numérico o cualitativo cuyo análisis permitirá *tomar decisiones, diseñar artefactos y controlar procesos de manera eficaz y fiable*. Esta actividad es lo que, a falta de un nombre mejor, llamamos **Matemática aplicada**. Cubre las áreas clásicas como la Física matemática y los Métodos Matemáticos para la Ingeniería, pero tiene hoy día contornos más amplios con el advenimiento del cálculo científico y la simulación numérica. La modelización, la simulación computacional, y el análisis de datos son herramientas esenciales en la ciencia y la industria modernas. La Matemática aplicada es simplemente **la Matemática de la realidad**, es decir, del mundo real, sea lo que sea lo que esta frase significa para cada lector individual.

Señalemos que hay aún otras visiones complementarias de las matemáticas: su aspecto cultural, su importancia en la enseñanza como vehículo del pensamiento racional, su importancia para comprender el mundo diario (las "matemáticas para el hombre de la calle"), su aspecto de juego intelectual (el reto de resolver un problema). La Matemática es al mismo tiempo la ciencia de lo exacto y el cálculo de lo probable. Es la ciencia del razonamiento abstracto y simbólico. Es también hoy día sinónimo de virtuosismo computacional, de capacidad y efectividad de procesar información, tan importante para el mundo que se gesta. Es el mundo del científico que trabaja con un trozo de papel y hoy también el mundo de la modelización, el cálculo y el control de procesos industriales. Todo ello forma también parte del múltiple legado de las matemáticas[2].

A continuación dirigimos nuestra atención hacia el pasado y presente de la Matemática Aplicada. El lector puede encontrar conveniente saltar en una primera lectura la información contenida en las notas a pie de página. Además, varias fórmulas famosas y ecuaciones importantes aparecerán aquí y allá en las páginas. ¡El propósito no es en absoluto que sean estudiadas como parte del texto! Es más bien recordar al lector iniciado su belleza y relevancia, y al mismo tiempo dejar claro que no existe ningún *camino real* (es decir, regio) de acceso a la Matemática: la divulgación tiene sus límites y una comprensión real de los temas aquí perfilados implica un estudio serio. En el capítulo final volveremos a tratar de las opiniones que se debaten y los hechos que sustentan tales opiniones.

2 Herederos de Galileo y Newton

Dos grandes figuras históricas fijaron el futuro *papel estelar* de las matemáticas en los momentos en que nacía la Ciencia moderna. GALILEO *lo formuló,* NEWTON *lo demostró.* No les faltaron precursores. Habría que recordar que en la Historia Antigua PITÁGORAS de Samos (569a.C.-475a.C.) sostuvo que *todo es número* y encontró la maravillosa conexión entre la Música y la Aritmética, mientras ARQUÍMEDES de Siracusa unió Geometría y Mecánica en el siglo III a.C. (m. 212 a.C.). Y un siglo antes de Galileo, el genio universal de LEONARDO DA VINCI *intuyó* el papel central de la Matemática en la Ciencia. Una pléyade de grandes matemáticos, los héroes de nuestro relato, los siguieron[3]. Se puede decir con Newton que los matemáticos que se ocupan de la aplicación de su arte otean el futuro desde los hombros de gigantes[4].

[2] sobre estos asuntos ver [44].

[3] En el recorrido histórico que sigue, los nombres de Galileo y Newton irán acompañados de otros matemáticos ilustres, a algunos de los cuales adjudicaremos un papel relevante. Tal selección, que nos ayudará a fijar los hitos principales y a conocer a los héroes de nuestra particular aventura, es sin duda injusta con otros personajes de la talla de Fermat, Leibniz o Gauss, de lo cual queremos dejar constancia y sólo la brevedad (el *estrecho margen* del que hablaba Fermat) y lo concreto de nuestro objetivo nos sirve de excusa, pues el propósito que tenemos en mente no es la historia de la ciencia.

[4] Tomado de una frase de Newton sobre sus predecesores en carta a R. Hooke, 1675: "If I have seen farther than others, it is by standing on the shoulders of giants". He tratado de incluir en el texto y notas algunas de las frases más famosas de matemáticos y científicos sobre la Matemática y su aplicación.

Procedamos por partes: es verdad que desde la más remota antigüedad las matemáticas han estado relacionadas, incluso motivadas, por problemas prácticos. La Aritmética se origina con las actividades de contar y sumar, la Geometría proviene de medir líneas, superficies y cuerpos. Pero también es verdad que la Matemática, como ciencia lógico-deductiva, tal como fue elaborada y nos fue legada por los griegos, de Pitágoras a Euclides, tuvo una base netamente intelectual, digamos ideal, que siempre ha conservado desde entonces y que es parte fundamental de la matemática pura, es decir, de las matemáticas en sí mismas. Este proceso intelectual vive en su propio mundo y no debe nada de su mérito o belleza a la posible utilidad o aplicación práctica, no más que un poema, una sinfonía o un cuadro. Un silogismo fácil y demasiado frecuente nos llevaría de aquí a concluir que la auténtica matemática vive esencialmente ajena a la aventura de la ciencia y la tecnología. *Este silogismo es falso* por mucho que haya sido sostenido por no pocos matemáticos, y nos proponemos demostrarlo usando la obra y las opiniones de las grandes figuras. Pues *la historia nos muestra que la simbiosis con la ciencia y la tecnología ha sido fundamental y fructífera y que las matemáticas deben mucho de su ser actual y de sus temas estrella a sus compañeras de aventura. Y viceversa.*

Los dos pilares. Como es bien sabido, la Ciencia moderna surgió en Europa al final del período del Renacimiento. No se basa sólo en las matemáticas. El pilar fundamental del edificio en germen fue formulado por el filósofo y político inglés Francis BACON hacia 1620 y consiste en el *método experimental*[5]. El objeto preferente de la filosofía se orienta hacia la Naturaleza, que debemos leer y comprender, y eventualmente controlar; la observación es el medio para la comprensión y el experimento es el test de nuestras predicciones. Las ciencias se formaron alrededor de este método, primero la física, luego las demás: biología, geología y química.

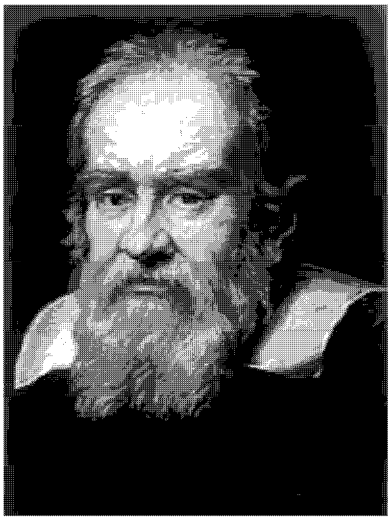
Las matemáticas son desde el principio **el otro pilar de las ciencias**. Fue Galileo GALILEI (1564-1642) quien más claramente señaló a principios del siglo XVII ese rumbo para las nacientes ciencias. Suya es la famosa cita tomada de su carta "Il saggiatore"[6] que reproducimos aquí en detalle:

"La filosofía está escrita en ese gran libro que constantemente está abierto ante nuestros ojos, el Universo, pero no puede entenderse a menos que se aprenda primero a comprender el idioma en que está escrito, a entender sus caracteres. Está escrito en el lenguaje matemático, y sus caracteres son triángulos, círculos y otras figuras geométricas..."[7].

[5] El método inductivo se presenta en su trabajo *Novum Organum* o *Nuevo Instrumento*, 1620.

[6] 1623, cf. Opere, VI, pág. 232; "El Ensayador".

[7] Las famosas palabras no suelen imprimirse en su italiano (toscano) original: *"La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri ne'*



GALILEO GALILEI

Galileo era un claro defensor del método experimental, al que contribuyó con sus famosas observaciones astronómicas y mecánicas[8]. Como hemos dicho, la actitud de Galileo tenía precedentes, siendo los más notables los de Pitágoras y Arquímedes[9] en la Antigüedad y el de Leonardo da Vinci (1452-1519)[10] un siglo antes, pero la formulación de Galileo fue decidida y su propuesta fue puesta en práctica, pues sucedió en el contexto histórico adecuado; corroyó las bases del aristotelismo y la escolástica dominantes hasta entonces en el mundo intelectual. Dio fruto en breve tiempo y los científicos nos vemos reflejados en él.

De hecho, las filosofías son poca cosa si se quedan en palabras y polémicas, si no son llevadas a término. La gloria del siglo XVII reside en una serie de grandes filósofos-científicos (llamados en aquel entonces *filósofos naturales*), quienes, sin olvidarse de la metafísica, se lanzaron decididamente en pos del conocimiento de la Naturaleza y de la invención matemática: René DESCARTES estudió los principios del arte de razonar, así como la mecánica y el universo; ligó la geometría al álgebra y escribió "El

quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi è impossibile a intendere umanamente parola, senza questi è un aggirarsi vanamente per un oscuro labirinto.

[8]Dejó escritas sus ideas sobre la física, las matemáticas y la ingeniería en el libro *Discursos y pruebas matemáticas acerca de las dos nuevas ciencias*, escrito en Florencia antes de 1633, pero sólo publicado en el extranjero en 1638 después de los problemas con la Iglesia. Las dos nuevas ciencias son la mecánica y la ciencia del movimiento. En 1995 la sonda espacial *Galileo* alcanzó Júpiter y con él los 4 planetas descubiertos por el sabio en 1610.

[9]Arquímedes es uno de los "grandes" de la Matemática Pura y Aplicada. Universalmente conocido por sus contribuciones a la Mecánica, que se puede decir que fundó como ciencia teórica, y a la Hidrostática (principio de Arquímedes), fue también un genial matemático que aplicó su intuición mecánica a la Geometría e inventó el "método de exhaustión" para el cálculo de áreas y volúmenes limitados por figuras curvas; este método implica aproximaciones sucesivas y es precursor del concepto de límite que tardará 19 siglos en salir a la luz. Su cálculo del número π fue un récord durante muchos siglos. También invento una notación para los números muy grandes. La matemática griega tuvo una brillante rama aplicada a la Astronomía con Aristarco de Samos y Eratóstenes de Cirene.

[10]Los intereses de Leonardo, un genio verdaderamente universal, abarcan la pintura y la escultura, la ingeniería y la arquitectura, la física y las matemáticas. Científico y visionario, dibujó los planos de un objeto volante (el precursor del helicóptero) y acuñó el término turbulencia en los fluidos. He aquí una cita pertinente de Leonardo: "Ninguna certeza existe donde no es posible aplicar la matemática o en lo que no puede relacionarse con la matemática". Por si quedaba duda de la opinión del gran hombre.

Discurso del Método"[11]; Blaise PASCAL escribió sus filosóficas "Pensées" pero también investigó los principios de los fluidos (como la presión), la geometría, el cálculo y las probabilidades[12]. Y análogamente hicieron Pierre de FERMAT, Edmond HALLEY, Christiaan HUYGENS y Gottfried W. LEIBNIZ, un matemático, lógico y filósofo del mayor renombre.

Estamos ya listos para conocer a uno de los caracteres y de los momentos más cruciales en la historia de la ciencia. En efecto, el siglo alcanza su culminación con la figura de Isaac NEWTON (1642-1727), quien demuestra el éxito indiscutible de la propuesta de Galileo aplicada a la mecánica. Ataca los problemas básicos debatidos durante el siglo y

ISAAC NEWTON

(i) concluye que el movimiento de cuerpos sólidos sigue una ley matemática simple que relaciona la segunda derivada del espacio (respecto al tiempo) con una entidad invisible *pero real*, la fuerza. En términos matemáticos, $\mathbf{F} = m\mathbf{a}$;

(ii) al aplicar esta teoría a los cuerpos celestes, concluye que se mueven en sus órbitas de acuerdo con la ley de atracción universal. En fórmulas, $F = Gmm'/r^2$.

Para estudiar matemáticamente los movimientos resultantes de estas leyes, descubre lo que nosotros llamamos Cálculo Infinitesimal y resuelve las ecuaciones diferenciales. Es más, la formulación misma de sus leyes no es posible sin los nuevos conceptos tomados del Cálculo Diferencial e Integral, que lleva los nombres de Newton y Leibniz, y que fue inventado combinando las intuiciones de la mecánica y de la geometría[13].

En 1687, en que se publica su trabajo monumental, los *Principia*[14], la mecánica queda sólidamente fundamentada sobre las mismas bases que tiene hoy día. La ma-

[11] *Le Discours de la Méthode*, Leiden, 1637, un trabajo importante en la historia de la ciencia. Su trabajo *Les Météores* es considerado el primer esfuerzo por poner el estudio del tiempo atmosférico sobre una base científica. Su más famoso dicho es sin duda el *"cogito ergo sum"*, "pienso luego existo".

[12] Y se ocupó de construir una máquina de calcular de la que volveremos a hablar.

[13] Para situar a Newton en la perspectiva apropiada hemos de combinar su formación matemática con el conocimiento astronómico que heredó de Tycho Brahe, Johannes Kepler y Galileo.

[14] *Philosophiae Naturalis Principia Mathematica*, es decir, "los Principios Matemáticos de la Ciencia".

temática no es sólo una herramienta indispensable, en realidad *es el idioma en que se concibe y expresa la Ciencia*, ésta es la razón del título del libro. Desde ese momento la descripción de la dinámica y la evolución de los sistemas mecánicos es una parte esencial de las matemáticas. Sigue un periodo de enorme desarrollo, durante el cual la matemática intenta cumplir este nuevo papel fundamental.

Newton es considerado generalmente el científico más influyente en la historia de la humanidad, cf. [38]. Permítasenos aportar algunos datos adicionales para entender bien la grandeza de su legado. Podemos anotar a su crédito los fundamentos de la mecánica y la astronomía, del cálculo diferencial e integral y las ecuaciones diferenciales; pero también estudió la naturaleza de la luz, puso los fundamentos a la óptica y contribuyó con notables adelantos técnicos como el telescopio de refracción. Además de todo esto, estudió los fluidos que se llaman hoy día newtonianos, explicó y calculó el funcionamiento de mareas por medio de la atracción lunar, computó la velocidad del sonido (y también se interesó por la teología, la alquimia y la astrología, rasgo bastante común de esos tiempos que no debe extrañarnos en un gran científico)[15]. Su prestigio entre sus contemporáneos era enorme y los filósofos más brillantes del siglo XVIII (Hume, Kant, Voltaire[16]) estudiaron su trabajo y creyeron posible extender su fabuloso éxito a todos los campos de la filosofía, tarea que ha resultado ser de una dificultad extrema. De hecho, todavía estamos ocupados en ella.

La inmensidad de la tarea de entender la Naturaleza no escapó a una persona tan penetrante como Newton, con todo su éxito. Una de sus opiniones más famosas dice como sigue: *"I do not know what I will look like to others; to myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me".*

3 El siglo de la Razón y de las Luces

Durante los tres siglos siguientes una parte de ese océano se ha visto colmado de verdad, ciencia y matemáticas. La ciencia y la tecnología, bases de la Revolución Industrial, han progresado con las teorías, razonamientos y experimentos. Como consecuencia, la sociedad del siglo XX ha cambiado más radicalmente con respecto al siglo XVII que todo lo que había pasado en varios miles de años antes, desde el

[15] "From the same principles, I now demonstrate the frame of the System of the World".

[16] Merece la pena recordar que los Principia fueron traducidos al francés por la amiga del último, la Marquesa de Châtelet, con su colaboración, en 1756. Mujer muy notable, la Enciclopedia Británica la describe como "Gabrielle-Émilie Le Tonnelier du Breteuil, Marquise du Ch., French mathematician and physicist who was the mistress of Voltaire". Sólo en el texto del artículo se entera uno de sus muchos logros.

inicio de las grandes civilizaciones agrícolas. El confort de la casa, el transporte, y las comunicaciones, la salud del ciudadano actual, descansan sobre bases técnicas completamente desconocidas para las personas del Siglo XVII.

Quienes prefieran contemplar el panorama de las matemáticas actuales, al final del largo camino, son invitados a saltar las próximas 3 secciones y proceder con las matemáticas del siglo XX. Más aún, quienes quieran sólo asomarse al futuro harían bien en avanzar hasta la sección 7. Para quienes se interesan por qué pasó entre tanto, el relato continúa en el comienzo del siglo XVIII. Empezando con el ya citado Leibniz, gran filósofo y rival de Newton en la famosa y un poco triste "disputa del cálculo", una serie de brillantes matemáticos (diríamos mejor físico-matemáticos), como la familia Bernoulli, Euler, D'Alembert,... aprovecharon el potencial del nuevo cálculo y formularon matemáticamente todo tipo de problemas mecánicos: problemas de disparo, problemas sobre la caída de los cuerpos, sobre el movimiento de los fluidos, de vibraciones mecánicas, de minimización,...

Los métodos infinitesimales son igualmente poderosos en su aplicación a la geometría, una disciplina que vive en simbiosis íntima con la mecánica. Los sabios estudian el Cálculo de Variaciones, un nombre para el cálculo de valores mínimos de los llamados "funcionales" que florecerá en el siglo XX como un capítulo fundamental del Análisis Funcional, por entonces ni siquiera previsto. Jean Le Rond D'ALEMBERT[17] estudió la vibración de las cuerdas y escribió la ecuación de ondas que lo llevó a descomponer una función en suma de ondas elementales, tarea también emprendida por Leonhard EULER (1707-1783), quien realizó la descomposición en suma posiblemente infinita de funciones sinusoidales. Euler es quizás el matemático más prolífico de la historia, hizo contribuciones fundamentales a la Geometría, el Análisis y la Teoría de Números, pero también a diferentes ramas de la Mecánica, la Elasticidad, la Hidrodinámica, la Acústica, y hasta la Música. Su latín no es difícil y sus libros de texto pueden leerse hoy con provecho y placer (¡preferentemente traducidos!). Vivió una gran parte de su vida en San Peterburgo, por lo que se le atribuye la fundación de la matemática rusa, junto con Daniel Bernoulli. El problema de las sumas infinitas preocupará a los matemáticos en el futuro próximo pero no en estos momentos de descubrimiento y euforia, y menos aún a L. Euler cuya

LEONHARD EULER

[17]Representante muy conocido de la *Ilustración* francesa, quien combinó una brillante carrera matemática con la publicación de la famosa *Enciclopédie*, juntamente con D. Diderot. ¡No todos los matemáticos viven en una nube!

intuición parece no tener límites.

Algunas de las glorias y penas de la matemática como idioma de la mecánica pueden observarse en el estudio de los fluidos. Una teoría sistemática escapó incluso al genio de Newton. De hecho, el aspecto más difícil de esta teoría consiste precisamente en encontrar las hipótesis matemáticas justas que permitan construir un modelo matemático, es decir, matematizarla *tal como realmente es*[18]. Hacia el año 1738 Johann y Daniel BERNOULLI establecen la ciencia teórica de la Hidrodinámica sobre la base idealizada de los llamados *fluidos perfectos*. El estudio fue continuado por Euler que escribe las famosas ecuaciones (1755)

$$\rho\,(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = 0, \qquad \nabla\cdot\mathbf{u} = 0,$$

(en la notación actual) cuya resolución analítica general ha resistido al paso del tiempo[19]. Es más, D' Alembert puso en evidencia las limitaciones de la idealización implícita en el concepto de fluido perfecto mostrando que un obstáculo sólido sometido a un "viento"perfecto no sufriría ningún *arrastre* neto y ninguna *sustentación* neta.

PIERRE S. LAPLACE

Esta dificultad nos devuelve al problema filosófico original, el papel de las matemáticas. De hecho, la dificultad se origina porque la mecánica teórica no trata de la Naturaleza, que escapa en su más pura esencia a nuestra curiosidad, sino que trata más bien del **modelo matemático** que nosotros nos podemos formar de ella. La concordancia experimental nos permite confirmar que una teoría es buena como modelo del mundo físico, pero nunca que es un modelo perfecto[20]. La modelización matemática es un aspecto fundamental de la matemática actual y clave de su posible utilidad.

A pesar del fracaso relativo con los fluidos, cuando termina el Siglo de las Luces una sensación de optimismo invade las mentes de los mejores matemáticos - mecánicos, como son Joseph Louis LAGRANGE, autor de la *Mécanique analytique*[21], y Pierre Simon LA-

[18]Recordemos aquí el dicho de Newton sobre su mecánica: *Hypotheses non fingo*, yo no me invento las hipótesis o axiomas.

[19]Guardan su misterio aún hoy día: la existencia de soluciones clásicas dados datos iniciales regulares en 3 dimensiones espaciales es todavía un problema abierto.

[20]Volveremos a este asunto al hablar de Einstein.

[21]Que describe las ecuaciones generales del movimiento, llamadas ecuaciones de Lagrange.

PLACE. El último publica su monumental libro *Mécanique céleste* (1788). Es también autor de la *Théorie Analytique des Probabilités* (1812), una de las más importantes referencias en el desarrollo de la teoría de las probabilidades. La ecuación de Laplace, $\Delta \mathbf{u} = 0$, es una de las más famosas de la Física[22]. Basándose en sus estudios mecánicos pensó que el universo funciona como un reloj (determinismo) y declaró que los problemas matemáticos más importantes estaban ya propuestos y resueltos, o a punto de ser resueltos en un corto tiempo. Afortunadamente, la Historia demostraría que el gran hombre erraba en este tema. ¿No recuerda ésto algunos recientes y acalorados debates sobre el fin de la Física o de la Historia?

4 El siglo XIX, el gran siglo de la ciencia

La contribución del siglo XIX a la Matemática, tanto pura como aplicada, es sorprendente por su novedad, por lo inesperado de su evolución y por su riqueza y amplitud de temas. Empecemos por las matemáticas que vinieron de la física.

• **La electricidad y el magnetismo:** De Michael FARADAY a J.C. Maxwell, experimentos y leyes parciales cubren un camino que cuenta con los nombres de Gauss, Ampère, Oersted, Biot, Savart, Lenz,... hasta llegar al (impresionante) sistema de ecuaciones diferenciales en derivadas parciales que relaciona los campos eléctricos y magnéticos (1863), obra cumbre de James Clerk MAXWELL[23]. Las ecuaciones de Maxwell (que no detallaremos en este momento por su complejidad, aunque sin duda merecen lugar de honor en este texto) son uno de los logros mayores de la Matemática en el siglo XIX. Gracias a James Maxwell una nueva rama de la ciencia, cuya existencia era insospechada un siglo antes, alcanzó el nivel de perfección matemática que Newton había otorgado a la mecánica. La teoría electromagnética tendrá profundas repercusiones no sólo sobre las ecuaciones diferenciales y el análisis funcional, sino además sobre la naciente topología (a través de conceptos como la homología)[24]. Elaborando las ecuaciones de Maxwell se llega a la ecuación de on-

JAMES C. MAXWELL

[22]Los ingenieros y científicos aplicados usan la transformada de Laplace.
[23]Publicación en forma final como *Treatise on Electricity and Magnetism*, 1873.
[24]Maxwell es considerado el físico teórico más importante del siglo XIX; Einstein opinaba que el

das, que es la herramienta que nos permite describir la propagación de los fenómenos electromagnéticos en forma de ondas, caracterizadas por tres parámetros: primero, la amplitud A; segundo, la velocidad c que depende del medio (y es por consiguiente constante en el vacío); tercero, la frecuencia de oscilación ω que es una cantidad que varía con el tipo de onda. En breve, y para una dimensión espacial la ecuación y su solución se escriben

$$u_{tt} = c^2 u_{xx} \quad \Rightarrow \quad u = A\cos(kx - \omega t + \phi),$$

donde u es la intensidad de la oscilación, $k = \omega/c$ se llama número de onda y ϕ es una constante, la fase, de la que no debemos preocuparnos por ahora, y los subíndices indican derivadas parciales. Pero veamos, ¿es tan necesaria esta fórmula para proceder? La respuesta es que sí, pues poco después, y como reflejo de la generalidad del parámetro ω en el modelo matemático, Heinrich R. HERTZ predice y descubre las ondas electromagnéticas fuera del rango visible (las ondas de radio, 1888), y Guglielmo MARCONI descubre la telegrafía sin hilos, es decir, la radio (1895), introduciéndonos así al mundo de las comunicaciones que son el alma del siglo XX. Y otra gran sorpresa: aparece una incompatibilidad con la mecánica de Newton sobre la que hablaremos en un momento. Quede dicho esto sobre las consecuencias de la formulación matemática en la evolución de la ciencia.

• **Los fluidos reales**, de Claude Louis NAVIER a George Gabriel STOKES, 1821 a 1856 y después. Las ecuaciones de Navier-Stokes describen los fluidos reales y gobiernan el comportamiento de los fenómenos atmosféricos (el clima, la Meteorología, la Hidrología, la futura Aeronáutica). La formulación correcta de las ecuaciones que describen el movimiento de los fluidos reales tardó por consiguiente unos 180 años tras los esfuerzos de Newton, las matemáticas profundas no se hacen en dos días. Una serie de brillantes matemáticos figuran entre los modelizadores, como S. POISSON y J. C. SAINT VENANT, así como el médico J. L. M. POISEUILLE. que investigó el flujo sanguíneo. Lord KELVIN y H. HELMHOLTZ ponen las bases para el estudio matemático de la vorticidad y los torbellinos. La comprensión matemática de los fluidos turbulentos, ya mencionados por Leonardo, es *todavía un problema abierto.*

Para no alargar excesivamente nuestro texto mencionaremos sólo dos teorías físicas más de gran importancia y repercusión matemática:

• **La Termodinámica** que estudia los intercambios de calor, adquiere una fundamentación matemática sólida con James JOULE, Saadi CARNOT, J. R. MAYER,... Tiene

trabajo de Maxwell representó la revolución más significativa en el estudio de la física desde Newton. La teoría de propagación de ondas es hoy día una de las ramas clásicas de la matemática aplicada, en sus múltiples variantes. Matemático excelente, Maxwell era defensor del método probabilístico en la Ciencia, que él aplicó al estudio de gases (distribución de Maxwell) y se le atribuye la frase: "la verdadera lógica del mundo es el Cálculo de Probabilidades".

una profunda repercusión sobre el cálculo en derivadas parciales y el concepto de diferencial exacta. Esta teoría incluye la famosa Segunda Ley de la Termodinámica (la ley del crecimiento de la entropía en el Universo), una ley fundamental en la ciencia. Mientras que su declaración matemática es simple, su interpretación práctica tiene implicaciones profundas que ocupan a generación tras generación de estudiosos[25].

• Por último mencionemos la **Mecánica Estadística**, asociada a los nombres de Maxwell, L. BOLTZMANN y W. J. GIBBS, que tallaron toda una rama de la Física Matemática basada en el Cálculo de Probabilidades, rama de las matemáticas que había permanecido un tanto al margen de esta aventura científica[26]. Esta idealización matemática del azar había sido elaborada en el fabuloso siglo XVII (ca. 1650) por B. Pascal, P. Fermat y C. Huygens para comprender los juegos de azar, y avanzada luego por BUFFON, BERNOULLI, DE MOIVRE y Laplace entre otros. De repente, el concepto de probabilidad cobra vida para la ciencia física a la hora de modelar el comportamiento de cantidades enormes de partículas[27]. Veamos por qué: las partículas están sujetas evidentemente a las leyes de la mecánica de Newton. Pero, dado que hoy se sabe que el número de moléculas de un gas por litro alcanza la fantástica cifra de $2,69 \times 10^{22}$ en condiciones normales (0^o C de temperatura y 1 atm. de presión)[28], es del todo imposible seguir sus trayectorias individuales. La mecánica estadística propone un comportamiento medio con efectividad sorprendente: de ella es inmediato predecir la relación de la temperatura con la energía y la presión para un gas perfecto, ¡y la predicción ideal resulta ajustada a los datos experimentales! La distribución de Maxwell-Boltzmann, $n = Ae^{-E/kT}$, es un objeto matemático tiene en mecánica estadística un papel tan importante como la distribución gaussiana en la ciencia estadística usual.

Cambiemos de escena para retratar a otro de nuestros héroes, una "vida ejemplar". Bernhard RIEMANN (1826-1866) es una de esas figuras sorprendentes cuya obra

[25]con consecuencias insospechadas: la entropía es hoy día un concepto central en la Teoría de Información tras el trabajo de C. Shannon, *The mathematical theory of communication*, Bell Syst. Techn. Journal **27**, pp. 379-423, 623-658 (1948).

[26]La tumba de Boltzmann en el cementerio central de Viena tiene como ornato su famosa fórmula de la entropía en mecánica estadística, $S = k \log W$, que puede considerarse una gesta del espíritu puro en la búsqueda de la comprensión de los secretos de la Naturaleza. El libro de Gibbs, *Elementary Principles in Statistical Mechanics*, publicado al final de su vida en 1902, jugó para la física estadística un papel similar al de Maxwell para el electromagnetismo

[27]Este no era un paso trivial. Boltzmann contó para ello con su creencia en la existencia de los átomos, que encontró fuerte resistencia en el momento por parte de científicos famosos como E. Mach. ¡Y estamos a finales del siglo XIX! La agria controversia afectó seriamente a la salud de Boltzmann.

[28]En los libros de química suele mencionarse la cantidad de átomos por cada mol = 22,4 l de gas, el llamado número de Avogadro, $6,022 \times 10^{23}$.

contiene lo mejor de la matemática pura y aplicada. El gran matemático alemán, muerto joven, es bien conocido como un gigante de la matemática más pura. Nos legó la hipótesis sobre los ceros de la "función zeta"(*Hipótesis de Riemann*) cuya demostración es quizá el problema abierto de las matemáticas más famoso al entrar el siglo XXI, tras la reciente resolución de la conjetura de Fermat. La hipótesis de Riemann afirma que las soluciones (o ceros) interesantes de la ecuación $\zeta(s) = 0$ están situadas sobre una misma línea recta en el plano complejo, precisamente la de ecuación $Re(s) = 1/2$. Esto se ha verificado para las primeras 1.500.000.000 soluciones[29]. Una prueba de que el aserto es verdad para toda solución aclararía muchos misterios, desde la distribución de números primos a cuestiones de física teórica. Riemann fue un investigador de mente geométrica que ligó la suerte del análisis complejo a las transformaciones conformes y pensó en los espacios generales de varias dimensiones definidos a partir de su geometría local[30]. Hoy día llamamos a esas *geometrías riemannianas* y son la base a partir de la cual se construye la física teórica.

BERNHARD RIEMANN

Pues bien, el mismo Riemann estudio la propagación de gases compresibles y llegó a la conclusión de que el modelo matemático[31], entendido en el sentido de las soluciones clásicas, era contradictorio (porque preveía líneas características que se cortan, y sobre las cuales las variables físicas - densidad, presión y velocidad - tomarían valores distintos simultáneamente). Sin embargo, aventuró que la teoría era correcta si *se cambiaba radicalmente el punto de vista* y se admitían como soluciones de una ecuación diferencial funciones que no sean derivables, ni siquiera continuas. Ante tal atrevimiento, tan típico de las mejores matemáticas de los siglos XIX y XX, recordamos de nuevo a Newton: Riemann no se inventaba esa teoría. La teoría de las *ondas de choque* es hoy día un tema fundamental de la dinámica de gases y de su aplicación a la aeronáutica, y es por ello una de las áreas más activas de investigación matemática en ecuaciones en derivadas parciales, ... y de la ingeniería.

La Evolución Interna. Pero, incluso tras el elogio de Riemann, esta visión sería totalmente injusta si no tuviera en cuenta la evolución interna de las matemáticas,

[29]Para los curiosos de las fórmulas, $\zeta(s) = 1 + 1/2^s + 1/3^s + \cdots$.

[30]Su famoso artículo "On the hypotheses which lie at the foundations of Geometry", 1854. En alemán *"Ueber die Hypothesen, welche der Geometrie zu Grunde liegen"*, 1854, publicado en 1868.

[31]Un sistema de ecuaciones en derivadas parciales no lineal de tipo hiperbólico, para quien desee el detalle.

que habían llegado a un alto nivel de madurez tras 300 años de intenso desarrollo. Solo comentaremos aquí muy brevemente este importante capítulo, pues es más conocido por el público matemático. Varios son los temas estrella, tan inesperados como cargados de futuro: geometrías no euclídeas de J. C. F. GAUSS[32], Janos BOLYAI y N. I. LOBACHEVSKI, fundamentación del cálculo infinitesimal de Augustin L. de CAUCHY, la teoría de funciones de Karl WEIERSTRASS, la lógica matemática de George BOOLE, la teoría de conjuntos de Georg CANTOR, por citar sólo un nombre al lado de cada gran capítulo[33].

CARL F. GAUSS

JOSEPH FOURIER

Existen campos de investigación en que las matemáticas toman claramente el relevo a la física en la tarea de extraer el jugo de un concepto. Esto sucede con el problema de representación de una función como una suma de funciones simples, resuelto por Brook TAYLOR y Colin MCLAURIN para las sumas de potencias y planteado por Daniel Bernoulli (1753) y Leonardo Euler para las sumas trigonométricas que aparecen en las ecuaciones de ondas y el calor. Es gracias a la insistencia de Joseph FOURIER (1822)[34] que los matemáticos se adentran en la aventura de dar un

[32]El "Príncipe de los Matemáticos", quizá el matemático más sobresaliente y conocido de la historia. Hizo contribuciones fundamentales a la teoría de números, al álgebra, a la geometría diferencial, a la geometría no euclídea; la distribución más popular de probabilidad lleva su nombre, así como uno de los teoremas de integración más famosos de la física matemática.

[33]Queremos dejar constancia expresa de la incomodidad que nos causa pasar tan de prisa por temas tan importantes de la Matemática, sin los que muchas de las páginas que seguirán no tendrían sentido.

[34]Escrito de 1807, memoria presentada a la Academia de Ciencias de Paris y publicada en 1822.

sentido riguroso a las sumas infinitas de funciones trigonométricas generales

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\omega x) + b_n \sin(\omega x)\}.$$

Éste es el origen de un área mayor de la teoría de funciones, conocida como Análisis de Fourier. La tarea estaba cargada de grandes dificultades y tuvo grandes éxitos. Así, cuando Paul DU BOIS RAYMOND construyó (1873) una función real continua y periódica cuya serie de Fourier no converge puntualmente, parecía que algo iba realmente mal en el análisis matemático de los fenómenos oscilatorios. Tras cuidadoso examen, tres opciones se planteaban al investigador:

(i) modificar la noción de función,

(ii) modificar la definición de convergencia,

(iii) reemplazar la base de senos y cosenos por candidatos mejores.

Es mérito notable de la comunidad matemática que *los tres caminos* hayan sido explorados con éxito asombroso[35]. El teorema fundamental de sumación de series de Fourier se debe a L. CARLESON, 1966[36], y necesita útiles como *la convergencia en casi todo punto*, los espacios L^2 y la maquinaria del análisis del siglo XX.

El Contexto Social. Es interesante decir dos palabras sobre la evolución social de la ciencia en el siglo XIX. Éste es el siglo en que las revoluciones industrial, burguesa y democrática se asientan en Europa trayendo consigo la extensión de los estudios científicos e industriales tanto en universidades como en otros centros especializados[37], con lo que aumenta exponencialmente el cuerpo de profesores investigadores. Los avances son tan impresionantes que el final de siglo vuelve a encontrar a los matemáticos en franco optimismo, si uno se fía de la historia escrita por el geómetra alemán Felix KLEIN[38]. Otra característica de este período es la profunda separación que se manifiesta entre matemáticos, físicos e ingenieros, consecuencia del enorme

[35] He aquí dos citas de Fourier para animar el debate sobre Matemática Pura contra Aplicada: La primera es "Las ecuaciones del diferencial de la propagación de calor expresan las condiciones más generales, y reducen las preguntas de la física a problemas de análisis puro, y éste es el objeto apropiado de la teoría". Ahora la segunda: "El estudio profundo de la naturaleza es la fuente más fecunda de descubrimientos matemáticos".

[36] *"On convergence and growth of partial sums of Fourier series"*, Acta Math. 116 (1966), pp. 135–157.

[37] Muchas de las Escuelas de Ingenieros se fundan en España en esa época, como las de Montes y Caminos en 1834.

[38] *Lectures on the development of mathematics in the 19th century.* He aquí una cita de Klein: "los grandes matemáticos como Arquímedes, Newton o Gauss siempre unieron teoría y aplicaciones en igual medida".

crecimiento de sus campos de estudio. Tal separación, a veces divorcio, tendría consecuencias profundas sobre la evolución de las matemáticas en el siglo XX, e incluso sobre el mismo concepto de matemática.

5 Un cambio de siglo revuelto

En todo caso, el cambio de siglo es espectacular tanto en física como en matemáticas. En éstas aparecen en el firmamento figuras extraordinarias como Henri POINCARÉ (1854-1912) y David HILBERT (1862-1943), que marcarán profundamente las matemáticas del siglo XX. Pero una gran parte del brillo en retrospectiva se debe a que el cambio de siglo fue una *época de crisis*, pues las evidencias de fenómenos fuera del gran esquema se acumulaban.

HENRI POINCARÉ

DAVID HILBERT

• El experimento de Michelson-Morley (1887) prueba que la velocidad de la luz es efectivamente constante (independientemente del sistema de referencia inercial), como predecía la teoría ondulatoria basada en las ecuaciones de Maxwell. El modelo mecánico del mundo de Euclides-Newton tiene por primera vez una gran grieta.

• La observación de las partículas suspendidas en los gases revela un movimiento altamente irregular, el movimiento browniano (Robert BROWN, 1827). Este es un golpe para la geometría de Euclides basada en puntos, rectas y curvas regulares (al menos regulares a trozos).

• Las sorpresas de la teoría de funciones llevan a la teoría de conjuntos (Georg Cantor) que junto con la lógica (George Boole, Gottlob FREGE, Giuseppe PEANO) son la base de un intento de fundamentar las matemáticas rigurosamente de una vez por todas. Las matemáticas proponen a la ciencia los conceptos de teoría *coherente*[39] y *completa*. Surgen las escuelas y las disputas: logicismo (Alfred N. WHITEHEAD y Bertrand RUSSELL[40]), intuicionismo (Luitzen BROUWER) y formalismo (David Hilbert). Las paradojas (de Russell, de BURALI-FORTI, de RICHARD) siembran un caos notable en los espíritus menos fuertes.

• No existen útiles analíticos ni computacionales para abordar las complejidades de las ecuaciones de los medios continuos, como los fluidos. En consecuencia, las matemáticas prácticas de la ingeniería se sumen en una serie de aproximaciones y recetas que las divorcian de la teoría.

• Pero incluso el tema clásico de la integración general de las ecuaciones del movimiento para tres o más cuerpos celestes se muestra imposible[41]. A grandes males grandes remedios: H. Poincaré propone los métodos cualitativos y abre las puertas a la geometría algebraica y la topología (llamada entonces *Analysis Situs*, 1895). Pero al tiempo descubre con sus métodos teóricos una tremenda complejidad escondida en el modelo matemático (que son los sistemas dinámicos). Uno de estos monstruos son las órbitas homoclínicas que sembrarán de *caos* la mecánica celeste cuando Poincaré sea bien comprendido (lo que llevó bastantes décadas). Para mejor medir la estatura de nuestro héroe valga la siguiente cita: "en sus cursos en la Facultad de Ciencias de París desde 1881, y de la Sorbona desde 1886 Poincaré cambiaba de tema cada año, tocando la óptica, la electricidad, la astronomía, el equilibrio de los fluidos, la termodinámica, la luz y la probabilidad".

• Agreguemos algunas notas más optimistas. Así, la teoría de la integración de funciones se ve coronada por los trabajos de E. BOREL y H. LEBESGUE[42]. En adelante el cálculo posee un concepto de integral (la integral de Lebesgue) donde el proceso de tomar límite es natural, el análisis funcional puede crecer (espacios de Hilbert) y el famoso problema de DIRICHLET[43] tiene solución (en un sentido aún visto como raro). El precio a pagar es la construcción de una teoría matemática sofisticada que los estudiantes de ciencias e ingeniería deben estudiar y absorber, o al menos han de aprender a convivir con ella[44].

• Descubrimientos importantes de naturaleza matemática ocurren en otras ciencias y darán fruto en el próximo siglo. El Científico ruso Dmitri MENDELEYEV

[39] *consistent* en inglés.

[40] su famoso libro *Principia Mathematica* data de 1910.

[41] como expone H. Poincaré en su libro *Méthodes nouvelles de la mécanique céleste,* Paris, 1899.

[42] La importantísima contribución a la teoría de la integración figura en su tesis doctoral, *Intégrale, longueur, aire,* Universidad de Nancy, 1902.

[43] Nombrado en honor a P. L. Dirichlet, el primero que probó que la serie de Fourier converge bajo ciertas condiciones

[44] parafraseando a J. von Neumann. *Ad astra per aspera*, dice el adagio latino.

encontró el orden en el caos de los elementos químicos y propuso la Tabla Periódica en 1869, que es hoy día la base del tratamiento físico-matemático de la Química. Por otro lado, el monje, botánico y experimentador de las plantas austriaco, Gregor J. MENDEL formuló las leyes racionales de la herencia, poniendo así los fundamentos matemáticos de la ciencia de la Genética[45].

6 El siglo XX, un siglo de maravillas

A estas alturas, esperamos haber comunicado al lector la impresión de la profunda simbiosis de la Matemática con la Física, de sus sorprendentes y en muchos casos inesperadas interacciones. La historia tal simbiosis incluye ya aplicaciones tecnológicas avanzadas, preludio de lo que será el nuevo siglo. La explosión de la Matemática y la Ciencia en el siglo XX hace aconsejable reducir nuestro texto a algunos de los temas más importantes. Un rasgo sobresaliente es la matematización progresiva de las demás ciencias, que aparecen ya como nuevos horizontes para la Matemática Aplicada.

Nuevas matemáticas que nos llegaron de la Física

El comienzo del siglo XX es testigo de dos grandes revoluciones en la manera de concebir el mundo físico, que cambiaron de forma radical el "universo newtoniano". Comprobado el hecho de que la luz no se comporta como era esperado, la teoría que lo explica trae consigo consecuencias dramáticas sobre nuestro concepto de espacio-tiempo, que afectan en la práctica a la Astronomía y al comportamiento de las partículas que se mueven deprisa. Por otra parte, en el extremo de lo muy pequeño, se observó que los átomos, moléculas y partículas subatómicas tampoco obedecen a las leyes de comportamiento tan cuidadosamente observadas por los entes macroscópicos, aunque por otras razones. Son *dos grandes revoluciones cuya más íntima esencia se expresa en fórmulas matemáticas.* Examinemos con algún detalle el surgir de ambas teorías.

• **La Teoría de la Relatividad**. Albert EINSTEIN, el Hombre del Siglo según *la revista Time* (año 2000), propuso las dos versiones de la relatividad: en 1905[46] (la relatividad especial) y en 1916 (la relatividad general). Esperamos no sorprender al lector al afirmar que en ambos casos se trata de una profunda reflexión sobre las matemáticas que sirven de base a la Física.

La relatividad especial tiene como precursores a LORENTZ, Poincaré y MINKOWSKI, que estudiaron el grupo de invariancia que corresponde a la nueva geometría del

[45] *Versuche über Pflanzenhybriden,* ("Experimentos con híbridos de plantas"), publicado en 1886.

[46] 1905 fue el *annus mirabilis* para Einstein. En tres artículos separados explicó el efecto fotoeléctrico, el movimiento browniano y la teoría de la relatividad. Es improbable que tal hecho vuelva a repetirse.

espacio-tiempo. La relatividad general usa los conceptos geométricos que Riemann elaboró más de un siglo antes como un puro *Gedankenexperiment*, es decir experimento mental, sobre las "hipótesis que subyacen a los fundamentos de la geometría", y que fue desarrollado por la escuela de geometría diferencial italiana de RICCI, LEVI-CIVITA y BIANCHI. La relatividad será un gran campo de juego de la geometría diferencial en el siglo XX. De las ecuaciones de Einstein se llegará al Big Bang y a los agujeros negros (OPPENHEIMER y SNYDER, 1939; PENROSE y HAWKING). Todo un ejercicio de matemática pura como modelo de una rama de la física.

ALBERT EINSTEIN

Conviene sin embargo no olvidar la otra cara de la Relatividad: desde la primera confirmación experimental de Lord A. EDDINGTON en 1919 incesantes experimentos han servido para confirmar (mejor diríamos, con la modestia de Einstein, no refutar) la teoría de la Relatividad. Pues en la ciencia real no se inventan las hipótesis[47].

Hagamos una pausa para echar una mirada a algunas de las fórmulas principales. En Septiembre de 1905 Einstein publicó un corto artículo en que demostró la fórmula fundamental $E = m\,c^2$ sobre la equivalencia matemática de masa y energía, que se ha convertido en un clásico de la cultura popular del siglo XX. Por otro lado, las leyes de transformación de la Relatividad Especial, que reemplazan a las leyes de transformación galileanas a velocidades relativas altas, conocidas como las leyes de transformación de Lorentz, son :

$$x = \gamma x' + \gamma v t', \qquad t = \gamma t' + \frac{v}{c^2}\gamma x',$$

donde la constante γ se llama factor de dilatación del tiempo. Depende de la velocidad relativa v y viene dado por la expresión: $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Por consiguiente, la

[47]He aquí una opinión significativa de Einstein sobre las matemáticas: "Mathematics deals exclusively with the relation of concepts to each other without consideration of their relation to experience. Physics too deals with mathematical concepts; however, these concepts attain physical content only by the clear determination of their relation to the objects of experience", de *The theory of Relativity*, 1950. Las opiniones de Einstein son tanto más interesantes si se tiene en cuenta que, contrariamente a otras grandes figuras en la historia de la Física, como Newton o Maxwell, no fue matemático excepcional, por lo menos técnicamente. Dejó sin embargo un legado impresionante a las matemáticas a través de sus teorías.

suma de velocidades sigue la sorprendente regla

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}},$$

muy en contra de lo que estamos acostumbrados a creer (es decir, $u = u' + v$).

La fórmula más conocida de Einstein es sin duda $E = m\,c^2$, que forma con la fórmula cuántica de Planck, $E = h\,\nu$, toda una nueva visión de la energía al principio del siglo. La energía había sido uno de los conceptos clave de la evolución de la física y las matemáticas que la acompañan en el siglo XIX, y se ve sometida a profunda revisión matemática en los comienzos del siglo XX. Precisamente, los *quanta* (o cuantos) son nuestro próximo tema.

• **La Mecánica Cuántica** describe el comportamiento de la materia y la luz a la escala atómica. En palabras del gran físico R. FEYNMAN *"Things on the very small scale behave like nothing you have any direct experience about"*. En particular, asistimos a otra enorme brecha en el hasta entonces perfecto edificio de la mecánica newtoniana. El segundo recorrido mágico[48] del comienzo del siglo XX nos lleva de la hipótesis de los quanta de MAX PLANCK, 1900, a la ecuación de SCHRÖDINGER (1926) pasando por N. BOHR, L. DE BROGLIE, W. HEISENBERG y P. A. M. DIRAC. El acceso al mundo atómico queda codificado en la maravillosa ecuación

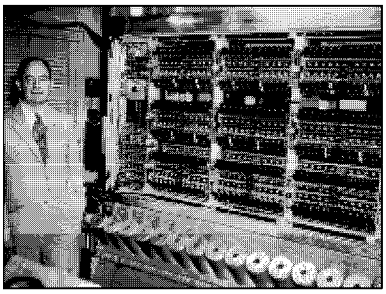
$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x,y,z,t)\psi,$$

donde $\hbar$ es la constante de Planck reducida, $\hbar = h/2\pi$, $i = \sqrt{-1}$, Δ es el operador laplaciano y $V = V(x,y,z,t)$ es el potencial. Todo ello parece realmente un trozo de la Cábala, y en el momento inicial se dudaba de qué representaba exactamente la variable $\psi(x,y,z,t)$ llamada "función de onda". Tal es el poder de la Matemática, estos físicos geniales habían encontrado un trozo del Código Matemático del Universo pero aún habían de interpretar qué significaban las variables. En 1928 Max BORN propuso la interpretación probabilista, donde $|\psi(x,t)|^2$ es la densidad de probabilidad de encontrar la partícula en el lugar x en el instante t, y aunque es mayoritariamente admitida, hay quienes se resistieron, siguiendo a Einstein en eso[49]. Porque la Mecánica Cuántica es un desafío fundamental a la manera previamente admitida de mirar el mundo, al determinismo tradicional y a la causalidad. Se puede decir que el determinismo está basado en el supuesto de que "el conocimiento exacto del presente permite calcular el futuro". ¿No es ése el sueño de las ciencias exactas, y no es cierto que la Mecánica Cuántica subvierte esa creencia? Ponderando el problema,

[48]Cita homenaje a "The Magical Mystery Tour", Lennon y McCartney, 1967.
[49]Suyo es el famoso comentario: *"God does not play dice"*, "Dios no juega a los dados".

W. Heisenberg encontró en 1927 la respuesta siguiente: *"no es la conclusión [de la hipótesis determinista] lo que es falso, sino la hipótesis inicial"*.

Dejando al lado el mundo de las interpretaciones, debemos informar que esta teoría, aun estando basada en el más alto nivel de abstracción matemática, es confirmada por todo un siglo de experimentos. La parte mágica, que tanto abunda, tiene un momento estelar cuando Dirac, usando la formulación relativista, propone la existencia de los positrones (1932) porque "las ecuaciones admiten el cambio de signo con respecto a la solución que describe el electrón",... y el positrón fue debidamente descubierto[50] por los físicos experimentales poco después (Anderson y Blacket, 1932-33). Dirac predijo la existencia del antiprotón que fue confirmado por Segrè en 1955, y también el monopolo magnético, pero esta vez su existencia ha quedado sin confirmación hasta el momento presente. Las predicciones de Dirac son un ejemplo notable, de ninguna manera único, en que el modelo matemático va delante de la evidencia experimental[51]. ¿No nos recuerda todo esto a Hertz?



J. V. NEUMANN

La cosecha matemática de la Mecánica Cuántica no es escasa: la teoría de operadores autoadjuntos en espacios de Hilbert con su correspondiente teoría espectral son desarrolladas por John VON NEUMANN (Janos v. N., 1903-1957), uno de los genios más polifacéticos del siglo[52], con el objeto de dar sentido a los operadores que aparecen en la ecuación, operadores laplacianos y demás. Su teoría se basa en el trabajo precursor de S. BANACH y los expertos italianos en cálculo de variaciones, pero la

[50]¿O deberíamos decir mejor "encontrado" o "'reconocido"?

[51]Por otro lado, la ciencia basada solamente en argumentos o analogías matemáticos puede ser mala ciencia. Un ejemplo: existe una tendencia matemática a afirmar que en el reino de partículas ciertas simetrías matemáticas son "ley" de la naturaleza. En particular, debería ser entonces correcta la ley de conservación de la paridad, que especifica que las partículas elementales y sus imágenes especulares *deben* comportarse idénticamente; en 1956-57 tres sino-americanos T. D. Lee, C. H. Yang, y C. S. Wu conjeturaron primero y probaron después que hay procesos subatómicos que violan esa ley.

[52]J. von Neumann, *Mathematische Grundlage der Quantenmechanik,* "Fundamentos Matemáticos de la Mecánica Cuántica", Springer, 1932. La trayectoria de Von Neumann recorre las áreas más diversas de la Matemática pura y aplicada: en su juventud modificó el sistema Zermelo-Fraenkel de la teoría de conjuntos, creó las álgebras de v.N. en teoría de operadores, es el padre de la Teoría de Juegos y lo veremos luego en el Instituto para Estudios Avanzados de Princeton como uno de los padres del primer gran ordenador moderno. Después de la guerra se ocupó de la hidrodinámica, de los métodos numéricos (Monte Carlo, estabilidad para los esquemas en diferencias finitas), la teoría de autómatas, y así sucesivamente.

Mecánica Cuántica tiene sus caprichos: necesita unos objetos de la segunda generación, los "operadores lineales no acotados en espacios de Hilbert". Estamos pues en el borde o más allá de los temas de la licenciatura en Matemáticas, lo cual es información interesante para quienes sostenían *que toda matemática útil ha de ser muy fácil*[53]. Junto con el Cálculo de Variaciones, la Mecánica Cuántica ha sido cantera inagotable de problemas para el Análisis Funcional, rama de las matemáticas que toma vuelo propio.

Por otra parte, el comportamiento anómalo de las partículas cuánticas respecto a las clásicas tiene aspectos matemáticos simples y relevantes, como su distinto comportamiento estadístico, que lleva a las distribuciones de Bose-Einstein y Fermi-Dirac que "corrigen" a Maxwell-Boltzmann.

Las matemáticas que vinieron de la ingeniería

• **La Aeronáutica**. Tras los impresionantes avances de la física matemática del siglo XIX y en particular de la mecánica de fluidos, pudiera parecer que un problema antiguo como el del vuelo, que ya había ocupado a Leonardo da Vinci, debería estar resuelto. Y los experimentos con globos habían tenido éxito un siglo antes[54]. Además la teoría de la variable compleja y de los flujos potenciales y vorticosos había obtenido un notable progreso. Pero con todo este progreso, el vuelo propulsado (por un motor) no era entendido ni practicado, y un desanimado Lord Kelvin reconocía a finales de siglo XIX que el sueño del vuelo propulsado era quizá imposible[55]. Es entonces cuando *el método experimental es reivindicado* por los hermanos Wilbur y Orville WRIGHT, fabricantes de bicicletas y consumados experimentadores, que logran volar en un artefacto propulsado en las inhóspitas playas de Kitty Hawk, Carolina del Norte, en la desapacible mañana del 17 de diciembre de 1903. Es el nacimiento de la Aeronáutica. La reacción de los teóricos fue fulminante y a la altura del desafío. Durante el periodo 1905-10 los principales ingredientes matemáticos que faltaban al modelo teórico fueron comprendidos (N. E. ZHUKOVSKI, M. KUTTA, L. PRANDTL, S. A. CHAPLYGIN). Se trata de los conceptos de sustentación, circulación, capa límite, separación, régimen laminar y turbulento. Una ingeniería nace y nos llevará en 30 años más allá de la barrera del sonido. Y nacen ramas de la matemática aplicada, como la teoría de las perturbaciones singulares, la teoría de los flujos supersónicos y

[53]Me refiero en particular a las opiniones del famoso matemático inglés G. H. Hardy en su libro *A Mathematician's apology*, [15], que refleja puntos de vista muy distintos de los sostenidos en este artículo, ver especialmente sección 26. Es un libro muy conocido y de un gran interés. El tiempo no parece haberle dado la razón en el tema que nos ocupa. Debe tenerse en cuenta que en 1940 la relevancia práctica de las teorías sofisticadas como la Mecánica Cuántica podía muy bien no ser evidente, como lo es hoy para el lector avisado.

[54]Hermanos Montgolfier, 1783.

[55] *"heavier-than-air flying machines are impossible"*, dijo en 1895.

transónicos y la teoría matemática de la combustión[56].

Resistimos aquí a la tentación de detallar las otras ramas de la ingeniería que también han tenido una interacción activa con las matemáticas. Lo cual no significa en absoluto que ignoremos su importancia, trataremos el tema en la sección 8.

Grandes novedades que vinieron de las matemáticas

Las matemáticas han vivido el siglo XX muy pendientes del desarrollo interno de las ideas recibidas del fabuloso siglo anterior. Para más fortuna, el siempre difícil y en general fallido intento de prever las líneas del futuro contó con una confirmación en la famosa propuesta de D. Hilbert al II Congreso Internacional de Matemáticos, celebrado en París. En 23 problemas Hilbert resumía los principales retos con que se enfrentaban las matemáticas, desde las más puras a la física matemática[57], cf. la referencia [17]. Esos 23 problemas han sido de gran importancia en el transcurso de los años, pero otras líneas inesperadas han venido a complementarlos y competir por las candilejas. Señalemos tres desarrollos importantes entre tantos.

• **El cálculo de probabilidades**. Como respondiendo a la necesidad planteada por la mecánica cuántica, pero en realidad independientemente, Andrei N. Kolmogórov estableció en Moscú la probabilidad axiomática[58] sobre la teoría de conjuntos y la teoría de la medida, tarea a la que se asocian los nombres de P. Lévy en Francia y N. Wiener en EE.UU. Hemos de recordar aquí que Boltzmann fue un estudioso del movimiento browniano, que L. Bachelier escribió su tesis en Paris en 1900 en un intento (infructuoso de momento) de modelar los mercados financieros, y que Einstein recibió el premio Nobel en 1921 no por la teoría que le hizo famoso sino por sus estudios del efecto fotoeléctrico y... del movimiento browniano. Las cadenas de Markov habían sido estudiadas desde 1900 por A. A. Markov.

Hoy día la teoría de los procesos estocásticos, en particular los procesos de Markov, es una de las áreas predilectas de esta floreciente rama de las matemáticas, y el Cálculo de Itō es una herramienta esencial del análisis estocástico continuo que compite con el cálculo infinitesimal clásico de Newton y Leibniz. Todo este desarrollo era completamente desconocido, incluso insospechado, hace poco más de un siglo y se ocupa de *informarnos sobre los fenómenos aleatorios y su evolución probable*, es decir nos permiten *hacer predicciones sobre lo no exacto*. Como es ya usual en nuestro

[56]Más hacia la matemática teórica tenemos la teoría matemática de la explosión para las ecuaciones diferenciales no lineales, de tanta actualidad. Permítasenos agregar que, aunque la práctica de la ingeniería aeronáutica descansa en bases teóricas firmes, las matemáticas profundas involucradas están lejos de ser bien entendidas y la investigación es muy activa

[57]Debe decirse empero que éste último tema estaba relativamente mal representado, y Hilbert dedicó mucho esfuerzo al asunto en los años siguientes.

[58]Su libro titulado *Grundbegriffe der Wahrscheinlichkeitsrechnung*, "Fundamentos del Cálculo de Probabilidades", es publicado en 1933.

relato, se trata de un empeño no sólo académico, sino que tiene aplicaciones muy importantes en los procesos científicos, industriales y financieros.

• **El caos determinista**. El estudio del caos generado por las ecuaciones diferenciales, ya anunciado por Poincaré, cuyas matemáticas habían madurado gracias al impulso de diversos matemáticos, especialmente G. BIRKHOF, ha de esperar a la obra de un físico dedicado a los estudios del clima para adquirir el impulso definitivo. En efecto, se atribuye a Edward LORENTZ, del MIT, ese mérito [59]. Preocupado por el estudio de los procesos convectivos en la atmósfera propone un simple modelo no lineal consistente en 3 ecuaciones diferenciales ordinarias que no me resisto a copiar

$$\begin{cases} x' = -10x + 10y, \\ y' = 28x - y + xz, \\ z' = \frac{8}{3}z + xy. \end{cases}$$

Para esta elección de los parámetros (es decir, los coeficientes de la ecuación, que pueden ir variando en el problema) encuentra sorprendido que las trayectorias numéricas que produce su ordenador no convergen a ninguna situación periódica. El artículo de 12 páginas data de 1963. Surgen conceptos que llegarán al gran público, como *caos determinista* y *atractores extraños*, y toda una rama de las matemáticas tanto teóricas como experimentales, una gran novedad posible gracias al desarrollo de los ordenadores. Autores como S. SMALE y M. FEIGENBAUM se hacen célebres[60]. Entran en escena los *conjuntos fractales* de B. MANDELBROT[61], ya anunciados en la obra de G. JULIA en los años 20[62]. Hurgando en la historia se descubre como precursor la figura gigante de H. Poincaré que había previsto este caos en su cabeza.

El estudio de los procesos caóticos, fractales y turbulentos es una de las fronteras del pensamiento matemático actual.

• **Nuevos conceptos de solución en las ecuaciones diferenciales.** Hacia los años 30 era claro para muchos investigadores que el concepto clásico de solución era insuficiente para construir una teoría de las ecuaciones diferenciales que satisfaga las necesidades de las ciencias a las que se aplican. En efecto, es natural en esta disciplina plantear *problemas*, es decir conjuntos de ecuaciones y datos adicionales, que sean *bien propuestos*; siguiendo a J. HADAMARD ello quiere decir que tales problemas han de tener una solución, que ésta ha de ser única si se dan datos suficientes, y que además tal solución ha de depender continuamente de los datos. No se trata ya de que la solución sea clásica, pues ésta puede no existir o puede que no sea el concepto de solución cuya existencia resulta natural demostrar.

[59] Su famosa publicación *"Deterministic non-periodic flow"*, J. Atmos. Sci **20** (1963), 130–141.
[60] cf. Ian Stewart, *Does God play dice? The New Mathematics of Chaos*, Penguin, Londres, 1989.
[61] cf. B. Mandelbrot, *The fractal geometry of Nature,* 2nd ed., San Francisco, 1982.
[62] Su publicación data de 1918

Enfrentados con este reto, los matemáticos han desarrollado diversas nociones de *soluciones generalizadas* con significado físico. Quizá el ejemplo más notable haya sido el problema de minimización de energía de Dirichlet ya mencionado[63], motivación de los espacios de Hilbert. Otro ejemplo básico es el problema de Riemann de la dinámica de gases, ya mencionado. Un tercer problema similar lo afronta J. LERAY[64] en 1933 en el estudio de las soluciones de las ecuaciones de Navier-Stokes de los fluidos reales (viscosos) en el espacio tridimensional. Gracias al trabajo de los analistas funcionales (S. L. SÓBOLEV, L. SCHWARTZ,...) se introducen los conceptos de *solución débil* y *solución en el sentido de las distribuciones.* Resumiendo mucho, no se pide a las soluciones que posean todas las derivadas implícitas en la ecuación sino que cumplan con ciertos tests. Con los expertos en leyes de conservación (P. LAX, O. A. OLEINIK, S. N. KRUZHKOV) se llega a las *soluciones de entropía*, que no son siquiera continuas (y se recupera así el legado de Riemann, Rankine y Hugoniot y sus ondas de choque).

En nuestros días aparecen nuevos conceptos de solución para satisfacer las crecientes necesidades, como las *soluciones viscosas* de M. G. CRANDALL, L. C. EVANS y P. L. LIONS. L. CAFFARELLI extiende este concepto a los problemas de cambio de fase o frontera libre, donde la discontinuidad es parte fundamental del planteamiento matemático. Y la saga continua con las soluciones *mild*, soluciones de semigrupos, soluciones renormalizadas,...

Uno de los aspectos más llamativos de estos nuevos conceptos es su compatibilidad con las *soluciones numéricas* propias de los métodos discretos del cálculo numérico. Se halla así una sorprendente alianza de los conceptos abstractos y los numéricos contra "la rigidez de los clásicos". Por otra parte, el Análisis Funcional pasa a formar parte del currículo básico del matemático aplicado y el ingeniero.

Las matemáticas y la vida social: la teoría de juegos

La teoría de juegos analiza los "juegos", es decir, situaciones en que se da un conflicto de intereses. Parte de los juegos más simples, pasatiempos que pueden ser analizados completamente; de ellos se pasa a los "juegos reales" como el póker o el ajedrez, y de ahí a los complejos problemas de estrategias en áreas de enorme interés social como la economía o la política. Vemos en ello un gran paralelismo con el proceder del cálculo de probabilidades y la estadística, que pasan de los juegos de azar con cartas o bolas a la estadística industrial y social por un lado, y al comportamiento de los gases o los átomos por otro.

[63]Se trata de minimizar la integral de energía $\int_\Omega |\nabla u|^2 dx$ entre todas las funciones admisibles $u = u(x)$ definidas en un recinto del espacio Ω y que toman valores asignados en el borde de Ω; ∇u designa el gradiente de u. El problema de principio, crucial para la correcta solución, es qué se entiende por función *admisible*.

[64]Jean Leray publicó tres artículos sobre el asunto en 1933-34. El último es el *"Essai sur les mouvements planes d'un liquide visqueux emplisssant l'espace"*, Acta Math. **63**, 1934.

El primer teorema en teoría de juegos es atribuido a E. ZERMELO, fundador de la versión de la teoría de conjuntos ZF hoy tomada por estándar, y se titula "Sobre una utilización de la teoría de conjuntos en la teoría del ajedrez", 1913[65]. Yendo hacia atrás en el tiempo, el primer libro de las matemáticas de la competición parece ser de Augustin COURNOT en 1838[66]. Otro conocido matemático, E. BOREL, escribió sobre juegos de estrategia en el período 1921-27 y dio una prueba restringida del teorema del minimax, uno de los resultados más importantes de la matemática aplicada del siglo XX al decir de Casti [4][67].

Pero son dos grandes figuras quienes asientan las matemáticas de la competición en el siglo XX. Uno es J. VON NEUMANN, que demuestra en 1928 el teorema del minimax y analiza en su famoso libro con MORGENSTERN, 1944, los juegos cooperativos y de suma cero[68]. El otro es J.F. NASH[69] que en cuatro artículos fundamentales de 1950-53 establece la teoría de los juegos no cooperativos[70]. Los conceptos de equilibrio dominante y equilibrio de Nash son hoy día herramientas matemáticas básicas de la práctica económica y política (en sus diversas vertientes de elección social) y deberían ser mejor conocidos por el gran público. J. Nash recibió el Premio Nobel de Economía en 1994 y es uno de los pocos Premios Nobel Matemáticos, junto con los economistas J. TINBERGEN[71], L. KANTOROVICH y SELTEN[72].

La Economía Matemática desborda evidentemente el tema de los juegos, la competición y las estrategias, que forman el reino de las matemáticas de la llamada Microeconomía. Después hablaremos brevemente de las matemáticas del mercado financiero.

En la Teoría de la Elección Social es importante el Teorema de Imposibilidad de

[65] *"Ueber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels"*, 1913, pp. 501-504 en los Proceedings of the Fifth International Congress of Mathematicians, Vol. II (E. W. Hobson and A. E. H. Love, eds.), Cambridge University Press).

[66] El libro se titula "Recherches sur les Principes Mathématiques de la Théorie des Richesses", título de lo más prometedor. Hemos de mencionar La Teoría de la Evolución de Darwin, que toca en un sentido el tema con su selección natural, que produce situaciones de equilibrio.

[67] Sus cinco favoritos son la teoría de juegos, el teorema del punto fijo, el problema de parada de Turing, el método simplex y ... se ruega al lector que consulte el libro.

[68] "Theory of games and economic behaviour", J. von Neumann and O. Morgenstern,

[69] Famoso también por sus trabajos en geometría y en ecuaciones en derivadas parciales y por su azarosa biografía reflejada en un filme reciente.

[70] Entre ellos J. F. Nash, "Non-Cooperative Games", 1951, Annals of Mathematics. "Two-Person Cooperative Games", 1953, Econometrica.

[71] Tinbergen es importante en nuestro relato, pues fue uno de los primeros propulsores de la modelización matemática más allá de los confines de la física; T. vio que las aplicaciones de las matemáticas podían afectar a muy diversas áreas.

[72] Otros científicos galardonados que han aparecido en nuestro relato son Lorentz, Raleigh, Planck, Einstein, Bohr, de Broglie, Heisenberg, Schrödinger, Dirac, Born y Feynman en Física y Lord Russell en Literatura.

ARROW[73], que pone un límite a las capacidades de los sistemas axiomáticos de elección, aplicando a la ciencia social las ideas de los célebres resultados de indecibilidad e incompletitud de Kurt GÖDEL (1931) para la aritmética formal, uno de los resultados más notables de la Matemática del siglo XX[74]. El resultado de Gödel trata de la indecidibilidad intrínseca a todos los sistemas formales que incluyan la aritmética, tema de Lógica y Fundamentos de la Matemática de apariencia eminentemente pura y por ello de nula interacción con el mundo práctico si hemos de creer a los fervientes defensores del aislamiento esencial de las matemáticas puras. Pues bien, volveremos a hablar de él en el próximo tema, que trata de ordenadores, de la mano de otro de nuestros héroes, A. Turing.

7 Ingeniería y matemáticas en la última revolución del siglo. Los ordenadores y la matemática computacional

La realización práctica del viejo sueño de construir una máquina de calcular toma cuerpo en forma del moderno ordenador que acredita dos orígenes, la Tecnología y las Matemáticas, los cuales confluyen en un fantástico invento en el año 1946[75]. Por una parte tenemos el viejo proyecto de la máquina de calcular, pensada ya en el siglo XVII por B. Pascal[76] y G. Leibniz[77], y que debe tanto a Ch. BABBAGE a principios del siglo XIX[78] proyecto que es realizable en el siglo XX de forma eficiente gracias al avance de la electrónica: primero el tubo de vacío y luego una espectacular saga de progresos técnicos que nos llevan al semiconductor, a la miniaturización y al *chip*[79].

Pero el ordenador o computadora no nace como máquina de calcular pasiva, sino

[73]Kenneth J. Arrow, trabajo doctoral en 1948-49 publicado en *Social Choice and Individual Values*, 1951. En 1972 Arrow recibió el Premio Nobel de economía por sus contribuciones al estudio del equilibrio económico y la elección social.

[74]La incompletitud de los sistemas formales fue publicada en *"Ueber formal unentscheidbare Saetze der Principia Mathematica und verwandter Systeme"*, "On formally undecidable propositions of Principia Mathematica and other related systems".

[75]Con esta fecha hago referencia al ordenador ENIAC.

[76]Su *machine à calculer*, la *Pascalina*, se hizo famosa.

[77]Leibniz pensó en la dirección del álgebra, la lógica simbólica y el lenguaje universal. Recientes investigaciones históricas indican que una cierta primacía de tales máquinas calculadoras se debe a otro alemán, Schickard, 1623, pero su máquina no llegó a funcionar.

[78]Babbage trabajó toda su vida en un proyecto mecánico, la *Analytical Machine*, el precursor del moderno ordenador electrónico, con la notable ayuda de Ada Byron, Lady Lovelace, hija del poeta y matemática.

[79]El circuito integrado fue inventado por R. Noyce y J. Kilby en 1958.

que nace con un programa. Esta es la herencia de la lógica matemática, desde G. Boole con su álgebra al programa de formalización de las matemáticas de D. Hilbert, que lleva a la prueba de indecibilidad e incompletitud de Kurt Gödel en 1931 que destruye el sueño de Hilbert de una matemática de demostraciones automáticas. Ello provoca el interés de otro matemático genial, ALAN TURING (1912-1954), que traduce el programa de formalización al lenguaje de las máquinas, 1937[80], e inventa con Alonzo CHURCH la teoría de la computabilidad, años antes de que el ordenador viera la luz.


A. TURING

Sigue un momento histórico: el esfuerzo de guerra, el desciframiento del código alemán Enigma, ... Entra en escena von Neumann con la idea del programa en memoria, y se construye el ENIAC en 1946[81].

La computadora moderna surge como una máquina calculadora eficaz con cuatro características: es de utilidad general, electrónica, digital y programable; *las dos últimas propiedades se relacionan directamente con las matemáticas.* La primera computadora comercial, UNIVAC, funcionó en 1951. En estos 50 años se pasa de las grandes máquinas (armatostes) que manejan kilobytes o megabytes a los ordenadores personales con capacidad de decenas de Gigas y a la WWW. La dualidad en el mundo del ordenador continua en forma de la famosa pareja Hardware y Software[82].

El mundo computacional, un nuevo mundo para las matemáticas.
El mundo del ordenador está cambiando poco a poco la vida diaria del ciudadano: las transacciones bancarias, el correo electrónico, la reserva de pasajes, ... Su efecto sobre las matemáticas, menos conocido por el gran público, es aún más dramático. Aparecen por un lado las nuevas ramas de la Matemática Computacional teórica, como la teoría de la computabilidad y la complejidad y la teoría de autómatas y lenguajes formales. Pero todas las ramas de la matemática pura y aplicada se contagian de la repentina

[80] *"On Computable Numbers, with an application to the Entscheidungsproblem"*, Proceedings of the London Mathematical Society.

[81] Las siglas ENIAC significan *Electronic Numerical Integrator and Computer*, construido por J.W. Mauchly y J.P. Eckert en la Univ. de Pennsylvania; hoy es reconocido el trabajo pionero de J.V. Atanasoff. Mención merecen también el Colossus inglés, 1942, y las máquinas alemanas Z1 a Z4, cf. la ref. [25]. Todas estas máquinas tenían un propósito militar.

[82] Los ordenadores personales aparecen en 1977 y, en contra de las predicciones de los gurús, han ocupado la escena, gracias sin duda al progreso impresionante del hardware: un chip puede contener al final del siglo XX unos 10^9 transistores

capacidad para calcular efectivamente lo que antes era sólo imaginable, y este nuevo gusto se propaga como una infección (potente pero benigna) en la práctica cotidiana de las matemáticas: matemáticos, científicos e ingenieros calculan órbitas de satélites o trayectorias de sistemas dinámicos, distribuciones numéricas o series temporales de procesos reales, mapas climatológicos o estudios de singularidades, distribución de temperaturas en un alto horno o propiedades estadísticas de los ceros de la función Zeta de Riemann,... Y la finanza y la administración también calculan.

Entre los notables cambios acaecidos, las matemáticas tienen un papel importante en los procesos industriales u otros en que se combina la experimentación en laboratorio con las nuevas herramientas matemáticas: aparece la combinación de **modelización matemática - análisis matemático - simulación numérica y visualización - control,** que forma una herramienta de uso habitual en los más diversos campos: las comunicaciones, la predicción del tiempo, la astrofísica, la ingeniería minera, industrial, la industria del automóvil y del petróleo, los problemas medioambientales y la ecología, la economía y las finanzas, las comunicaciones, y en este momento la biología y la medicina, como veremos con algún detalle en la sección 8. Esta área de las matemáticas tiene como tarea *aproximar de una manera eficaz* las soluciones de modelos matemáticamente muy sofisticados y complejos. El interés por su desarrollo y aplicación da lugar a los grandes Institutos y Centros de Cálculo.

Los nuevos conceptos: modelo numérico, simulación numérica, experimento o exploración numérica, visualización dinámica,... se hacen de uso diario en el medio científico e industrial. El desarrollo de métodos de formulación numérica de los modelos continuos, como las ecuaciones diferenciales, es una rama fundamental de la matemática computacional (a saber, los métodos de diferencias finitas, y elementos finitos[83], los volúmenes finitos,...). El estudio de las propiedades y la convergencia de estos métodos constituye el Análisis Numérico, que tiene una conexión profunda con el Álgebra. Por otra parte, la capacidad de cálculo da nueva vida a la matemática discreta, como la teoría de grafos, con sus importantes aplicaciones (por ejemplo, a las redes telefónicas y en general al mundo de las comunicaciones).

Un nuevo paradigma de la ciencia. El broche final de esta evolución vertiginosa es el surgimiento de un nuevo paradigma científico en que la **Ciencia computacional** es el tercer componente básico del método científico, junto con la Teoría y el Experimento. Nos hallamos pues ante una alteración profunda de la herencia científica de Galileo y Newton, que la enriquece en la dirección de las matemáticas.

Esta nueva visión, que comenzó en la ingeniería y las ciencias físicas, se practi-

[83]Los elementos finitos son un ejemplo maravilloso del desarrollo de una herramienta matemático-numérica por el esfuerzo paralelo, pero separado, de matemáticos e ingenieros, cf. el interesante relato histórico de [2]. El fenómeno no es aislado, piénsese en la reciente historia de las ondículas o "wavelets". Estos ejemplos deberían llevarnos a pensar más en los beneficios de la comunicación.

ca hoy día intensamente en todas las ciencias, dando lugar a **nuevas disciplinas** o **subdisciplinas**, como la Física Computacional y la Dinámica de Fluidos Computacional, la Biología Computacional o la Química Computacional. Programas de las licenciaturas (incluso nuevas titulaciones), programas de investigación internacionales, centros de investigación, congresos y revistas prestigiosas confirman la relevancia del *tercer rostro* de la ciencia en los albores del siglo XXI. La ventaja del camino computacional queda perfectamente reflejada en la siguiente declaración de los Reviews in Computational Chemistry: *"As a technique, Computational Chemistry has the advantage of producing answers cheaply and quickly (compared to e.g. thermodynamic measurements)"*. Es decir que cuesta menos calcular que medir (y es fiable). Y añade otro aspecto importante, la capacidad para examinar lo hipotético: *"and [it works] for hypothetical structures, like transition states"*.

Lo anterior no se circunscribe a las ciencias clásicas, afecta incluso en mayor grado a la ingeniería y la ciencia económica. La novedad del cambio, que sucede ante nuestros ojos, es un reto de enorme importancia para el futuro de las matemáticas y resulta difícil de asimilar para muchos colegas. No hay nada malo en seguir anclado en un pasado glorioso,... pero se paga un precio. De la amplitud del panorama hablamos en la próxima sección.

8 Los retos y tendencias del siglo XXI. Matemáticas en las ciencias, la industria, las finanzas y la administración

En consonancia con los apuntes vistos de la reciente evolución de la matemática pura y aplicada, que combina la exigencia de una sólida teoría con una ambición universal, el panorama que ofrece el mundo de las matemáticas de cara al futuro es de una asombrosa variedad. Usando un idioma algo retórico, los expertos dicen que las matemáticas son *ubicuas*, está por todas partes, y *relevantes*, importan. La modelización matemática juega un papel mayor que nunca en la ciencia, la ingeniería, los negocios y las ciencias sociales.

Mencionaremos solamente algunos de los principales temas de aplicación que aparecen en la literatura, en los congresos, en los programas de los institutos de investigación. También hemos utilizado una serie de fuentes como [11, 12, 13, 27, 33]. En itálicas señalamos aspectos matemáticos relacionados para comodidad del lector.

• Mecánica celeste. Problemas de la ciencia aeroespacial. *Estabilidad y caos en sistemas dinámicos. Atractores extraños.* Mecánica de sólidos y fluidos en gravedad cero.

• Teoría de fluidos. Aplicación a la Meteorología y la Climatología. Ingeniería del océano. Problemas medioambientales complejos, recalentamiento global y otros temas geosociales. *Modelos de circulación global, modelos de equilibrio; modelización estocástica del clima; jerarquías de modelos de complejidad intermedia, como los modelos geostróficos.* Glaciología. Acústica y aplicación a la industria del sonido. Fluidos industriales, lubricación. Turbulencia. *Predecibilidad y caos. Estabilidad, bifurcación. Problemas de frontera libre.* Áreas de intersección, como la interacción fluido-estructura.

• Aeronáutica. Problemas de la hidrodinámica. Vuelo supersónico y transónico. Problemas de la combustión (propagación de llamas, detonación). *Ondas de choque y ecuaciones hiperbólicas. Capas límite y desarrollos asintóticos. Ondas viajeras.*

• Física fundamental. Las matemáticas del mundo atómico y de las partículas elementales. El modelo estándar, la electrodinámica cuántica, la cromo-dinámica cuántica. *Teoría de grupos, renormalización, teorías gauge, supersimetría, ecuaciones de Yang-Mills, instantones, dilatones, "branes",.... Geometrías y topologías exóticas en dimensiones superiores.*

• Astrofísica. Relatividad general, modelos estelares. Matemáticas de la física de plasmas, magnetohidrodinámica. *Ecuaciones cinéticas (Boltzmann, Fokker-Planck, Vlasov, ...)* .

• Ciencias de la tierra. Problemas de recursos y minería. Problemas de conservación del medio ambiente. Transporte de contaminantes en el aire y el suelo. Hidrología computacional. *Las ecuaciones de la extracción de petróleo, de la filtración en los suelos, de la difusión de contaminantes: sistemas no lineales de EDPs y problemas de frontera libre.* Matemáticas de los fenómenos sísmicos, *propagación de ondas, problemas inversos.*

• Ciencia de materiales. Modelado y simulación de materiales "composites", materiales magnéticos, polímeros, cristal y papel. Propagación de fracturas y otros mecanismos de fallos. *Elasticidad lineal y no lineal. Teoría de la homogeneización.* Transiciones de fase, crecimiento de cristales, superconductividad e histéresis.

• Nanotecnología. Ópticas integradas, redes ópticas. Electrónica y óptica cuántica. Técnicas de Nanoescalas en medicina, materiales porosos. *Acoplamiento de modelos con estados cuánticos, mesoscópicos y continuos. Teoría de Boltzmann semiclásica, ecuación de Wigner.*

• Ingeniería industrial. Procesos de la siderurgia, altos hornos. Prototipos de la industria automovilística (fluidos, aerodinámica, materiales y teoría de la fractura).

• Comunicaciones. Telecomunicación y redes ópticas: análisis, simulación, optimización, optimización de la tasa de transmisión, diseño de redes. Antenas, rádar y sónar. *Teoría de campos electromagnéticos.* Los hornos de microondas acoplan las ecuaciones de Maxwell con la teoría del calor de Fourier.

• Matemática Discreta. *Teoría de grafos, combinatoria.* Aplicaciones a la administración de empresas, programación de tareas, rutas,...

• Informática. *Lógica matemática, algoritmia, complejidad computacional. Paralelización.* Autómatas finitos, lenguajes formales, *álgebra.* Aprendizaje de máquina, minería de datos, inteligencia artificial, proceso del idioma natural.

El diseño de la computadora cuántica abriría un nuevo mundo a la computación.

• Control. Control óptimo, control robusto, control no lineal. Control predictivo. Sistemas de control "fuzzy". Redes neuronales. Detección y diagnóstico de fallos en los procesos industriales. Modelado y control de sistemas económicos. Programación con condiciones. Comunicación y control de sistemas híbridos distribuidos.

• Automatización y Robótica. *Geometría Algebraica y computación.* Visión por computadora y realidad virtual. Aprendizaje biológico y computacional.

• Teoría de la información. Codificación de mensajes, códigos correctores de errores. Las sorprendentes aplicaciones de *la teoría de números y el álgebra.* Proceso y compresión de imágenes. *Ondículas, fractales, teorías de EDPs no lineales.* Reconocimiento del habla y las imágenes.

• La estadística en la ciencia, la industria, la empresa y el gobierno. Estimación y tests de hipótesis, diseño de experimentos. Procesos estocásticos. Series temporales. Epidemiología. Control de calidad. Análisis de varianza. Análisis multivariante. Muestreo, votaciones.

• Teoría de Optimización y Programación Matemática. Programación entera, programación no lineal, programación convexa. Métodos iterativos. Optimización del diseño industrial. *Métodos numéricos, ecuaciones en derivadas parciales, cálculo de variaciones, combinatoria, álgebra lineal.*

• Problemas de transporte óptimo. Los problemas del tráfico (modelos continuos y discretos). Planificación de redes. El tráfico en la *Web.*

• Economía. La matemática financiera (valoración de opciones, comercio de derivados, riesgo,...) une *las ecuaciones diferenciales estocásticas con las ecuaciones en derivadas parciales y problemas de frontera libre.* Modelos para la economía global.

• Química. Química cuántica: *simulación de estructuras atómicas y moleculares a través de las ecuaciones fundamentales. Modelos de Schrödinger, Hartee-Fock, Thomas-Fermi, Born-Oppenheimer,...* Dinámica de reacciones, combustión. *Matemáticas de la nucleación, crecimiento de cristales y quemotaxis. La propagación de frentes, ondas viajeras, osciladores químicos. Caos.* Diseño de drogas.

Las Ciencias Naturales y la Medicina:

• Biología: Ecología matemática, epidemiología, biométrica, la bio-informática.

Matemática de la Genética, Filogenética computacional. La estructura y función del ácido nucleico. Evolución molecular. Proteómica. *Cálculo con ADN.* Alineación de secuencias, razonamiento borroso. Modelización matemática en biopolimerización.

• Medicina: interacción fluido-estructura como modelo para el flujo sanguíneo. Modelado y simulación de la función de otros órganos: cerebro, pulmones e hígado. *Auto-organización y geometrías fractales.* Asistencia computacional en cirugía. Farmacocinética, modelado del crecimiento de tumores. Neurociencia computacional. Matemática de las enfermedades infecciosas y difusión de epidemias. Órganos artificiales, modelado del sistema inmunológico.

• Tratamiento de imágenes en Medicina. Tomografía: tomografía computerizada, reconstrucción 3D de imágenes. *Transformadas de Fourier y Radon, problemas inversos.*

• Aunque la Matemática computacional (tomada aparte de la Informática) penetra todos los campos de aplicación, merece una mención por sí misma en este listado: métodos numéricos y códigos; algoritmos eficientes; aproximación, estimaciones (a priori y a posteriori) del error, métodos y modelos adaptativos, mallado, descomposición del dominio, análisis multiescala, cálculo numérico de procesos aleatorios,...

• Por otro lado, la Modelización Matemática en sus diferentes variantes (determinista, continua, discreta,...) plantea los problemas de validación de modelos y las técnicas para obtener y elaborar los datos en que se basa la validación (ver apartado de Estadística), así como el importante (y debatido) concepto de jerarquía de modelos, una manera progresiva de acercarse a la “ realidad ”que es hoy día parte integrante de la “caja de herramientas”del científico aplicado (los viejos idealistas con su la “verdad eterna”se revolverán en sus tumbas; ¿o quizá no?).

Detendremos aquí el listado y haremos una muy necesaria pausa con algunos comentarios. Se observará que la lista está solo ligeramente articulada por afinidad de temas; sin embargo, la interconexión íntima de las ramas de la matemática aplicada nos obliga a cometer repeticiones, o a poner un tema bajo uno de varios posibles títulos. Por otra parte, hemos dejado fuera diversos campos de aplicación: las teorías de los sistemas complejos, la autosemejanza en el mundo natural, la formación y reconocimiento de modelos (*patterns*) y el sistema de posicionamiento global (GPS), la matemática de los sistemas electorales; la arquitectura, la industria textil y la alimentaria también han llamado a la puerta de la matemática. Y hay una muy fuerte tendencia para que la Matemática juegue un papel importante en las artes visuales, como ya hace en la Industria del Ocio combinada con el progreso formidable de la tecnología de las computadoras. Y ¿cómo pude haberme olvidado de hablarles de la Teoría de Nudos, del Método Simplex de G. Dantzig, líder incontestado del uso de las matemáticas en las empresas, o del Filtro de Kalman? En conclusión, esta larga lista es incompleta, principalmente debido al conocimiento limitado del autor; pero espero que convencerá al lector de la variedad enorme de intereses de la matemática aplicada actual.

Me gustaría agregar una reflexión personal final sobre las tendencias profundas que veo bajo la diversidad anterior. Las matemáticas del porvenir serán mucho más **estocásticas** y **algorítmicas** de lo que fueron hasta el siglo XX, y la **modelización matemática** será considerada una parte esencial de la educación y la actividad matemática, junto con el cálculo y la simulación. Pero pase lo que pase, me parece que una **prueba** clara y completa, y tan elegante como sea posible, será siempre el meollo de nuestra ciencia, como ha sido desde tiempos del buen Euclides, y los matemáticos futuros todavía se entusiasmarán con **problemas y conjeturas,** y algunos de ellos al modo de Galileo **mirando al mundo** (o las estrellas). Y construirán, posados sobre hombros de gigantes del pasados, esos delicados, intrincados y huidizos objetos llamados **teorías,** algunas de ellas destinadas al olvido, unas pocas a la eternidad,..., o al uso diario. ¿Quién se maravilla ya de la sorprendente existencia de las ondas electromagnéticas llenando el aire, ahora que incluso se han vuelto una forma de contaminación? Pero basta de filosofía por el momento.

9 De los 23 problemas de Hilbert en 1900 a los problemas de Clay en 2000

Ya hemos señalado el profundo impacto que la lista de problemas propuesta por D. Hilbert en 1900 tuvo sobre sus contemporáneos y sucesores. Han pasado 100 años desde entonces y diversas iniciativas pretenden dar la réplica al gran hombre, cf. por ejemplo los libros de Arnold - Atiyah - Lax - Mazur, y de Engquist - Schmid[84] El miércoles 24 de mayo de 2000 se anunció en el Collège de France de Paris el Conjunto de los 7 problemas matemáticos que constituyen los *Millennium Prize Problems*, patrocinados por el *Mathematics Clay Institute.* Recordando a Hilbert pretendía reflejar 7 de los más importantes problemas abiertos de la ciencia matemática al comienzo del nuevo siglo [85]. Estos problemas recorren las diversas áreas las matemáticas puras y aplicadas y son

1. P versus NP (Teoría de la computación)
2. Conjetura de Hodge (Geometría algebraica)
3. Conjetura de Poincaré (Geometría y topología)

[84]Para más información ver el artículo de Jackson citado en las referencias finales. Ver también el vol. 3, no. 1 (2000) de la Gaceta de la RSME, artículo por J. L. Fernández y M. de León.

[85]la resolución de cada problema valdría al autor un premio de 1 millón de dólares. Toda la información sobre el premio y los problemas se puede obtener en la dirección `http://www.claymath.org/prize_problems`.

4. Hipótesis de Riemann (Teoría de números)

5. Existencia de Yang-Mills y Hueco de Masa (Física teórica)

6. Existencia y regularidad para las ecuaciones de Navier-Stokes (Mecánica de Fluidos y PDEs)

7. Conjetura de Birch y Swinnerton-Dyer (Geometría aritmética algebraica)

A riesgo de ser impertinente (pido disculpas al lector) desearía dar una impresión personal sobre esta lista que parece destinada a ser famosa e influyente. Afortunadamente, incluye problemas abiertos importantes en temas variados de la matemática pura y aplicada. Sin embargo, no hace suficiente justicia a la visión aquí expuesta de la matemática como lenguaje y herramienta básica de la ciencia y la ingeniería.

10 Ejemplos de nuevos cursos

Tras dos secciones consagradas a la enumeración, es tiempo de volver al trabajo. A continuación echaremos una ojeada más detallada a algunas de las novedades de la matemática actual. Entre las muchas opciones, tomaremos tres ejemplos: de las finanzas, las comunicaciones y la física fundamental.

• Matemáticas de la incertidumbre financiera y el riesgo

Un ejemplo notable de las aplicaciones prácticas de las matemáticas, desarrollado en los últimos decenios, es la llamada matemática financiera. Los nuevos instrumentos financieros de *futuros* y *derivados* se basan y a su vez motivan esta nueva rama de la matemática aplicada, la cual combina procesos estocásticos, ecuaciones en derivadas parciales y problemas de frontera libre. El resultado más famoso es el *modelo de Black-Scholes*[86] para el mercado de opciones, el cual reduce la valoración a la solución de una ecuación del calor (inversa en el tiempo). Me gustaría registrar esta reducción en la siguiente sucesión de fórmulas

$$\frac{dS_t}{S_t} = \mu dt + \sigma\, dB_t \quad \Rightarrow \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + b\, S\, \frac{\partial P}{\partial S} - r\, P = 0,$$

que pasa de una ecuación diferencial estocástica, representando la incertidumbre del azar, a una EDP determinista que permite la valoración del precio. Éste es un ejemplo sorprendente de *transferencia de conceptos y técnicas* hecho posible por la clave común matemática (y por el hecho de que F. Black era licenciado en Física Cuántica).

[86] F. Black, M.Scholes, *The pricing of options and corporate liabilities*, 1973. Merton y Scholes recibieron el Premio Nobel de Economía en 1997. ¡Una primera versión del modelo había sido propuesta por L. Bachelier en 1900! se tardaron siete décadas para llegar a un modelo realista y a que la aplicación ocurriese.

La inestabilidad inherente a esos mercados y las enormes repercusiones sobre la economía pública y privada hacen tanto más importante la aplicación del método matemático para intentar hallar la clave matemática que rige tales procesos y a reemplazar las reglas empíricas y la adivinación en la práctica financiera por matemáticas. Un reto para el nuevo siglo.

• Del análisis de Fourier a las ondículas

Hemos discutido hace un rato la situación creada en el análisis de Fourier cuando Du Bois Raymond halló un ejemplo de serie de Fourier no convergente, y queremos recordar aquí que la tercera opción para salir del atolladero consistía en cambiar la base de las funciones usadas en la representación. Esto es lo que hizo A. HAAR en 1909[87], resolviendo así la dificultad en principio. Podemos decir que éste es el origen remoto de las ondículas (wavelets), una idea que tardó un siglo entero en madurar. La investigación en este problema antes de la Segunda Guerra Mundial parece haber seguido un interés exclusivamente matemático sin ninguna aplicación en mente. Pero después de la guerra, ingenieros y científicos aplicados aterrizaron en la idea llevados por las aplicaciones, notablemente, a la teoría de la información de Claude SHANNON. En el futuro las dos líneas de pensamiento se unieron y el análisis de ondículas se ha convertido en una importante y fértil intersección de las fronteras de las matemáticas, el cálculo científico y el tratamiento de señales[88].

• Modelos matemáticos de la Física Teórica y la naturaleza de la materia

Las dos grandes revoluciones científicas en la Física del siglo XX, es decir la Relatividad y la Mecánica Cuántica, han impreso en esta ciencia una aún mayor conexión con la matemática pura. La Física se enfrenta con el desafío enorme de construir una teoría que una a ambos modelos en un todo coherente. Experimentales y teóricos han emprendido la búsqueda de la “teoría última”que explicaría todo, desde la constitución del átomo a los extremos más lejanos del Universo. Tal teoría está aún por llegar (y podría estarlo mucho tiempo) pero se han obtenido grandes logros (pues se hace camino al andar, como dijo el gran poeta). He aquí algunos hitos, todos ellos matemáticas profundas.

La Electrodinámica Cuántica (QED) fue desarrollada para describir la interacción electromagnética en el marco de la Mecánica Cuántica, y trata de las cargas y los fotones y usa los hermosos diagramas de Feynman. Una teoría matemáticamente coherente valió a sus autores, Julian SCHWINGER, Richard FEYNMAN y Sin-Itiro TOMONAGA el premio Nobel de Física en 1965. Por su lado la Cromodinámica[89]

[87] *“Zur Theorie der orthogonalen Funktionensysteme”, Math. Annalen* **69** (1910), pp. 331-371

[88] La mayor parte de esta información está tomada del libro [19], cf. también [16]

[89] El nombre hace referencia a la pintoresca denominación para la carga conservada, llamada “el color”.

hace un trabajo similar para describir la fuerza llamada “fuerte”que actúa entre los *quarks*, partículas postuladas por M. GELLMANN y G. ZWEIG en 1964 como los entes constituyentes de neutrones y protones. De las cuatro fuerzas básicas de la Naturaleza (gravitacional, electromagnética, débil y fuerte), las dos intermedias reciben una teoría unificada en 1967 con el trabajo de S. WEINGER, SH. GLASHOW y Abdus SALAM. *Simetría*, *gauge* y *renormalización* son las palabras clave en este mundo de alta matematización. Las ecuaciones de Maxwell, Schrödinger y Dirac ceden el lugar a las ecuaciones de Yang-Mills. El trabajo cristaliza en los primeros años 70 en el Modelo Estándar de partículas elementales, que explica la realidad atómica en términos de tres generaciones de quarks y *leptones*. Estas partículas actúan mutuamente a través de la teoría del grupo $SU(2)\times U(1)$ para la fuerza electrodébil y la de $SU(3)_{color}$ para la fuerza fuerte. La Matemática está por consiguiente en el puro centro del modelo, en forma de grupos de Lie, geometría diferencial (más específicamente, conexiones en fibrados) y ecuaciones en derivadas parciales.

Siguiendo adelante, las Teorías de Gran Unificación intentan combinar ambas teorías de grupos en una. En la Teoría de Cuerdas la vieja idea básica de las partículas puntuales es reemplazada por la idea de cuerdas vibrantes elementales. Al final del siglo XX la Teoría de Supercuerdas propone un modelo matemático para la unificación de todas las fuerzas, de todas las físicas. Le falta sin embargo comprobación experimental suficiente; sin ésta una teoría es simplemente una teoría. Y la búsqueda continúa. Este chorro de ideas ha motivado desarrollos matemáticos importantísimos, asociados a los nombres de matemáticos famosos como M. F. ATIYAH, S. K. DONALDSON y E. WITTEN.

Estos físicos creen pues que la combinación modelos-y-experimentos nos permitirá entender un mundo extraño en que la materia, el espacio, y tiempo no son lo que nosotros solemos pensar, dónde el espacio vacío está lleno de actividad e incluso podrían existir bastantes dimensiones espaciales adicionales (es decir, por encima de las 3 que vemos más el tiempo) arrugadas en distancias ridículamente pequeñas (la distancia típica sería de 10^{-35}m, por eso no las vemos, *voilà l'astuce*; pero nos dicen que vemos la matemática, y a su debido tiempo veremos las consecuencias).

11 Hechos y opiniones

En palabras de John MILNOR, *“pure mathematicians tend to judge any work in the mathematical sciences on the basis of its mathematical depth, the extent to which it introduced new mathematical ideas and methods, or it solves long standing problems”*. A lo que yo agregaría que las nuevas ideas y métodos deben ser juzgados por su productividad, y mencionaría como importantes cualidades la elegancia de la prueba y la visión o intuición. Continúa así: *"However, when mathematics is applied to other branches of human knowledge, a quite different question must be asked first: to what extent does it increase our understanding of the real world"*[90].

Hubo en épocas no muy remotas un movimiento de separación en las matemáticas que parecía alejar cada vez más a los cultivadores de ambos géneros, puro y aplicado (en la medida en que se puede hablar de una separación que en los mejores casos nunca ha sido neta). Y no debemos olvidar el rechazo de muchos científicos puros contra un tipo de matemática aplicada más atenta a la ganancia que a la exigencia científica, y, al contrario, el rechazo de muchos científicos aplicados hacia los mundos excesivamente artificiales (y aburridos) de cierta matemática pura. Afortunadamente, presenciamos hoy día una serie de sucesos simultáneos - a saber, la explosión de vitalidad de la matemática pura, los éxitos de las matemáticas en la formulación y resolución de los problemas clave de la física contemporánea, la economía y la industria, y la variedad insospechada de aplicaciones de todas las ramas de las matemáticas. Todo ello está alterando profundamente la visión de ambos campos, que tienden a confluir en uno, en la mejor tradición del pasado. Este esfuerzo generoso no es nuevo, como expresan las palabras del notable matemático ruso del siglo XIX P. L. CHEBYSHEV: "Unir la teoría y la práctica conduce a los más favorables resultados; no sólo la práctica se beneficia, también las ciencias se desarrollan bajo la influencia de la práctica que revela *nuevos temas* a la investigación, así como *nuevos aspectos* de viejos temas"[91]. La importancia de la teoría para la práctica viene descrita en estas bellas palabras de Euler: *"La généralité que j'embrasse, au lieu de'éblouir nos lumières, nous découvrira plutôt les véritables lois de la Nature dans tout leur éclat"*[92].

Es para los profesionales un gran misterio el que las partes pura y aplicada de las matemáticas sean caras de la misma moneda. Que ambas no son exactamente lo mismo queda muy bien reflejado en las palabras de Albert Einstein: "Hasta donde las leyes de matemática se refieren a la realidad, no son exactas; y en cuanto son exactas no se refieren a la realidad"[93]. Pero el ideal y la práctica se unen con resultados sorprendentes. Es famosa la frase de E. WIGNER que se asombraba de la "efectividad de las matemáticas en las ciencias más allá de lo razonablemente esperable", literalmente, *"the unreasonable effectiveness of mathematics in the natural sciences"*[94].

[90] Ver las Notices de la Amer. Math. Soc., 1998.

[91] Tomado de [20]. Énfasis nuestro.

[92] En traducción algo libre, "La generalidad con la que opero, en lugar de despistarnos, nos descubrirá las verdaderas leyes de la Naturaleza en todo su esplendor". La frase figura en la tapa de la revista Archive Rat. Mech. Anal.

[93] Tomado de *Geometry and Science*, 1921. Incluido en *Sidelights of Relativity*, Dover, 1983. Traducción propia

[94] Conferencia dada en New York, 1959. Publicada en la revista *Comm. Pure Applied Math.* **13**

HACER Y ENSEÑAR MATEMÁTICAS HOY. Pasamos a comentar los cambios en la manera de "hacer matemáticas", especialmente cuando son aplicadas. La emergencia de la *era del ordenador* ha dado nuevas alas a las matemáticas, *¡podemos calcular!* La capacidad de *cálculo eficaz, rápido y barato* se ha hecho disponible al principio del siglo de XXI para el científico y en medida creciente para el hombre común, y la sociedad pide cada día más. Ello plantea retos y reflexiones.

Los teoremas siempre serán teoremas y una deducción lógica sigue siendo la llave de la correcta comprensión, pero la vía al descubrimiento nunca será ya la misma, como tampoco lo es el *día después:* la implementación numérica es ahora punto importante en muchas de las matemáticas (en todas las matemáticas aplicadas). No se trata de abjurar de Euclides, se trata de desarrollar la parte de Euclides inventor de algoritmos. Los efectos sobre la enseñanza son de lo más drástico, como es de suponer, pero todavía están siendo desarrollados[95].

Con ello llegamos a un importante tema de debate, ¿es la nueva forma de hacer y aplicar las matemáticas meramente instrumental o genera nuevas matemáticas? Este es un debate tan viejo al menos como Arquímedes, que utilizaba la mucha mecánica que sabía para inventar pruebas geométricas o conceptos completamente nuevos. Sostenemos pues que los nuevos campos son fuente inagotable de nuevos problemas, nuevas intuiciones, o visiones sorprendentes de viejos temas que dábamos por perdidos o por agotados. Repasemos tan sólo algunas de las páginas anteriores para ver la sorprendente cosecha geométrica de las teorías de partículas de Donaldson, Witten y compañía. O las consecuencias del poder de cálculo sobre las disciplinas más puras como la teoría de números o el álgebra.

LA MODELIZACIÓN. Un rasgo importante de las matemáticas aplicadas modernas es la modelización matemática, el arte de idear *representaciones sensatas* de los más diversos fenómenos del mundo real en términos matemáticos, basadas en *hipótesis racionales* que simplifican la realidad para hacerla calculable. J. L. LIONS, el matemático francés recientemente fallecido que tanto contribuyó a la presente relevancia de las matemáticas en el mundo industrial europeo, dijo en 1991: *"Ce que j'aime dans les mathématiques appliquées c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir"*[96]. Y añadió: *"De toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure."*[97]

(1960), 1-14.

[95]Internet está poblada de propuestas didácticas maravillosas; junto a otras abominables, claro está

[96]" Lo que me gusta de las matemáticas aplicadas es que ambicionan dar una representación del mundo de los sistemas que permita comprender y actuar".

[97]"De todas las representaciones la matemática, cuando es posible, es la mejor y la más flexible".

Hemos de recordar que un modelo es sólo un modelo y refleja la realidad de la forma contradictoria que Einstein describía. Pero es todo lo que nosotros tenemos, a menos que consideremos un modelo mejor (o incluso una jerarquía de ellos). Esta es la gloria y la debilidad de la modelización, aspecto crucial de la matemática aplicada actual. El público que presencia el acalorado debate sobre las predicciones de los modelos matemáticos del clima acerca del calentamiento global en la Tierra sabe cuán importante es el problema y debe comprender *cuán difícil es llegar a conclusiones nítidas y fiables manejando evidencias parciales, basadas en modelos parciales y apoyadas por enormes bases de datos de compleja interpretación*, y huyendo de juicios a priori por muy verosímiles que parezcan. Pero es también claro que toda conclusión no basada en números y modelos fiables es pura ideología. Lo que nos permite apreciar el mérito de los modelizadores gigantes del pasado, como Newton, Maxwell, Einstein y el grupo cuántico.

PROMESAS Y PLAZOS. Como hemos apuntado, una enorme parte de las mejores matemáticas se ha originado para explicar aspectos del mundo físico, pero rara vez las consecuencias dramáticas de las matemáticas han sido inmediatas. La formulación de los procesos físicos en clave matemática al gusto de Galileo exige un proceso de maduración que tiene sus reglas y ritmos, que van desde varios años a varios siglos[98].

En un nivel más especulativo, el conocido matemático y escritor científico Ian Stewart afirma que es posible que las matemáticas sean eficaces "porque representan el lenguaje subyacente del cerebro humano". Con lo cual invertimos la apuesta de Galileo, quizá entendemos el mundo en clave matemática porque esa es la clave de nuestra mente. Pero ese es un debate distinto.

PUNTOS PARA UN DEBATE. Resumiré a continuación las opiniones básicas que me he formado en años de estudio y curiosidad por el mundo de la matemática. Espero que sea mínimamente útil en el eterno y necesario debate:

- Sólo las buenas matemáticas pueden ser buenas matemáticas aplicadas. Las Matemáticas Aplicadas como arte diferente y separado de la Matemática propiamente dicha, simplemente no existen[99]. Pero al poner las matemáticas a trabajar, la aplicación las cambia, las enriquece y les abre nuevas vías.

- La Matemática sólo es aplicada de verdad si ataca un importante problema de la ciencia, la tecnología, la economía, o más generalmente, de la sociedad. Ya hemos visto cuán variados estos problemas pueden ser.

- Si bien podemos llegar a juzgar con cierto grado de fiabilidad qué es importante hoy, la tarea de predecir qué rama de la matemática será importante mañana

[98] Sería una bendición si la administración y las autoridades educativas fueran conscientes de este hecho en su toma de decisiones.

[99] Tomo en parte esta idea radical de A. Rényi, [34], quién la atribuye en su relato a Arquímedes.

(la llamada planificación estratégica) excede la capacidad de las personas sensatas, salvo que simplemente contestemos: "las buenas matemáticas importarán" o "las matemáticas del mundo real importarán siempre". Las hipótesis autorizadas y opiniones sobre temas específicos son humanas y pueden ser útiles como orientación personal, pero cuando se trata de decisiones y prioridades la prudencia es de rigor.

- Desde un perspectiva histórica no se puede afirmar que los grandes matemáticos vivan en una torre de marfil de teorías desconectadas de toda realidad. No decimos que no puedan hacerlo, o que no les resulte interesante, necesario, incluso natural en muchos momentos vivir en la abstracción absoluta; afirmamos que, vista en perspectiva, su actividad ha sido un factor esencial en la comprensión que hoy tenemos del mundo.

- Está además la interesante cuestión de filosofía: es un hecho bien atestado que al enfrentarse a un enigma matemático, al matemático "aplicado" le gusta construir y comparar modelos adecuados, y ansía *resolver el enigma* preciso planteado sea cual sea el daño temporal que se inflija a la perfecta deducción lógica, mientras su colega "puro" se deleita en la prueba lógica; sólo la *demostración* gobierna sus días.

Así pues, ¿son lo mismo las matemática puras y las aplicadas? o más cuidadosamente formulado, ¿son lo mismo en el fondo? Dejemos al amable lector que juzgue por sí mismo. Ya saben mi opinión (más o menos), pero me permito agregar en un tono más relajado una cita de Yogi Berra[100]: "En teoría no hay ninguna diferencia entre teoría y práctica; en la práctica, sí que hay".[101]

12 Breve apunte sobre las Matemáticas en España

España tuvo en un momento dado de la Edad Media tardía un papel importante en la transmisión de la cultura árabe a Occidente e incluso hubo un rey en Sevilla[102] que escribió poesía y promovió las matemáticas (el saber astronómico). Al Andalus, la España musulmana, tenía sólidos intereses científicos, en particular en medicina y astronomía, con sabios de renombre como AZARQUIEL (o Al-Zarkali, activo en Toledo) quien compuso tablas astronómicas. El sistema de numeración indio basado

[100]Famoso jugador de béisbol americano, muy conocido por sus cómicas pero atinadas salidas. Esta es la frase original: *"In theory, there is no difference between theory and practice; in practice, there is"*.

[101]He aquí una (medio) broma sobre las diferentes formas de ver las matemática: los ingenieros dicen que las ecuaciones aproximan la realidad, mientras los físicos piensan que la realidad aproxima las ecuaciones; por su lado, los matemáticos se asombran ante la idea de que exista una conexión entre "sus" ecuaciones y la realidad (y se enojan no poco si se les insiste).

[102]Alfonso X el Sabio.

en la posición ya estaba en uso en Al Andalus en el siglo IX. [103] Después de la toma por los Cristianos (1085 d.C.), Toledo, la ciudad de las tres culturas - cristiana, árabe y judía - fue durante siglos un gran centro de saber con su Escuela de Traductores que vertieron al latín los trabajos de autores griegos y árabes[104]. En otra dirección, el mallorquín Raimundo LULIO (Ramón Llull) desarrolló en su *Ars Magna* un entero arte de razonamiento algorítmico en que podemos ver un temprano precedente del Álgebra de Boole y la lógica de las computadoras (Llull, que vivió en el siglo XIII, es al mismo tiempo uno de los clásicos más antiguos de la lengua catalana). Un siglo más tarde, los mapas náuticos llamados *portulanos* de Mallorca eran la cima del arte, y los nombres de SOLER y CRESQUES son muy conocidos. El último, un judío, participó en la organización de la escuela náutica portuguesa que fue el origen del descubrimiento del camino a las Indias alrededor de África, e, indirectamente, también de América.

Luego las cosas fueron a peor por largo tiempo. Las fundadas esperanzas del tardo Medievo y primer Renacimiento fallaron en España, y la matemática (y las otras ciencias) han tenido un humilde devenir durante siglos. Mientras la literatura española y arte están con la crema de la creación mundial desde el siglo XVII hasta nuestros días, está claro que ningún nombre español aparece en los libros de texto afamados en que se aprenden las matemáticas, elementales o superiores. Hay en tales textos numerosos conceptos y resultados nombrados en honor a autores de las diversas naciones con gran tradición científica: franceses, ingleses, alemanes, italianos (e Italia era un país católico), en tiempos más recientes rusos y americanos,.., como también son frecuentes los ejemplos países que, debido a su tamaño y las circunstancias no jugaron un papel tan prominente en la Historia, pero que sí están en el Libro de la Ciencia. Durante estos siglos de desarrollo glorioso, de Galileo a Einstein, no se mencionan nombres españoles. ¿Pudo la historia haber sido diferente? El rey Felipe II comprendió la necesidad de la ciencia y creó una *Academia Matemática* en Madrid (1582) bajo la dirección de Juan de HERRERA, el arquitecto de El Escorial, pero la institución no tomó cuerpo y dejó de existir unos años después, mientras que iniciativas similares dieron nacimiento en el extranjero a la *Royal Society* en Inglaterra, la *Académie des Sciences de Paris* en Francia, y así sucesivamente. Ha habido sin duda ejemplos de ilustres hombres digno de mención, como PEDRO CIRUELO, OMERIQUE, JORGE JUAN y ECHEGARAY, pero son autores aislados, una escuela nunca tomó raíz hasta muy recientemente y ningún gran teorema salió de sus esfuerzos. Hubo en el siglo XVIII un gran esfuerzo de los gobiernos ilustrados por afianzar en el país el amor

[103]La primera escuela andalusí de matemáticas parece haber sido la de Maslama al Magriti, es decir, de Madrid, que floreció en el siglo X en Córdoba. Puede considerarse la primera escuela en la Península en todos los tiempos, y tuvo numerosos discípulos. En el siglo XII el rey Almutamán de Zaragoza fue un notable matemático.

[104]El monasterio de Sta. María de Ripoll en Cataluña también tenía una biblioteca mundialmente conocida.

al estudio y la industria y España participó en la medición del meridiano terrestre, pero las consecuencias matemáticas fueron reducidas.[105] ¿Cuáles son las razones? Difícil cuestión, pero señalemos que durante siglos se prohibió a los estudiantes y profesores españoles viajar y aprender en los países extranjeros, una regla de seguridad bastante estricta que previno con éxito contra la heterodoxia, y al tiempo contra la ciencia y el progreso.

Éste no es lugar para un estudio detallado de la Historia, para lo cual dirigimos al lector a los especialistas[106], así que procederemos señalando cómo se ha llegado en fecha muy reciente a un presente bastante halagüeño. España pareció surgir de su profundo letargo matemático en la primera mitad de este siglo y la figura del insigne J. REY PASTOR sirve como referencia a un esfuerzo notable de poner al día a nuestro país basado en las únicas ideas que podían funcionar: el estudio en los grandes centros del extranjero y la importación de las matemáticas que realmente existen en la comunidad mundial, que es la única que tiene real sentido en la ciencia, al menos en la nuestra. Este método había tenido un éxito fulgurante en la creación de la matemática norteamericana y todo indicaba que había de funcionar en nuestro país. Sin embargo nuestra funesta historia se encargó de disgregar el notable esfuerzo, que daría frutos abundantes en tierras americanas, personificados en figuras como L. SANTALÓ. Con alguna muy honrosa excepción, que la hubo, la actividad matemática hasta los años 60 volvió al ritmo del pasado.

Poco a poco, sobre todo a partir de los años 70, comienza por fin el despertar de España a lo que podríamos llamar la realidad matemática. Tras una década de esfuerzo ingente de una generación que aprendió en las fuentes originales, que enseñó en sus clases los textos más actuales, que organizó seminarios de investigación y que viajó o mandó a sus jóvenes alumnos al extranjero, que empezó a publicar en las revistas internacionales reconocidas y a participar en los grandes eventos, llegan a partir de los años 80 los años dorados de la *creación original*, lo que se traduce en las mil facetas de la vida matemática auténtica y que se reflejan (aunque no se resuman) en la palabra *publicación*: las mejores revistas empiezan a recibir artículos de autores españoles, primero tímidamente, luego en cascada[107]. Las señales de los buenos tiempos se hacen múltiples e inequívocas, y podemos concluir que "España ya no es diferente". Los indicadores oficiales nos permiten poner cifras a esta evidencia de cambio. De ellos se deducen dos hechos que inicialmente han sorprendido a muchos:

(a) Que las matemáticas españolas han pasado de un lugar muy modesto en 1980

[105] El lema de la Academia de Ciencias portuguesa resume el espíritu de esta época: *Nisi vtile est qvod facimvs stvlta est gloria.* "Si lo que hacemos no es útil, tonta es la gloria".

[106] Como Juan Vernet, cuyo trabajo [46] se usa en los párrafos anteriores.

[107] En esta coyuntura conviene evocar las palabras de Galileo sobre la Ciencia que le atribuye B. Brecht en su "Vida de Galileo": *La Ciencia tiene un solo mandamiento: contribuir a la Ciencia.*

(menos del 0.4 % de la producción mundial según la base de datos ISI[108]) a una posición honorable en el momento, inmediatamente después de EE.UU., Alemania, Inglaterra, Francia, Rusia, Italia, Japón y Canadá, con una producción en revistas importantes que se ha multiplicado por un factor de más de 10 y representa en 2001 una proporción mundial de más de 4,18 % (ISI).

(b) Que en el análisis comparativo de la ciencia española, la Matemática figura entre las especialidades bien situadas.

Para más información sobre la investigación matemática en España en el último decenio referimos al lector al informe [21], que refleja en gran detalle los avances realizados.

Otra consecuencia del estado creativo de la matemática española es la presencia de numerosos y valiosos libros de texto y monografías de investigación en prestigiosas colecciones. Digamos además que España, que ha alcanzado una sólida posición en la investigación, también cuenta con una tradición en educación matemática, con un papel muy influyente en el ICMI[109].

Finalmente, la tendencia hacia los aspectos computacionales y aplicados de las matemáticas, junto con el énfasis en las matemáticas como herramienta por excelencia en la modelización, es ahora fuertemente sentida en una comunidad anteriormente ligada casi en exclusiva al pensamiento matemático abstracto. Abrir las ventanas al ancho mundo de ahí fuera es un reto enorme en pro de la salud de nuestra matemática y del bienestar de generaciones futuras, y todos los esfuerzos son bienvenidos. ¡Dejemos entrar el aire fresco!

13 Conclusión

Llegamos al fin de nuestro viaje. Hemos dicho al principio del relato que la "Matemática Aplicada"es la Matemática del "Mundo Real". Puede quedarle al lector cierta duda sobre la esencia de tales conceptos, y se preguntará si han sido suficientemente aclarados en el texto. No ha sido nuestro propósito examinar a fondo este problema más bien filosófico. Siguiendo la práctica usual de los matemáticos aplicados, poco partidarios del exceso de teorización, o quizá movidos por la inmensidad de la tarea y la premura de tantos nuevos hallazgos, hemos seguido una *aproximación constructiva* a ambos conceptos y hemos intentado mostrar su contundente relevancia en la gestación de la sociedad actual y su papel en el futuro que se vislumbra. Lo que

[108] *Institute for Scientific Information.*

[109] *The International Commission on Mathematical Instruction*, presidida durante años por el matemático español Miguel de Guzmán.

no excluye que otros se ocupen de tales temas con un espíritu más discursivo.

Recordando a Galileo, me gustaría concluir así: el *Libro de la Naturaleza* se abre ante nosotros para que lo admiremos con su infinita, cambiante y sorprendente belleza; las matemáticas como lenguaje de la ciencia están ahí para que comprendamos la Naturaleza, y nos permiten además utilizarla y explotarla, estando este aspecto final cargado de promesas y peligros, como todo lo humano. Espero que los matemáticos de hoy día realicemos nuestra parte en el esfuerzo de comprender y mejorar la Sociedad de la Información que nos ha tocado ver nacer. En la era de los ordenadores y la información, *la Realidad está en el Número*, como habría gustado a Pitágoras. O por lo menos un enorme pedazo de ella la explican y la reproducen los números, con la ayuda de nuestros amigos científicos y tecnólogos, y de los ordenadores.

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El apéndice histórico refleja ideas del autor sobre el presente de la Matemática española tomado con mínimos cambios de la referencia [45], sección primera. Más sobre el mismo asunto en [44]. Interesantes fuentes en español son los *Boletines de SEMA*; la *Gaceta de la RSME* (cf. el vol. 3, 1 (2000)) y la *Revista Española de Física*, vol 14, no. 5, consagradas al estado de la Matemática con ocasión de la celebración del Año Mundial Matemático. Cf. también [1, 8, 18, 27, 32, 42]. Las ilustraciones están tomadas del sitio web *The MacTutor History of Mathematics Archive*, de la Univ. de St Andrews, un notable archivo biográfico cuya lectura me ha sido de gran utilidad. Finalmente, la lista de referencias que sigue refleja lecturas del autor durante la preparación de este texto y no significa en modo alguno una selección de las mejores lecturas disponibles.

Referencias

[1] V. ARNOLD, M. ATIYAH, P. LAX, B. MAZUR, "Mathematics: Frontiers and Perspectives", AMS Publications, 2000.

[2] I. BABUSKA, *Courant Element: Before and After*, in "Finite Element Methods", edited by Krizek, Neittaanmäki and Sternberg, M. Dekker Inc, New York, 1994.

[3] G. I. Barenblatt, *George Keith Batchelor and Daniel George Crighton, Applied Mathematicians*, Notices American Math. Soc., vol 48, no 8 (2001), 800–806.

[4] J. L. Casti, "Five Golden Rules", John Wiley, New York, 1996. "Five More Golden Rules", John Wiley, New York, 2000.

[5] G. Chaitin, "The limits of mathematics", Springer, Singapore, 1998.

[6] B. Cipra, "What is happening in the Mathematical Sciences", vols. 1–4, Amer. Math. Soc, Providence, RI.

[7] COMAP, "Las Matemáticas en la vida cotidiana". Addison Wesley - Universidad Autónoma de Madrid. 1998. Versión inglesa: S. Garfunkel et al., "Introduction to Contemporary Mathematics", W.H. Freeman & Co, New York, 1988.

[8] B. Engquist (Editor), W. Schmid (Editor), "Mathematics Unlimited - 2001 and Beyond", Springer Verlag, Berlin, 2001.

[9] R.P. Feynman, "Feynman Lectures On Physics (3 Volume Set)", Reading, Mass., Addison-Wesley Pub. Co. [1963-65] ,

[10] R.P. Feynman, "Six Easy Pieces: Essentials of Physics Explained by Its Most Brilliant Teacher", Helix Books, 1995 (Addison-Wesley Longman, 1996), and "Six Not-So-Easy Pieces: Einstein's Relativity, Symmetry, and Space-Time", Addison-Wesley Pub., Reading, Mass. 1997.

[11] I. Fonseca et al., "The Impact of Mathematical Research on Industry and vice versa", Round Table at 3rd European Congress of Mathematics, Barcelona, july 2000.

[12] A. Friedman et al., "Mathematics in Industrial Problems", (a 10 volume collection), IMA Volumes in Mathematics and its Applications, Springer-Verlag (1988-1998).

[13] A. Friedman, J. Lavery, "How to Start an Industrial Mathematics Program in the University", SIAM Report, Philadelphia 1993

[14] J. Gleick, "Chaos: Making a New Science", Penguin Books, Nueva York, 1987.

[15] G.H. Hardy, "A Mathematician's apology", Cambridge, 1940.

[16] E. Hernández, G. Weiss, "A first course on wavelets. With a foreword by Yves Meyer", Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996.

[17] "Mathematical Developments arising from Hilbert Problems", Proceedings of Symposia in Pure Mathematics, XXVIII, Amer. Math. Soc, Providence, 1976.

[18] A. Jackson, *Mathematical challenges of the XXI century*, Notices Amer. Math. Soc., vol. **47**, no 10 (2000), pp. 1271-1273.

[19] S. JAFFARD, Y. MEYER, R.D. RYAN, "Wavelets, tools for Science and Technology", SIAM, 2001.

[20] A.I. KHINCHIN, "Mathematical Foundations of Information Theory", Dover, 1957 (Papers appeared in 1953, 1956 in Uspekhi Mat. Nauk in Russian).

[21] "La investigación matemática en España", informe elaborado por el Comité Español para el Año Mundial de las Matemáticas (CEAMM2000), coord. por C. ANDRADAS y E. ZUAZUA, Madrid, 2002.

[22] M. KLINE, "Mathematical Thought from Ancient to Modern Times", Oxford Univ. Press, 1972.

[23] M. KLINE, "Mathematics. The loss of certainty", Oxford Univ. Press, Oxford, 1980.

[24] J. P. MAURY, "Galileo, el mensajero de los astros", Claves, Ed. B.S.A., Barcelona, 200.

[25] N. METROPOLIS, J. HOWLETT, G. C. ROTA, EDS., "A History of Computing in the Twentieth Century", Academic Press, 1980.

[26] P. A. MEYERS, "Encyclopedia of Modern Physics", Academic Press, San Diego, 1990.

[27] Mitteilungen der Deutschen Math. Vereinigung (es decir, *Notices of the German Math. Union)*, 2-1998. Contiene un informe sobre el futuro de las Matematicas por D. MUMFORD, A. FRIEDMAN, L. LOVÁSZ, YU. MANIN, G.C. ROTA, R.B. JENSEN y R. PENROSE.

[28] K. MORIYASU, "An elementary primer in Gauge Theory", World Scientific, Singapore, 1983.

[29] J. MUÑOZ SANTONJA, "Newton, el umbral de la ciencia moderna", col. La Matemática en sus personajes, vol. 3, Nivola ed., Madrid, 1999.

[30] S. NASAR, "A beautiful Mind: A Biography of John Forbes Nash, Jr", Simon & Schuster, New York, 1998. En castellano "Una mente prodigiosa. Historia de John Forbes Nash", Grijalbo Mondadori, 2001.

[31] I. NEWTON, "Philosophiae Naturalis Principia Mathematica", Pepys, London, 1687. En castellano: "Principios matemáticos de la Filosofía Natural", Alianza Ed., Madrid, 1987.

[32] OBRA COLECTIVA, "Fotografiando las Mathematics", Carroggio SA de Eds, Barcelona 2000.

[33] J.T. ODEN ET AL., "Research directions in computational mechanics", Report of the US National Commitee on Theoretical and Applied Mechanics, Washington DC, 2000.

[34] A. Rényi, “Dialogues on Mathematics”, Holden-Day, San Francisco,1967.

[35] Revista Española de Física, vol 14, no. 5, año 2000. Special number “La Física y las Matemáticas”.

[36] J. M. Sánchez Ron, “El siglo de la ciencia”, Taurus, Madrid, 2000.

[37] M.M. Schiffer, L. Bowden, “The role of Mathematics in Science”, The Math. Assoc. of America, New Math. Library vol. 30, 1984.

[38] J. Simmons, “The scientific 100”, Citadel Press, Kensington Publ. Corp, Nueva York, 1996.

[39] S. Singh, “The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography”, Doubleday & Company, 1999.

[40] W. Smith, Mapping and Sequencing the Human Genome: A Beginner's Guide to the Computational Science Perspective, “ACM Crossroads Student Magazine”, `http://www.acm.org/crossroads/ xrds4-1/genome.html`.

[41] D. Stauffer, H.E. Stanley, “From Newton to Mandelbrot, A Primer in Theoretical Physics”, Springer, Berlin, 1991.

[42] I. Stewart, “The Problems of Mathematics”, Oxford Univ. Press, 1992.

[43] M. Tanur, F. Mosteller et al., ”Statistics: A Guide to the Unknown”, Brooks/Cole, 1989.

[44] J.L. Vázquez, *Las Matemáticas y los objetivos del año 2000. Un llamamiento a los matemáticos españoles*, Gaceta de la Real Soc. Matemática Española, vol. 3, 1 (2000), pp. 9-22. Cf. also `http://dulcinea.uc3m.es/ceamm`.

[45] J.L. Vázquez, *Mathematical Events in Spain in the Year 2000*, Intelligencer, Springer-Verlag, July 2000, pp. 12-14.

[46] J. Vernet Ginés, “Historia de la Ciencia Española”, Editorial Alta Fulla, 1998.

[47] M. S. Waterman, “Introducción a la Biología Computacional”, Chapman & Hall, 1995.

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UNIBERTSITATE-GIZARTE
ELKARGUNEA
ENCUENTROS
UNIVERSIDAD-SOCIEDAD

Recopilación de Intervenciones en los

"Encuentros Universidad-Sociedad"

EL PAPEL DE LAS MATEMÁTICAS EN LA EMPRESA
MATEMATIKAREN BETEKIZUNA ENPRESAN

"UNIBERTSITATE-GIZARTE ELKARGUNEA"
HITZALDIEN BILDUNA

Palacio Euskalduna – Bilbao, 21 de Febrero de 2000

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UNIBERTSITATE-GIZARTE ELKARGUNEA

"El papel de las Matemáticas en la Empresa"
"Matematikaren betekizuna Enpresan"


EL PAPEL DE LAS MATEMATICAS EN LA EMPRESA
MATEMATIKAREN BETEKIZUNA ENPRESAN
ELKARGUNEA
ENCUENTROS

Participantes en el debate sobre

“El papel de las Matemáticas en la Empresa”

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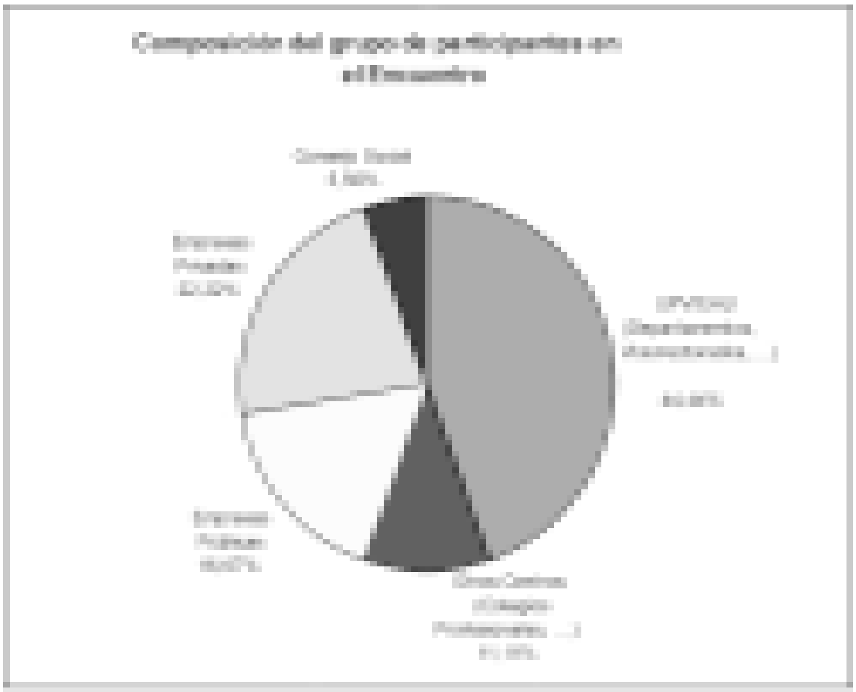

El cuestionario de satisfacción ha sido respondido por 22 de los asistentes (64,70 %).

RESULTADOS DEL CUESTIONARIO (notas medias)

1. ORGANIZACIÓN DE LOS ENCUENTROS

   a. La comunicación e información previa a las Jornadas: 6,95

   b. La duración: 8,18

   c. El horario: 8,09

   d. Medios utilizados: equipos, material, traducción simultánea,... 7,88

2. CONTENIDOS

   a. Contenidos tratados: 7,59

   b. Cantidad temas abarcados: 7,04

   c. Claridad, amenidad en la exposición: 7,45

   d. Ponente: 8,50

3. SUGERENCIAS

   a. Valoración general del Encuentro: 7,75

Indice

“El papel de las Matemáticas en la Empresa”

Pedro Larrea, Presidente del Consejo Social

Buenos días a todos. Bienvenidos a este Encuentro y muchas gracias por su asistencia. Una asistencia cualificada y equilibrada en el sentido de que aproximadamente la mitad de los aquí presentes procedéis del mundo académico y estáis involucrados en actividades docentes y de investigación en torno a las matemáticas y el 50% restante somos gente del mundo de la empresa, tanto pública como privada. Esta composición así de equilibrada promete dar bastante juego en el coloquio de este Encuentro.

Tras una breve presentación del ponente, a cargo de Manuel Tello, Presidente de la Comisión de Relaciones con la Sociedad, Alfredo Bermúdez de Castro expondrá su ponencia, en torno a 30 ó 40 minutos, a partir de esa hora tenemos tiempo hasta las dos, para debatir. Yo os pido que habléis de manera absolutamente libre, siguiendo algún hilo conductor que os haya parecido interesante seguir.

La importancia del tema no hay que enfatizarla. Por propia experiencia profesional yo puedo asegurar que muchas veces me he quedado bloqueado por falta de una mayor formación matemática en el desarrollo de algunos asuntos. En cualquier caso, hay aquí suficientes representantes de la empresa como para podernos confirmar si esto es así. Espero que la parte académica sepa vender a la empresa hasta qué punto la utilización de los instrumentos de análisis matemático pueden permitir una gestión más profesional de las organizaciones empresariales.

Os recuerdo que la documentación entregada contiene un resumen de la introducción que va a realizar el ponente, una publicación que recoge las intervenciones de la jornada anterior de estos Encuentros y el programa de las charlas que el Comité del País Vasco ha organizado para la celebración del Año Mundial de las Matemáticas durante el mes de marzo. También quiero recordar que podéis hacer llegar al ponente las preguntas que deseéis y en el momento que queráis hacerlo, haciendo uso de unas hojas que a este efecto tenéis encima de la mesa.

Por último, decir que a mi derecha está el profesor Bermúdez de Castro. También nos acompaña el profesor Lezaun, que es el Coordinador del Comité en el País Vasco para los eventos del Año Mundial de las Matemáticas y, como sabéis, profesor de la UPV/EHU; a mi izquierda está la Secretaria Técnica del Consejo, Pilar Elorrieta y, el profesor Tello, Presidente de la Comisión de Relaciones de la Sociedad y Decano de la Facultad de Ciencias.

Sin más cedo la palabra al profesor Tello.

Manuel Tello, Decano de la Facultad de Ciencias y Miembro del Consejo Social

Buenos días. La Jornada de hoy se va a desarrollar en torno a la ponencia que va a dar el profesor Bermúdez de Castro. El profesor Bermúdez de Castro es Catedrático en la Universidad de Santiago de Compostela. Inició su formación postgraduada en París en el laboratorio que en aquel momento dirigía el profesor Lyons, posteriormente Presidente de la Academia de Ciencias francesa. Es interesante recalcar que los franceses llaman laboratorio a un departamento de matemáticas donde se investiga en matemáticas. Al terminar su Doctorado volvió otra vez a Santiago donde se dedica a trabajar en ámbitos diversos de las matemáticas. Para los fines de este encuentro debemos destacar su dedicación a las aplicaciones de las matemáticas a temas de desarrollo industrial e impacto económico de los entornos. Así, ha trabajado en temas relacionados con la industria metalúrgica, con la industria aerospacial y con la industria medioambiental haciendo análisis de impacto medioambiental, sistemas de corrección, etc. El trabajo profesional del profesor Bernúdez de Castro nos demuestra que el mundo de las matemáticas no es simplemente el de suministrar una herramienta que necesitan las demás áreas del conocimiento sino, que es una ciencia que tiene contenido en sí misma y que, además en las sociedades avanzadas tiene un interés creciente para el sector productivo. Desde su incorporación a la Universidad de Santiago ha dirigido una docena de tesis doctorales y ha formado parte, prácticamente, de todas las iniciativas que han surgido en España y muchas de las que han aparecido en Europa relacionadas con la Aplicación de las Matemáticas a Entornos Industriales. Así pertenece a la Sociedad Española de Métodos Numéricos e Ingeniería, y a la Sociedad Española de Matemática Aplicada, desde su creación. Además es miembro de varias Comisiones Europeas dedicadas exactamente al mismo tipo de problemas. Entre ellas destaca, su participación en una red europea de excelencia (son redes que se crean para intentar incrementar la competitividad de Europa) dedicada a las matemáticas, a la computación y a la simulación para la industria. Y lo que le da valor a su personalidad como docente y como científico es el hecho de, dedicándose a temas tan aplicados no deja de ser valorado en ámbitos mucho más restringidos a ciencias básicas. Una prueba de ello es la reciente concesión del Premio de la Real Academia de Ciencias Exactas, Físicas y Naturales. Esto quiere decir que también los temas aplicados pueden ser de valor, pueden tener valor y además, en una sociedad innovadora, deben de tener valor. Y sin más les dejo con él, que es quien realmente nos va a dirigir la sesión.

Conferencia

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

En primer lugar quiero agradecer la invitación. Cuando hace unos días los profesores y amigos Mikel Lezaun y Luis Vega contactaron conmigo para ver si participaba en estas Jornadas, debo confesarles que experimenté sensaciones contradictorias. Por un lado el tema me agradaba mucho, pero por otro lado suponía un desafío para una persona que es de profesión matemático y que por lo tanto está más acostumbrado a hablar de ecuaciones y de teoremas, que a participar en un foro como éste, con una presencia importante de personas que, conociendo las matemáticas, no son realmente especialistas... En fin, creo que de todas formas la misión del ponente no es más que recopilar una serie de elementos para suscitar un debate y estoy seguro que la alta cualificación de todos ustedes va a suplir todas las carencias de mi intervención.

La charla va a tener dos partes: en una primera voy a hablar, de forma muy general, de las aplicaciones de las matemáticas, de cuál es la situación en nuestro país y de qué cambios habría que introducir para mejorarla; porque yo creo que se puede mejorar. En la segunda parte voy a intentar ilustrar algunos aspectos de la primera contándoles experiencias que ha llevado a cabo nuestro grupo de investigación en la Universidad de Santiago de Compostela con algunas empresas. De manera que la primera idea clave es que las matemáticas son útiles para el sistema productivo. Yo diría que son útiles en dos vertientes bastante diferentes, la primera se deriva del hecho de que estudiar matemáticas forma a la persona y le confiere una serie de aptitudes genéricas como son: capacidad de análisis, capacidad para detectar y analizar problemas y por lo tanto para intentar resolverlos, capacidad para comunicar, para formalizar, es decir, lenguaje, rigor, etc. Yo creo que esto es algo de lo que todos los aquí presentes somos conscientes y, sin duda, es lo que lleva a algunas empresas, a contratar matemáticos, no necesariamente en puestos de matemáticos, no necesariamente para hacer cálculos, sino para integrarse en equipos en el ámbito, por ejemplo, de la organización. Pero, además de esta vertiente hay otra de la que yo querría hablar más aquí, y es que las matemáticas sirven para resolver problemas tecnológicos y, en este sentido, entroncan con algo que hoy resulta fundamental en la empresa moderna: la innovación. Las matemáticas, como luego intentaré ilustrar con algunos ejemplos, constituyen un soporte esencial, una valiosa herramienta, para resolver problemas tecnológicos en el ámbito de la innovación; pero también es bien conocido que las matemáticas ayudan a tomar decisiones en el ámbito de la planificación, de la organización de la producción, de la planificación financiera; en el mundo de las modernas finanzas, las matemáticas constituyen un soporte fundamental para la evaluación del precio de los “productos derivados”. Pero, además de ser útiles en la industria, en la empresa en general, las matemáticas son útiles para mejorar la calidad de vida y reducir el impacto que las actividades económicas producen en el medio ambiente. Aquí tenemos tres ejemplos que ilustran este epígrafe que encabeza la transparencia; son ejemplos de índole muy diversa. En primer lugar, cuando uno entra en un hospital y se somete, por ejemplo, a una exploración radiológica, detrás de ese aparato hay matemáticas. El análisis de Fourier está detrás de muchos dispositivos que se utilizan hoy en radiodiagnóstico; los aparatos que se emplean para corregir la miopía con láser utilizan matemáticas. La medicina yo creo que es un ámbito donde en los próximos años las matemáticas van a estar muy presentes. Pero las matemáticas permiten también hacer cosas que están presentes en nuestra vida cotidiana, como es el caso de la predicción meteorológica. Cuando uno observa el mapa del tiempo a través de la televisión todas las noches, el llamado mapa de isobaras, generalmente desconoce que ha sido producto de una simulación en ordenador del comportamiento de la atmósfera. Ese mapa y otros que ayudan a los meteorólogos a predecir el tiempo a corto y medio plazo, se obtienen al resolver un modelo matemático constituido por ecuaciones. Lo mismo ocurre en el estudio de la evolución del clima; cuando se dice “si las actividades humanas no permiten reducir las emisiones de dióxido de carbono en un cierto porcentaje, el planeta se va a calentar tantos grados en el años 2100”, es porque existen modelos matemáticos que están indicando este tipo de evolucion. Un último ejemplo que enlaza con un tema de gran interés social: las matemáticas ayudan a evaluar el impacto ambiental. Con ayuda de modelos matemáticos se puede saber si las emisiones de ciertos vertidos, por ejemplo en una ría, van a ser toleradas por el ecosistema o, por el contrario, van a producir aumentos inadmisibles de la contaminación.

Bueno, pues, a pesar de todo esto, a pesar de que las matemáticas son realmente útiles, tanto para el sistema productivo como para mejorar la calidad de vida de los ciudadanos, en España la presencia de matemáticos en la empresa es escasa, a diferencia de países de nuestro entorno como Reino Unido, Francia o Alemania. Los matemáticos están poco en las empresas; los encontramos, fundamentalmente, en la docencia, en la investigación básica, y aunque es cierto que cada vez hay más empresas que contratan a matemáticos yo creo que no los contratan para hacer cálculos, no los contratan para hacer matemáticas en sentido estricto, los contratan la mayoría de las veces para tareas relacionadas con la informática de gestión. Las razones de esta situación son diversas; algunas son de carácter general, podrían aplicarse no sólo a los matemáticos, sino también a físicos, a químicos... Entre ellas está una falta de tradición científica y tecnológica. Los científicos tenemos conciencia clara de que en España ha habido recientemente un enorme avance en la investigación; en el caso de las matemáticas es clarísimo: en los últimos 20 o 25 años ha habido un desarrollo extraordinario, pero también somos conscientes de que no hay nombres relevantes en la historia de las matemáticas que sean españoles. Tenemos, por tanto, una falta de tradición científica que se traduce en una posición débil en el campo de la innovación tecnológica. Es cierto que, como decía, esta situación está cambiando.

Probablemente esta falta de tradición científica lleve a que la mayoría de la población no perciba la importancia que los avances científicos tienen en su bienestar. Antes les ponía un ejemplo en el ámbito de la medicina: estoy convencido de que cuando alguien va a hacerse una exploración radiológica a un hospital va a pensar que el aparato que tiene delante está relacionado con los ordenadores, con la informática; sin embargo, yo creo que no es consiente de que detrás de ese aparato hay algoritmos matemáticos bastante sofisticados. Por lo tanto, resulta lógico que después ese ciudadano no se muestre muy sensible a dar su apoyo a la investigación matemática. En mi opinión ésta es una de las causas por las que en este país no ha habido ningún programa para el desarrollo de la investigación matemática, como los que existen en otros campos de la ciencia o la tecnología La situación se refleja también en la poca presencia que los temas matemáticos, y en general los científicos, tienen en los medios de comunicación; de nuevo aquí hay una diferencia con los países de nuestro entorno.

Y por último, algo que afortunadamente está cambiando, y es que no existe una tradición de colaboración entre los centros de investigación y el sistema productivo. En este sentido yo creo que foros como éste del Consejo Social ayudan sin duda a que la situación cambie.

Pero además de estas causas de carácter general, yo creo que existen otras que son más específicas, más particulares de las matemáticas; en las que los matemáticos tenemos una responsabilidad especial. Me refiero por ejemplo a la orientación de los planes de estudio de las Facultades de Matemáticas que, en mi opinión, han estado muy polarizados hacia la matemática pura. Nos hemos preocupado de formar profesionales para que investigasen, lo cual ha dado sus frutos: un indicador como son las publicaciones realizadas por españoles en matemáticas en los últimos años, se ha incrementado considerablemente y ha llegado a porcentajes realmente importantes. Sin embargo hemos descuidado otros aspectos, muy importantes también por su transcendencia social, como son las aplicaciones. Lo peor es que esto no sólo ha ocurrido en las Facultades de Matemáticas, sino también en las Escuelas de Ingeniería y ahí el asunto es mucho más grave porque los ingenieros deberían estar especialmente sensibilizados por las aplicaciones de las matemáticas. En muchas ocasiones las matemáticas en la Escuelas Técnicas han jugado un papel de selección: puesto que son asignaturas difíciles, entonces las suspende mucha gente y eso permite que sólo continúen los mejores.

Con objeto de respetar el horario y no superar la media hora o cuarenta minutos, voy a hablar de la modelización de manera muy esquemática. En primer lugar quisiera hacer una observación: como todos sabéis, las matemáticas son muy diversas y en una charla como ésta resulta imposible referirse a todas ellas; por ejemplo no voy a hablar de la estadística, pero creo que es obvio el papel que tiene esta disciplina en muchos ámbitos; tampoco de la matemática discreta y sus relaciones con la informática; yo voy a hablar, fundamentalmente, de la modelización en ingeniería, por lo tanto me voy a restringir a un ámbito importante pero no el único de la matemática aplicada.

La idea es bien sencilla: el mundo físico se rige por unas leyes que se pueden formular con matemáticas de modo que éstas son, de alguna forma, el lenguaje de las ciencias experimentales; tradicionalmente de la física, pero hoy día también de la biología o las ciencias sociales. Entonces, estas leyes constituyen modelos que permiten simular el comportamiento de los dispositivos o procesos industriales. Generalmente los modelos están constituidos por sistemas de ecuaciones; esas ecuaciones tienen unas incógnitas que son las magnitudes físicas que representan el fenómeno objeto de estudio. Por ejemplo, si se desea conocer lo que pasa en el interior de una caldera de carbón de una Central Térmica, todos somos conscientes de que medir, observar dentro de un recinto donde tienen lugar flujos de gases complejos, temperaturas elevadas, etc., es algo realmente difícil. Entonces una alternativa consiste en recurrir a modelos matemáticos: la física y la química nos han proporcionado ecuaciones que gobiernan fenómenos como el movimiento de los fluidos, la transferencia de calor, las reacciones químicas de la combustión, etc. Pues bien, todas estas ecuaciones se pueden resolver hoy día con ayuda, eso sí, de ordenadores muy potentes y de métodos matemáticos sofisticados. Uno puede determinar, efectivamente, cuál es la temperatura en cada uno de los puntos de una caldera, cuál es la concentración de los diferentes gases, saber si la reacción de combustión progresa adecuadamente o no, si se producen asimetrías en la caldera que puedan provocar un calentamiento excesivo de alguna de sus paredes y por lo tanto escoriaciones, etc. Bueno, es claro que el uso de modelos matemáticos es una herramienta que permite al ingeniero diseñar y también operar adecuadamente una caldera de una Central Térmica. Este ejemplo resulta de alguna forma paradigmático, ya que se puede repetir en muchos otros ámbitos de la industria. Ahora me referiré a algunos ejemplos concretos.

Esta transparencia incluye algunos que son bien conocidos. Por ejemplo, el cálculo estructural con elementos finitos. Se trata de una herramienta que se ha incorporado perfectamente a la industria. Hoy día, el diseño de un avión, tanto en los aspectos estructurales como aerodinámicos, se lleva a cabo en el ordenador antes de construir maquetas y hacer ensayos en túnel de viento. Cabe pensar en lo que esto supone de ahorro de tiempo y dinero, y su importancia en el ámbito de la industria moderna que debe situarse en una economía abierta donde es preciso desarrollar los productos a gran velocidad para mantener la competitividad.

Antes me referí a aplicaciones de las matemáticas a otros campos diferentes al de la empresa industrial. En esta línea querría mencionar, aunque sea de pasada, uno que está experimentando un gran desarrollo en los últimos tiempos: el de las finanzas; concretamente el cálculo de los precios de los llamados productos derivados entre los que se cuentan las famosas “opciones sobre acciones”.

Esta ecuación que ven ustedes en la transparencia se utiliza para calcular el precio de una “opción europea”. Como pueden observar se trata de una ecuación en derivadas parciales que en este caso concreto se puede resolver “a mano” haciendo cálculos matemáticos. Esto es lo que hicieron Black, Merton y Scholes, los dos últimos Premio Nobel de Economía en el año 1997. Pero hoy día hay productos financieros mucho más sofisticados que estas opciones europeas, como las opciones americanas, o las asiáticas y si uno quiere calcular su precio tiene que resolver ecuaciones de este estilo; en muchas ocasiones esto no es posible hacerlo a mano y entonces es necesario ir a métodos numéricos y utilizar el ordenador como herramienta de cálculo.

Después de todo lo que llevo dicho está claro que hay una gran relación entre el modelado matemático y la informática, de hecho si el modelado matemático es cada vez más útil se debe, indudablemente, al avance de los ordenadores. En efecto, los modelos que rigen los procesos que interesan en ingeniería se conocen desde finales del siglo XIX; sin embargo la mayoría son ecuaciones complicadas que hasta la introducción de los ordenadores sólo se podían resolver en casos muy especiales de carácter académico y de escaso interés industrial. El avance de los ordenadores, su abaratamiento, factor muy importante, y el desarrrollo de nuevos métodos de cálculo permiten resolver modelos complejos en tiempos de cálculo razonables. Por lo tanto, los ordenadores están permitiendo que la simulación numérica sea realmente una herramienta útil, y no sólo para la gran empresa. Hace 10 o 15 años, tan solo la gran industria utilizaba modelización matemática. La razón es que, en aquella época, los ordenadores que permitían resolver en tiempos de cálculo razonables los modelos eran superordenadores, por lo tanto, dispositivos muy costosos que requerían instalaciones sofisticadas. Sin embargo, en la actualidad, muchos de estos modelos se pueden resolver en ordenadores personales y esto significa una auténtica revolución porque, la herramienta ya está al alcance de la pequeña y mediana empresa.

Esta transparencia es un poco más técnica y recoge lo que sería la metodología de la simulación y el modelado matemático. Lo primero que hay que hacer es, evidentemente, analizar los fenómenos que uno pretende simular. Por ejemplo, si se quiere estudiar la combustión en una caldera de carbón, hay que pensar, en primer lugar, cuáles son los fenómenos fisico-químicos involucrados. El análisis del proceso va a permitirnos detectar los fenómenos fisico-químicos y por tanto los modelos matemáticos que rigen su comportamiento. El paso siguiente es la construcción del modelo. Después hay una etapa importante que es el análisis matemático de este modelo, quizás uno de los aspectos clásicos del quehacer de los matemáticos. De ella podemos obtener una valiosa información, sobre todo cualitativa, sobre el comportamiento del sistema. Pero las cosas no paran aquí porque ahora existen métodos numéricos y ordenadores potentes que hacen posible resolver la mayoría de los modelos. Este proceso va a producir cantidades ingentes de números y es necesario tener unos sistemas de representación que los haga fácilmente aprehensibles. De nuevo la informática proporciona una herramienta fundamental para la visualización de estos resultados. Finalmente todas estas etapas tienen que ir acompañadas de una validación del modelo, comparando sus predicciones con medidas experimentales. La experimentación sigue siendo necesaria aunque la simulación numérica permita reducirla: en primer lugar se necesita para determinar los parámetros característicos de los materiales, como puede ser un módulo de elasticidad en un cálculo de estructura, pero también, en segundo lugar, la validación de un modelo requiere contrastar sus resultados con medidas experimentales.

Voy a pasar a la segunda parte de la charla, en la que voy a presentarles algunas experiencias que hemos llevado a cabo en mi grupo de investigación, en la Universidad de Santiago, para algunas empresas. Voy a hablar en primer lugar de dos aplicaciones en la industria del aluminio: la simulación de cubas electrolíticas y de coladas; después, del modelado de calderas de carbón de centrales térmicas; a continuación, del diseño de electrodos metalúrgicos y, por último, de la evaluación del impacto medioambiental de vertidos en el mar.

En esta transparencia se recoge de forma esquemática el proceso de producción del aluminio. El aluminio se obtiene a partir de un mineral llamado bauxita, que es hidróxido de aluminio; la bauxita se transforma en óxido de aluminio, en alúmina, en los llamados hornos Bayer. A continuación, ésta sufre un proceso de reducción electrolítica mediante un proceso que fue patentado simultánea e independientemente por Hall en Estados Unidos y por Héroult en Francia, a finales del siglo pasado. Se trata, por lo tanto, de un proceso muy antiguo que a lo largo del tiempo se ha ido mejorando. Una vez que el aluminio en forma líquida se obtiene en las cubas electrolíticas, se transporta a otra nave de la factoría, para proceder a su solidificación en forma de lingotes de diferentes secciones y tamaños, como son las llamadas “placas” y “tochos”. Voy a hablarles brevemente de la simulación de una cuba electrolítica y también del proceso de solidificación de coladas.

Esto son dos fotografías de fábricas donde se muestran series de cubas electrolíticas. Típicamente, en una serie hay decenas de ellas conectadas eléctricamente en serie. Veamos un poco el interior. Aquí tenemos una sección transversal de una cuba; se compone fundamentalmente de un recipiente de acero en cuyo interior se dispone un bloque conductor de la electricidad que lleva embutido una barra de acero y que constituye el cátodo. El otro polo, el ánodo, es de materiales carbonosos y a él llega la electricidad a través de unas agujas metálicas. La cuba contiene además una serie de materiales, ladrillos refractarios y aislantes, para evitar una fuga de calor excesiva. El conjunto deja libre un recipiente en el que se introduce la alúmina disuelta en un electrolito llamado criolita. Entonces, al paso de la corriente eléctrica continua, la alúmina se descompone y se produce el aluminio que queda en fase líquida, depositado en el fondo de la cuba. Posteriormente se extrae y es transportado a la planta de colada, para proceder a su solidificación. Entro un poco en estos detalles porque me interesa contarles cuál es la problemática que tiene el ingeniero de una fábrica de aluminio y que le plantea en un momento dado al matemático. Estas cubas consumen gran cantidad de electricidad, siendo éste un factor muy importante en los costes de producción. Por ello es muy importante conseguir un rendimiento energético óptimo. Pero, además, estas cubas se deterioran con el tiempo y cada cierto tiempo, algunos años, es necesario rehacerlas lo que conlleva también costes importantes. En resumen, el objetivo es doble: conseguir un buen rendimiento energético y alargar la vida de la cuba. Esta última depende, entre otros factores de la forma que tiene el "talud". El talud es un material sólido que se produce en las paredes de la cuba como consecuencia de la solidificación del baño electrolítico. Su posición y tamaño, determina, no sólo el funcionamiento de la cuba sino también su vida, y ello por razones que van a entender inmediatamente. En primer lugar, si el talud fuese demasiado grande y se metiese bajo la "sombra del ánodo", la corriente eléctrica se vería obligada a girar para evitarlo, porque es un material no conductor, y ello produciría una fuerza electromagnética que generaría inestabilidades en la superficie de separación entre el aluminio líquido del baño electrolítico. Estas inestabilidades pueden ocasionar cortocircuitos al tocar el aluminio el ánodo, lo que haría bajar el rendimiento eléctrico de la cuba. Por otro lado, si el talud fuese demasiado pequeño dejaría desprotegida la llamada "gran junta" y el baño, que es muy corrosivo, se infiltraría hacia la parte inferior de la cuba destruyéndola. Lo que el ingeniero desea saber es cuál es el comportamiento térmico de la cuba y, por las razones apuntadas, que forma va a tener el talud. Este conocimiento puede llevarse a cabo mediante simulación numérica. Uno puede escribir un modelo matemático del comportamiento de la cuba, en la que hay fenómenos de transporte de electricidad que se pueden modelar con las ecuaciones de Maxwell y de transferencia de calor que se puede modelar con la ecuación de Fourier; la resolución de esos modelos va a permitirnos representar las temperaturas, y en particular, saber cuál es la superficie del talud, cuál es su posición. Por ejemplo, los ingenieros de Inespal, que es la empresa para la que hemos hecho este estudio, se plantearon en algún momento, modificar el material del bloque catódico para mejorar el rendimiento energético, pero querían saber si ese cambio en el bloque podría alterar el equilibrio térmico de la cuba y modificar la posición del talud. Pues bien, con ayuda de programas de simulación numérica desarrollados por nosotros, fueron capaces de introducir las modificaciones necesarias en la geometría y en las propiedades del aislamiento térmico para que el talud tuviese la forma adecuada.

No voy a entrar en los detalles pero, aunque sea unos segundos, quiero mostrarles el modelo matemático utilizado; como ven hay una serie de ecuaciones que se resuelven mediante algoritmos matemáticos. Todo termina con la escritura de un programa y la elaboración de una aplicación que tiene una ventana principal con una serie de botones y menús desplegables, análoga a las que presentan las aplicaciones que utilizamos frecuentemente en la informática personal. Esto permite a los ingenieros introducir los datos con facilidad y al final obtener resultados como pueden ser los que les muestro en esta transparencia, donde uno puede ver por ejemplo las isotermas en una sección de la cuba, la posición del talud, etc. etc.

Otro proceso, también de la industria del aluminio, en cuya modelización hemos trabajado es la solidificación de coladas. El aluminio líquido se vierte en un molde que dispone de un falso fondo y que está refrigerado por agua que circula en su interior. Entonces comienza a solidificarse en las zonas en contacto con el molde y una vez que el aluminio sólido alcanza un espersor suficiente para contener al que todavía está líquido, se hace descender el falso fondo hasta completar la pieza colada.

Por cuestiones en las que no voy a entrar sobre la forma de cristalización del aluminio, es interesante utilizar una alternativa desarrollada en la antigua Unión Soviética, ya en los años 60, que es la colada electromagnética. En esta colada electromagnética el molde físico al que antes hacíamos referencia, se reemplaza por un molde electromagnético, es decir, por una bobina por la que se hace circular una corriente alterna y que va a generar un campo electromagnético que hace levitar el metal. En este caso son muchos los problemas tecnológicos que aparecen y en los que la simulación numérica puede aportar soluciones. Interesa, por ejemplo, determinar cuál es la intensidad de corriente que debe de atravesar la bovina para obtener una pieza de un tamaño determinado.

Por otro lado, existen problemas debido a las deformaciones que tiene lugar en el proceso de solidificación, por ejemplo, el llamado “talón” sufre una fuerte deformación que después obliga a cortar una parte importante de la pieza colada que hay que refundir. Por otro lado, debido también a que el metal al solidificarse se contrae, es necesario determinar la forma del molde para que, al final, la sección de la pieza sea, por ejemplo, un rectángulo perfecto. Pues bien, a todo este tipo de problemas se puede responder con ayuda de programas de simulación numérica.

Simplemente un apunte sobre la validación de los modelos. Aquí tenemos una comparación de las temperaturas calculadas por el modelo con las obtenidas mediante termopares en coladas reales, en la empresa Inespal. Naturalmente, este excelente acuerdo se obtiene una vez que uno ha hecho un ajuste.

Paso rápidamente a otro ejemplo. Se trata de un proyecto que estamos desarrollando para la empresa Ferroatlántica I+D, una empresa del sector de las ferroaleaciones. En particular, la fábrica situada en el polígono industrial de Savon, cerca de A Coruña, produce silicio-metal que se utiliza para fabricar aleaciones aluminio-silicio. Esto que tienen aquí es un horno de arco que como vemos es un recipiente cilíndrico con tres electrodos que transportan la corriente eléctrica trifásica de alta intensidad. En su extremo inferior se produce un arco eléctrico que libera la energía necesaria para romper la molécula de cuarzo y producir el silicio. La empresa Ferroatlántica, a lo largo de los últimos años, ha desarrollado un nuevo electrodo para este tipo de hornos, que está alcanzando un enorme éxito a nivel mundial, pues permite reducir los costes de producción del silicio en un porcentaje importante. Este electrodo, llamado ELSA, es de tipo compuesto ya que consta de un núcleo de grafito y una corona de pasta Soderberg y constituye una alternativa rentable a los electrodos llamados “precocidos”, formados al unir piezas que se compran a fábricas especializadas. Por el contrario, en el ELSA, la corona exterior se produce in situ, al cocerse una pasta compuesta esencialmente de coque y brea. La idea es que los electrodos precocidos son más caros que la pasta Soderberg, de manera que si podemos cambiarlos por esta corona de pasta estamos abaratando los costes.

En este caso se trataba de simular el comportamiento de este electrodo en sus diferentes aspectos termoeléctricos y termomecánicos. El proceso es como sigue: cada electrodo está rodedo de una corona de cobre por el que entra la corriente eléctrica. Ésta se distribuye por el interior del electrodo y después produce un arco eléctrico en la punta que desencadenará la reacción química de reducción. Uno de los problemas que se plantean es controlar la posición de la isoterma de cocción de la pasta. Si esta isoterma de cocción está demasiado baja, como el electrodo se hace descender periódicamente porque se consume por la punta, corremos el riesgo que la pasta que todavía está líquida, sin cocer, se vierta al exterior. Por otro lado, aunque infrecuentes, a veces se producen roturas del electrodo, originadas por tensiones debidas al peso o a gradientes térmicos.

Con ayuda de la simulación uno puede saber cómo se distribuye la corriente y ver que, por ejemplo, ésta tiene tendencia a entrar hacia el grafito porque es mejor conductor compitiendo de esa forma con el “efecto piel” que era una especie de mito entre los ingenieros del sector, pues creían que el grafito no conducía porque el efecto piel hacía que la corriente se fuese hacia el exterior del electrodo. Esto lo cuento porque puede ser una anécdota representativa del proceso de colaboración con la empresa donde muchas veces la simulación numérica ayuda a eliminar falsas creencias sobre el funcionamiento de los dispositivos.

En esta transparencia se muestra la isoterma de cocción de la pasta y en esta figura vemos las tensiones principales que se producen en la zona de los “nipples”. Este programa ha sido alimentado, previamente, con las temperaturas obtenidas mediante un modelo de transmisión de calor. Aquí podemos ver, por ejemplo, un punto de concentración de tensiones; este resultado está de acuerdo con la experiencia pues buena parte de las roturas de los electrodos se producen a este nivel.

Bien, de este otro tema ya les hablé algo hace un rato. Este es un proyecto que hicimos en el marco del Plan de Investigación y Desarrollo Electrotécnico para la empresa ENDESA, ya hace algunos años. Su objetivo era simular el proceso de combustión de carbón en la caldera de una Central Térmica. En esta transparencia puede verse una caldera; esta es la zona del hogar donde tiene lugar la combustión del carbón pulverizado y esta especie de dedos son los conductos que desembocan en los quemadores. Nosotros desarrollamos un código de simulación que permite, por ejemplo, representar las velocidades de los gases. Aquí tienen las zonas de entrada de los quemadores y cómo es el flujo en el interior de la caldera. Este flujo va a ser determinante para el reparto de la energía y, por tanto, para la distribución de temperaturas en el interior del hogar. Se pueden obtener transparencias como ésta que representa una sección horizontal a nivel de un piso de quemadores; observen las estructuras complicadas de los flujos, esos torbellinos que se producen, por ejemplo, detrás de los chorros y que pueden suponer zonas de remanso donde, eventualmente, puede haber puntas de temperatura y escoriación. En esta transparencia se pueden ver no sólo las velocidades indicadas por flechas sino también las isotermas, las zonas rojas corresponden a las zonas más calientes. Éste es un caso en que las zonas calientes están hacia el interior y eso significa un buen comportamiento.

También el modelo matemático suministra las concentraciones de las diferentes especies químicas que constituyen los gases; por ejemplo, aquí se representa el dióxido de carbono. Vemos cómo a la entrada los valores en azul son menores y después, en las zonas donde se produce la combustión, aumenta su concentración.

Uno podría plantearse, por ejemplo, ¿qué ocurre si se cambian los caudales de aire en los quemadores? ¿Qué ocurre cuando uno de los quemadores está inactivo? ¿Qué consecuencias tiene para el funcionamiento de la caldera?.

Y ya para terminar, una aplicación en la que también estuvimos involucrados, en este caso para la Consejería de Medio Ambiente de la Xunta de Galicia. Se trataba de estudiar el impacto producido por el vertido a través de emisarios submarinos de aguas residuales de origen urbano o industrial en las rías gallegas. En esta transparencia pueden verse las corrientes en la ría de Vigo. Con ayuda de estas corrientes, que han sido obtenidas con modelos matemáticos, se puede saber cómo se distribuyen en la ría las sustancias contaminantes. Aquí tenemos resultados de una simulación; en concreto se representa con diferentes colores la concentración de la "demanda bioquímica de oxígeno" producida por vertidos a través de un emisario submarino situado cerca de la ciudad de Vigo. Analizando estas gráficas uno puede saber, por ejemplo, si se alcanzan valores peligrosos en zonas de cultivos marinos, o en zonas de baños.

Este es un ejemplo de simulación numérica. Pero actualmente estamos aplicando también técnicas de teoría de control, a problemas de gestión de plantas de tratamiento. Supongamos que a una ría se vierten simultáneamente aguas residuales desde diferentes puntos porque hay varias poblaciones o varias industrias. Supongamos que cada uno de ellos tiene una planta de tratamiento aneja, con unos costes que pueden ser diferentes en función del tipo de vertido o de la tecnología empleada. Entonces es posible considerar un modelo global para ver cuál es la situación de la ría cuando están todos vertiendo simultáneamente e intentar gestionar el sistema de manera optima, es decir, establecer una función coste de depuración, y unas restricciones sobre el nivel de contaminación en determinadas zonas y determina cuál es el grado de depuración que hay que hacer en cada planta para que el coste global de tratamiento sea mínimo.

Y ya voy a terminar, porque creo que me he pasado un poco del tiempo asignado, presentando unas conclusiones. Podemos decir que los modelos matemáticos son una herramienta valiosa para la concepción y el diseño de dispositivos y procesos en la industria. Que la ingeniería moderna emplea cada vez más estas técnicas, conocidas a veces bajo el nombre de CAE (Computer Aided Engineering o Ingeniería Asistida por Ordenador). El uso de modelos permite acortar y abaratar el proceso de diseño y, en consecuencia, la salida al mercado de un producto con lo que esto representa de ventaja en una economía competitiva.

Una cuestión importante a la que antes me referí es que la simulación numérica está cada vez más al alcance de las empresas medianas y pequeñas ¿por qué? Pues, esencialmente, porque los ordenadores resultan cada vez más rápidos y menos costosos. Hoy día con un ordenador personal es posible utilizar muchos de las programas de simulación a los que me he referido en esta charla.

Finalmente quisiera decir que los matemáticos pueden colaborar: por una parte, en la escritura de los paquetes informáticos de simulación (porque hemos dicho que ahí lo fundamental son métodos de cálculo para resolver los modelos), pero también en su utilización. Los matemáticos pueden incorporarse a equipos en la industria formados también por ingenieros y físicos, y desempeñar un papel importante en el uso de estas herramientas que a veces, esto es innegable, resultan un poco sofisticadas. Es preciso conocer un poco más que el manejo mecánico del paquete como una caja negra. Los conocimientos básicos que configuran este campo son, por una parte, la modelización, lo que significa esencialmente conocer la física de los medios continuos, y por otra las ecuaciones en derivadas parciales que son los modelos fundamentales en este ámbito. Además, por supuesto, hay que conocer los métodos numéricos para poder resolverlas. Con estos ingredientes tenemos un perfil profesional que puede moverse en este campo y como digo colaborar para el desarrollo de la ingeniería en las industrias.

Y nada más, muchas gracias por vuestra atención.

Pedro Larrea, Presidente del Consejo Social

Bien, muchas gracias, Alfredo. Damos comienzo ahora al debate. Vuelvo a recordar que tenéis por ahí unas hojas para formular preguntas que luego trataremos de ordenar temáticamente a fin de facilitar las respuestas. Entre tanto, yo querría abrir el fuego haciendo una primera pregunta de tipo muy simple, muy mercantil.

Desde vuestra experiencia en Santiago, ¿cómo se establece el contacto con la industria, es decir, cómo vendéis vuestro producto (en el sentido más noble de la expresión), cómo vendéis vuestras capacidades, vuestro saber hacer, cómo conoce la industria cuáles son vuestras capacidades, habilidades para esa cooperación, para esa ayuda?. En definitiva ¿cómo se establece el vínculo oferta-demanda?

Afredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de Compostela, Facultad de Matemáticas

La pregunta me parece muy pertinente e interesante, es un tema que yo no he tocado pero ya esperaba que saliese en el debate. Si no recuerdo mal, cada uno de estos proyectos de los que os he hablado ha surgido de manera diferente. Desde luego, con carácter general, puedo decir que ha sido muy importante la existencia de instituciones u organismos de transferencia, como la Fundación Empresa-Universidad Gallega (FEUGA), que supongo que será análoga a otras que existen en esta Comunidad Autónoma, y la Oficina de Transferencia de Tecnología de la Universidad de Santiago.

Por ejemplo, las relaciones con Inespal surgieron a través de FEUGA: los ingenieros de la fábrica de A Coruña ya estaban sensibilizados por estas cuestiones de modelización porque en aquellos momentos todavía había una presencia en la compañía de ALCAN, la empresa de aluminio canadiense, que utilizaba la modelización para el diseño de las cubas. Por tanto, ellos tenían ya una motivación previa, y se dirigieron a la Fundación que entró en contacto con nosotros. Después, también pongo otro ejemplo, las relaciones con Ferroatlántica surgieron indirectamente a partir de la colaboración con Inespal. En un momento dado, la Universidad, ya no recuerdo con qué motivo, puso un stand en una feria donde se mostraban una serie de proyectos, entre ellos el nuestro con Inespal. Casualmente un ingeniero de Ferroatlántica que también sentía la necesidad de hacer estudios del electrodo con técnicas de simulación, vió el stand y se puso en contacto con nosotros.

En otras ocasiones algún colega de la Universidad pasa a trabajar en una industria y nos encarga un proyecto, etc. Yo creo que los caminos son muy variados, pero en líneas generales quiero destacar que todos estos organismos de interfaz desempeñan un papel muy importante.

José A. Jainaga, Director General de SIDENOR

Da la impresión que desde que salieron las calculadoras de bolsillo a los chavales se les ha olvidado multiplicar y dividir. Bueno, se les ha olvidado o no saben ya multiplicar y dividir. Con el desarrollo de la informática y de los ordenadores ahora hay tratamientos matemáticos que están al alcance de todo el mundo, incluso de particulares, pero sin embargo se dicen todos los días barbaridades en los periódicos o en las conversaciones sobre una cosa tan sencilla como es una media aritmética o sobre cuándo la diferencia entre dos medias es significativa o no, etc… y nunca se habla de probabilidades, uno se pregunta: ¿realmente hasta dónde se enseña a la gente?, ¿cómo se puede garantizar que la herramienta no sobrepasa al individuo? es decir, ¿somos capaces de conocer los límites de la herramienta?, ¿sabemos cómo utilizarla?, ¿no estarán contribuyendo los ordenadores a que la gente sea todavía más inculta en temas matemáticos?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

La pregunta me parece muy interesante porque toca una cuestión clave. Efectivamente nos encontramos con herramientas informáticas, en este caso yo he hablado de herramientas de simulación, que son aparentemente fáciles de usar. Y cuando digo que son fáciles de usar me refiero a que en cursillo de tres o cuatro días la empresa suministradora se compromete a explicar cómo hay que desplegar los menús e introducir los datos, y cómo después obtener unos resultados y visualizarlos. Efectivamente, esto da una imagen de que las cosas son demasiado sencillas. Podemos sacar la impresión de que el ordenador es infalible y de que todos los resultados que produce son correctos y conformes a la realidad. Aquí hay un gran peligro porque estas herramientas no son tan fáciles de utilizar como pudiera parecer a primera vista, al menos si uno quiere obtener resultados fiables.

Realmente es necesario saber un poco lo que hay detrás para poder ser críticos con los resultados. No sirve con que a uno le den un cursillo de una semana para utilizar un paquete de elementos finitos en cálculo estructural, hay que saber qué significa un comportamiento elástico o plástico, un cálculo estático o dinámico o un modelo de contacto; en ese sentido yo creo que la participación de los matemáticos puede ser útil. Los matemáticos en modo alguno van a suplantar a los ingenieros en una empresa, pero sí pueden ayudarles a elegir los modelos, a hacer los cálculos y a criticar y analizar los resultados.

Juan Andrés Legarreta, Director Gerente EUSKOIKER y Profesor de la Escuela Técnica Superior de Ingenieros Industriales y de Telecomunicaciones de Bilbao, UPV/EHU

¿Se deben incluir dentro de las “matemáticas” áreas como la geometría, informática o estadística?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de Compostela, Facultad de Matemáticas

Sí, sí, por supuesto, como indiqué a lo largo de mi intervención yo he hablado de una parcela de la matemática aplicada que es la que conozco; creo que se trata de una parcela muy importante en la ingeniería, pero indudablemente no es la única. La semana pasada con ocasión de una tesis doctoral estuvo por Santiago, un colega ingeniero que trabaja en este campo. Me decía que, en su opinión, el futuro de la modelización en ingeniería pasa por los modelos estocásticos y los criterios de diseño basados en conceptos probabilísticos. En el ámbito de la estadística, el análisis de datos tiene también una gran importancia en la industria. Muchas industrias tienen cantidades ingentes de datos extraídos de sus procesos de producción. Estos datos contienen mucha información, pero hay que desenmascararla mediante tratamientos matemáticos adecuados. De la informática creo que ya he hablado bastante. Es una disciplina imprescindible para el cálculo matemático. Sin ordenadores los modelos no podrían resolverse.

Eva Ferreira, del Departamento de Economía Aplicada III, Facultad de Ciencias Económicas y Empresariales Bilbao, UPV/EHU

¿Qué tipo de equipos de universitarios suelen estar en esos proyectos a los que te has referido, por ejemplo, matemáticos, ingenieros, algún otro grupo más y qué formación tienen los interlocutores en las empresas?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Veamos, en nuestra experiencia, inicialmente hemos organizado equipos dentro del Departamento, por tanto equipos de matemáticos; además, nuestros interlocutores en las empresas han sido ingenieros. Tengo que decir que esto ha sido posible, en alguna medida, porque desde hace tiempo nos hemos preocupado de estudiar física. Si esto no hubiese ocurrido es indudable que desde el primer momento habríamos tenido que contar con nuestros colegas de otras áreas científicas o técnicas. Para establecer el diálogo con la empresa es fundamental tener una idea de los fenómenos que se quieren modelar. Generalmente esto significa saber física, pero si uno quiere colaborar con personas que están en el mundo de las finanzas, pues habrá que saber de finanzas. A este respecto yo pienso que los matemáticos, por el tipo de formación que recibimos, estamos bastante bien preparados para aprender otras materias y esa tarea debemos hacerla porque caso contrario se establece una barrera que impide la comunicación y por tanto la colaboración.

Después, con el proyecto en marcha, necesitamos datos experimentales de los materiales y en ese momento hemos involucrado a personas de otros departamentos de la Universidad. Así por ejemplo, en el proyecto con Ferroatlántica del que les hablé también está trabajando un equipo del Instituto de Cerámica y otro del Área de Electromagnetismo, ambos de la Universidad de Santiago.

Con respecto a la formación de los ingenieros de las empresas, debo decir que, por supuesto, nos hemos encontrado con excelentes profesionales pero tengo la sensación de que algunos adolecen de un cierto déficit de conocimientos científicos básicos. Yo comprendo que hay que optar y que resulta difícil cubrir al mismo tiempo la enseñanza de conocimientos básicos y otros muy especializados, pero creo que en estos últimos años, con las reformas de planes de estudios, se han descuidado los primeros. Me refiero por ejemplo a que los contenidos de matemáticas y física han tendido a reducirse; yo estoy acostumbrado a trabajar con ingenieros franceses que, como todos sabéis, tiene una formación básica muy sólida, y echo de menos esa formación en España.

Juan Andrés Legarreta, Director Gerente EUSKOIKER y Profesor de la Escuela Técnica Superior de Ingenieros Industriales y de Telecomunicaciones de Bilbao, UPV/EHU

Me gustaria plantear una observación, y es una cierta discrepancia del aserto anterior suyo cuando decia que “en las Escuelas Técnicas las matemáticas se han convertido más en un elemento de selección que en una herramienta de aprendizaje propiamente dicha”.

En el año 1964 se produjo en España una drástica reforma de las enseñanzas técnicas. Por decisión ministerial, todas las carreras de Ingeniería Superior dejaron de tener los cursos Selectivo e Iniciación más cinco años de carrera y pasaron a tener, como el resto de las licenciaturas, únicamente cinco cursos. Esto supuso, entre otras cosas, una reducción importante en la formación científica básica que necesita un ingeniero. Hasta entonces las Escuelas de Ingenieros en España habían permanecido fieles a la inspiración de las Escuelas Francesas.

A finales de los años 70, unos 15 años después, todas las Escuelas de Ingenieros cambiaron sus planes de estudios de 5 a 6 años de docencia. En esos años se había impartido menos formación tanto en matemáticas como en otras ciencias básicas. Si incluimos la Estadística y la Informática dentro del campo de las matemáticas y tomando como ejemplo el Plan de Estudios de la Escuela de Bilbao, nos encontramos con un incremento en la formación matemática del Ingeniero Industrial que alcanzaba el 28% del total de su formación. A esto se había llegado por el convencimiento del profesorado de la necesidad de una sólida formación matemática en los futuros ingenieros.

El prestigio que han podido alcanzar los ingenieros españoles que han ido por ejemplo a Estados Unidos para realizar cursos de especialización ha estado basado en la preparación científica que tenían y, especialmente, matemática. Y por lo menos en la Escuela de Bilbao y en las Escuelas en las que he estado y conozco, en todas se hace la misma valoración sobre la necesidad de una sólida formación matemática. Yo creo que no se puede mantener el que las matemáticas en las Escuelas se han convertido más en un elemento de selección que en una herramienta fundamental para resolver problemas de ingeniería. Cualquier docente y cualquier ingeniero docente en una Escuela es consciente de la necesidad de la formación matemática porque si no la hay, no puede haber un conocimiento científico y técnico riguroso.

Juán José Anza, del Departamento de Matemática Aplicada, Escuela de Ingenieros Industriales y de Telecomunicaciones de Bilbao, UPV/EHU

Soy profesor del Departamento de Matemática Aplicada pero soy ingeniero industrial, lo cual ya significa algo de las cosas que estamos hablando. En este segundo cuatrimestre estoy impartiendo una asignatura que se llama Ampliación de Análisis Numérico, donde lo que tratamos es la resolución de las ecuaciones diferenciales, y la introducción al curso coincide en un 80% con la introducción que ha hecho usted hoy. Es otro dato para saber cómo estamos trabajando puesto que toda esta importante evolución que hay con el tema de ordenadores, etc., también afecta a las Escuelas de Ingeniería.

Los departamentos de matemática aplicada en las Escuelas de Ingenieros para mí son un campo de confluencia de matemáticos y de ingenieros. Cuando hablamos de matemáticas no debemos de pensar solamente en las matemáticas de las Facultades de Ciencias Exactas que son muy importantes sino en las matemáticas que se utilizan como herramienta en otras Escuelas, en otras Facultades, por ejemplo en las Escuelas de Ingeniería. Quizás, también porque otro de los asuntos que se ha tratado hoy es la formación de equipos, y la formación de equipos precisa interlocución, y la interlocución precisa sensibilidad por las cosas, etc. En resumen estoy muy de acuerdo con todo lo que se está diciendo aquí y mi intervención es sólo para ampliar campos, que en estos asuntos estamos mucha gente haciendo estas cosas.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Gracias. Me parece muy oportuna vuestra intervención para matizar algo que yo dije con carácter general y que probablemente hoy día ya no sea cierto, como vuestros ejemplos muestran. Yo tengo la impresión, pero, claro, es una impresión que quizás ya esté un poco obsoleta, de que hace algunos años los matemáticos que acudieron a las Escuelas Técnicas a explicar matemáticas lo hicieron de una forma muy parecida a como explicaban las matemáticas en las Facultades. Esto creo que hizo mucho daño, porque cuando hablo con ingenieros de mi edad, o incluso más jóvenes, tienen un recuerdo de aquellas asignaturas de matemáticas como de algo completamente estratosférico, es decir, algo que ellos no relacionaban con su ingeniería. Sufrieron las matemáticas porque las tenían que sufrir, tenían que aprobar aquellas asignaturas para hacer la carrera y realmente es una experiencia triste que al cabo de los años, cuando interaccionan con nosotros, se den cuenta de que las matemáticas son útiles; entonces se plantean, bueno, ¿por qué estas cosas no nos las contaron? Está claro que hoy día las circunstancias están cambiando y que hay una sensibilidad distinta en el profesorado de matemáticas de las Escuelas Técnicas. Nosotros, por ejemplo, en el Departamento de Matemática Aplicada de la Universidad de Santiago, impartimos asignaturas de la Facultad de Química, de la Escuela de Óptica, de la titulación de Ingeniería Química, etc. y hemos hecho un gran esfuerzo por aproximar la docencia a las necesidades reales y a las sensibilidades de estos Centros.

Javier Barrondo, Director de Planificación y Selección de IBERDROLA

No se trata solo de percepción. Hay muchos matemáticos en la enseñanza, incluso en las Escuelas de Ingeniería. En consecuencia se está tendiendo cada vez más a enseñar matemática especulativa en detrimento de la matemática aplicada con la pérdida de imagen asociada.

¿No habría que forzar más la opinión de la Empresa?

¿Qué papel puden o deben jugar las Fundaciones en la relación Universidad-Empresa?

¿Hasta qué punto falta marketing y comunicación al exterior?

Quería hacer un matiz previo en la línea que estaba marcando el ponente: las asignaturas las hace el profesor, es decir, son un poco a imagen y semejanza de quien las da. El ponente ha dicho algo así como “Que hay una sensación general de que los que estudian matemáticas sólo son para enseñarlas o para crear más matemáticas” Y esto ha sido una realidad, hay un montón de gente de exactas dando clase incluso en las Escuelas de Ingeniería. En consecuencia en las Escuelas de Ingeniería durante unos años se ha hecho una matemática especulativa, una matemática mucho más pura en contra de la matemática aplicada de la ingeniería. En este sentido iba mi pregunta ¿qué podemos hacer? ¿qué piensa el ponente?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Pues, yo creo que un papel muy importante. Yo no sé si nuestros amigos aquí presentes de las empresas, conocen la situación de los departamentos universitarios, de matemáticas o de otra cosa, es decir, qué medios tienen, qué misiones se les encargan, etc. Los profesores tenemos que dar clase, eso todo el mundo lo tiene claro, pero en los últimos años se nos piden además otras cosas. Se nos pide que hagamos investigación; incluso parte de nuestro salario está en relación con la investigación que producimos. Pero, además, en los últimos tiempos se nos pide que hagamos transferencia, se nos pide que intentemos acercar la universidad a la sociedad; y casi siempre sin que se nos den los medios necesarios. En muchos departamentos universitarios no existe una secretaría que sirva de soporte cuando, por ejemplo, uno tiene que mecanografiar un documento para presentar a una empresa. Las Universidades, tal vez por el sistema de gobierno que poseen, tienen una gran dificultad para discriminar a la hora de asignar recursos y decir, por ejemplo: "puesto que la misión de la Universidad también es hacer transferencia a las empresas de los resultados de la investigación, entonces vamos a apoyar a los departamentos que la hacen". Veamos otro ejemplo. Existe un problema serio y es que la asignación de profesores, con todas las excepciones que ahora me podáis relatar, se hace por criterios de docencia exclusivamente; se dota una plaza de profesor en un departamento solo si hay unas horas de docencia por cubrir. Ahora bien, esto es un poco absurdo: si la Universidad tiene una serie de misiones, y modernamente, no sólo la docencia sino la investigación, la transferencia, los desarrollos para las industrias, etc., pues entonces la asignación de recursos tendría que tener en cuenta todas esas misiones.

A mí me parece importante que la Universidad asuma este papel con todas sus consecuencia, si es que realmente cree que debe asumirlo. Yo veo que en este momento ya existe un discurso institucional, que incorpora estas actividades, pero después la política del día a día no lo sigue. Creo que en este sentido el Consejo Social debe hacer una labor para cambiar este tipo de hábitos.

Volviendo al tema de la pregunta, creo que las Fundaciones y los Centros de Transferencia de Tecnología de la universidades son de gran utilidad, pero si se quiere impulsar a los equipos de investigación a trabajar para las empresas hay que dotarlos de recursos singulares siempre que estén dispuestos a hacer el esfuerzo.

Carlos Bertrand y David Maza del Departamento de I+D, SIDENOR

¿En qué medida absorben hoy en las empresas los ingenieros y físicos el papel de los matemáticos? Y en este sentido, ¿qué podrían aportar, realmente de diferencia los matemáticos para convencer a los responsables de selección de personal de las empresas de que tienen que contratar también a este tipo de profesionales?. Otra pregunta más ¿se puede pensar que estamos en un momento en el que las matemáticas se están ya acercando a la fase de diseño industrial a todo nivel (esto tendría que ver con los ejemplos de modelización que has puesto), pero todavía están alejadas del momento del proceso productivo en sí, algo así como el día a día de las operaciones donde todavía se emplea mucho el método de prueba error y seguir funcionando?.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Con respecto a lo primero, vamos a ver, es una pregunta un poco difícil de responder y algo comprometedora para mí. Ingenieros, físicos, también matemáticos, bueno... depende. Me explico: un físico que tenga una buena preparación en matemáticas, como generalmente tiene una buena preparación en modelización, es indudable que puede hacer un gran papel. Ahora, yo tengo alguna experiencia en manejar paquetes de simulación, por ejemplo en mecánica de fluidos o en mecánica de sólidos no lineal, bueno y existen aspectos realmente sofisticados que uno necesita conocer cuando utilice ese paquete; me refiero a los que tienen que ver con los algoritmos numéricos, con el proceso de cálculo; para éstos, en principio, los matemáticos están mejor preparados.

Pero esta pregunta me lleva también a plantear otro tema que de alguna forma esbocé en mi exposición: ¿es adecuada la formación que le damos a los matemáticos en las Facultades? Yo estoy convencido de que los matemáticos pueden jugar un papel complementario del de los ingenieros o los físicos en temas como la simulación numérica donde el cálculo es fundamental, pero siempre y cuando en las Facultades de Matemáticas se haga un esfuerzo por adaptar los planes de estudio. Si queremos colocar a los matemáticos en este sector es necesario que los planes de estudio incluyan, por ejemplo, la mecánica de los medios continuos o el electromagnetismo. En este sentido, por si puede servir de ejemplo, puedo contaros nuestra experiencia en la Universidad de Santiago. Hace unos 8 años creamos una especialidad que se llama “matemática aplicada” y que pretende formar profesionales en el campo de la modelización matemática en ingeniería y que incluye entre sus asignaturas estas materias. Si ustedes me hablan de matemáticos que no conocen los modelos de la mecánica, la transferencia de calor, etc. probablemente sea preferible incorporar físicos o ingenieros, pero si me hablan de matemáticos que tiene conocimientos de modelización e informática, entonces creo que tenemos un profesional realmente interesante para las empresas. En este sentido también lanzaría un mensaje a mis colegas matemáticos: en mi opinión sería muy conveniente que este aspecto de la matemática como herramienta de modelización se incorporase a toda la enseñanza desde el principio, es decir, que cuando uno está explicando en los primeros cursos de la carrera los teoremas de existencias de las ecuaciones diferenciales, también se informe a los estudiantes de que ciertas ecuaciones diferenciales son modelos para un circuito eléctrico o un sistema mecánico, etc.

Josu Sagastagoitia, Director Gerente de Metro Bilbao

¿Cuál sería la aportación de un matemático a una empresa de servicios tan concreta como puede ser una empresa de explotación ferroviaria, un metro como el metro de Bilbao?. ¿Qué es lo que puede aportar un matemático, en un caso como éste?.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Voy a atreverme a contestar aunque intuyo que el campo de aplicación, en este caso al Metro, no es precisamente en el que yo trabajo. Indudablemente hay unas disciplinas, relacionadas con la matemática discreta, teoría de colas, transporte, programación entera, teoría de grafos, etc. que suelen incluirse en la llamada “investigación operativa”, y que son adecuadas para todos los problemas de opimización de rutas, de gestión de coches, etc. que pueda haber en el Metro, es decir, sin lugar a dudas un experto en estadística e investigación operativa sería muy conveniente en una empresa como la suya de explotación de un sistema de transporte.

Antonio Corral, Director de Area, Consultora IKEI

¿Hacia dónde se dirige hoy la investigación matemática? ¿Qué tipo de problemas se tratan de resolver? ¿No ocurre que la investigación tecnológica es la que está marcando el camino?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Es una pregunta que como “matemático aplicado” me resulta difícil de contestar. Yo puedo hablar de investigaciones en curso en torno a los modelos, a las ecuaciones en derivadas parciales, a los métodos numéricos, temas en los que en estos últimos años hay una gran actividad, pero las fronteras de la investigación matemática más innovadora tal vez estén en otros campos, más relacionados con la topología o con el álgebra. En ese sentido, con motivo del Año Mundial, se encargó a un Comité que estableciese un listado de problemas que presumiblemente van a ocupar el quehacer de los matemáticos en este siglo; algo análogo a lo que hizo Hilbert en el famoso Congreso de principios del siglo XX. Ahí figuran una serie de problemas abiertos alguno ya propuesto por Hilbert, y todavía sin cerrar. Son problemas que se encuadrarían en la matemática pura y ahora aprovecho, ya que hemos hablado tanto de la matemática aplicada, para decir que la sociedad no debe olvidarse de la investigación básica, porque la investigación básica de hoy es la base para la investigación aplicada del futuro. Cuestiones que en principio han sido de lo más abstracto y alejado de la realidad, con el paso del tiempo han tenido aplicaciones muy importantes. Pensemos en temas de teoría de números, de criptología que hoy en día se utilizan en las telecomunicaciones y que aparecieron en el ámbito de la matemática pura; campos como la teoría de grupos, o la geometría de Riemann, que ha sido tan importante para la teoría de la relatividad; en resumen, la investigación básica hay que mantenerla y el llamamiento que hago a mis colegas de las Facultades es que intenten preservarla a toda costa. Es necesario que en los planes de estudio haya una vía para que la gente pueda al acabar hacer una investigación matemática de altura. Ese es un objetivo al que no debemos renunciar. Yo creo que el problema es encontrar un equilibrio: la sociedad nos exige que al mismo tiempo que financia esa investigación básica, que va a tener unos retornos a más largo plazo, obtenga resultados a más corto plazo para algunas demandas del día a día. Creo que muchas veces a los matemáticos nos ha faltado esa capacidad para acercarse a problemas que a lo mejor desde ciertas corrientes se consideraban secundarios, de segunda fila, porque estaban más relacionados con el cálculo y menos, a lo mejor con las teorías abstractas y porque resultaban menos interesantes desde el punto de vista de la investigación. Insisto, pienso que es una cuestión de equilibrio y ese equilibrio también lo deben recoger los planes de estudios.

Pedro Mª Altuna, Miembro del Consejo Social en representación de ELA

¿Hasta qué punto cuando la Administración o las grandes empresas anuncian sus previsiones económicas sobre el IPC, magnitudes macro, crecimientos, beneficios, etc. hay rigor matemático detrás de todo eso o más bien se hace con criterios propagandísticos?.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Es una pregunta a la que yo no le puedo responder. Está claro que los modelos econométricos, son fundamentales para la predicción económica y estoy seguro de que en el Ministerio de Economía se utilizan con mucha frecuencia, o en el Banco de España, etc. etc., eso es indudable. Ahora bien, es obvio que los resultados que proporciona un modelo matemático dependen de los datos con que se alimenta, tanto en el caso de la ingeniería como en el caso de la economía: si uno está haciendo el cálculo de una estructura y resulta que introduce mal el módulo de Young del material, las tensiones que calcule el modelo serán incorrectas. Y esto ocurre igual en la economía. Evidentemente el hecho de que este tipo de técnicas estén muy distantes de los conocimientos habituales de los ciudadanos, permite que sean más manipulables. Y a veces diciendo “No, no, porque esto lo dice el modelo X”. El modelo X no es un absoluto, depende de cómo lo haya utilizado.

Luis Vega, Departamento de Matemáticas, Facultad de Ciencias, UPV/EHU

A propósito del tratamiento de datos en las empresas, ¿es verdad que hay carencias?, ¿cómo se podrían solventar?

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

En este tema del tratamiento de datos yo antes os contaba la experiencia que tuve con Ferroatlántica. Su director me dijo en una ocasión: “Mira yo tengo aquí muchos datos obtenidos día a día en la planta, estoy convencido de que esos datos encierran mucha información, pero no sé cómo extraerla; intuyo que eso es una labor de matemáticos, tu qué opinas?” Yo les puse en contacto con colegas del Departamento de Estadística y elaboraron un proyecto que en este momento ya está dando resultados. Creo que este tema del análisis de datos es común a muchas empresas industriales y no siempre se sabe que hay unos profesionales con conocimientos para extraer información a través del análisis de esos datos. Por supuesto también en las Administraciones Públicas. Así, por ejemplo, actualmente se están empleando estadísticos en los hospitales para el tratamiento de datos. No sé si esto responde a tu pregunta, Luis...

Luis Vega, Departamento de Matemáticas, Facultad de Ciencias, UPV/EHU

Me gustaría preguntar a los empresarios que hay aquí, si esto es una realidad o es algo que nos imaginamos y también qué es lo que demandan.

Agustín Berasaluce, Subdirector General del Departamento de Investigación Comercial, BBVA

Yo puedo contestar a esa pregunta. Nosotros tenemos mucha información del comportamiento de nuestros clientes y de cómo operan con nosotros, por qué canales, qué productos, y evidentemente es una labor muy importante. Yo creo que todos estos temas son muy importantes para muchas industrias, en general, y para todas aquellas en particular que tiene muchos clientes; para todas las empresas de servicios, desde servicios financieros, hasta servicios de telefonía por ejemplo. Yo creo que todas aquellas empresas que tienen muchos clientes y prestan servicios, estamos en un grado de desarrollo intermedio y a mí una cosa que también me extraña de este tema es que todos los softwares, todas las sistemáticas de análisis para usuarios de datos, son extranjeras y normalmente americanas. Es decir, esto es algo que está viniendo de otros países y nosotros estamos utilizándola como usuarios y con desarrollos internos salvo algunas cosas específicas que estén haciendo, que estén haciendo ad hoc para alguna empresa. Pero no se sabe, es un poco una visión del tema.

Yo lo que quería preguntar es si el papel de las matemáticas en la empresa aumenta, pues yo creo tenemos las empresas que contratar muchos más matemáticos y eso era un poco también se solapa alguna pregunta que ha habido antes de cómo se posicionan los licenciados matemáticos porque creo que mucha de la exposición se está haciendo muy dirigida hacia la investigación o hacia, digamos, la gente que se dedica a investigar dentro de la Universidad, pero un licenciado de la Universidad con muchas inquietudes y muchas lagunas respecto a lo que es el mundo de la empresa.

Para que se contraten más matemáticos yo creo que la propia Universidad debe posicionar a sus individuos, no sé, hacia áreas concretas. La pregunta concreta era: ahora hay un boom de determinados sectores, la nueva economía, internet, los mercados financieros, son sectores que están demandando bastante análisis numérico, cualitativo y también bastante sofisticado y a mi me gustaría saber si, en conjunto, hay un planteamiento de la ciencia matemática, decir, ¡oye!, mis licenciados que salen de aquí van a estar bien posicionados en estas áreas y van a poder competir con los físicos, los economistas o lo que sea. ¿Cómo se plantea la propia Universidad que sus licenciados puedan estar presentes en todo este mundo de la nueva economía?.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Bueno, como la pregunta es muy general pediría que otros contertulios interviniesen. Yo voy a referirme a dos aspectos que has tocado. Por un lado, la parte relacionada con la informática. Todos sabéis que en España existe una titulación de Ingeniería Informática, pero también hay otras ingenierías que están muy cerca de la informática y sobre todo de las comunicaciones; me refiero naturalmente a la Ingeniería de Telecomunicación. También físicos y matemáticos tienen en sus planes de estudio varias asignaturas de informática donde los estudiantes aprenden lenguajes de programación, sistemas operativos, informática gráfica, etc. Por tanto es éste un campo donde los matemáticos pueden competir perfectamente con otros titulados, incluidos los titulados de informática.

Después tocaste un campo que es el de las modernas finanzas donde yo sinceramente creo que tenemos ventaja frente a otros titulados. Creo que en temas como la gestión de riesgo, de las carteras, la evaluación de los productos financieros "derivados", todos ellos tan importantes hoy día, se requieren matemáticas y matemáticas nada sencillas que incluso superan los contenidos que se imparten en la Licenciatura. Un estudio a fondo de estas cuestiones pasa por la teoría de la probabilidad, los procesos estocásticos, las ecuaciones en derivadas parciales y su resolución numérica, etc. Estos temas son, de principio a fin, matemáticos, por eso creo que los matemáticos tienen ventaja incluso frente a los economistas que, en general, tiene una formación matemática probablemente suficiente para lo que era el contexto tradicional de la carrera de económicas pero creo que demasiado escasa para abordar este tipo de problemas. Una anécdota a este respecto, en Inglaterra hay grupos de matemáticos que estaban haciendo investigación en mecánica de fluidos que se han pasado a trabajar en matemática financiera; ¿por qué?, pues en buena medida porque esas ecuaciones de la mecánica de fluidos, son practicamente las mismas que aparecen a la hora de valorar una "opción". Me comentaba un colega francés, de la Universidad de Lyon, que daba un curso de modelización en física e ingeniería y que un año incorporó tímidamente unas cuantas lecciones sobre matemática financiera. Pues bien, de la noche a la mañana, no sólo se incrementó el número de sus alumnos sino que empezó a ver como los titulados, se empezaban a colocar en la banca.

Mikel Lezaun, Director del Departamento de Matemática Aplicada y Estadística e Investigación Operativa, Facultad de Ciencias, UPV/EHU y Coordinador del Comité en el País Vasco para la celebración del Año Mundial de las Matemáticas

Quisiera hacer algunos comentarios desde mi perspectiva como profesor de la Facultad de Ciencias.

En la exposición que nos ha hecho Alfredo, las colaboraciones industriales que ha presentado, son con grandes empresas y se puede llegar a pensar que estos proyectos sólo pueden hacerlos estas empresas de gran tamaño. Sin embargo, yo creo que hay muchas posibilidades de colaboración y de participación, también, en proyectos con pequeñas empresas y esto dentro del Programa Nacional de I+D. En este sentido, nosotros tenemos una experiencia de colaboración con una empresa de ingeniería agroalimentaria que, entre otras actividades, desarrolla tecnología para el cultivo hidropónico en invernadero. Esta empresa de ingeniería es INKOA y en el proyecto también participa un centro de investigación agraria, NEIKER, y una empresa productora, BARRENETXE, S. Coop.

En este proyecto, efectivamente, ha habido que hacer una modelización del clima en un invernadero siguiendo todos los pasos que ha comentado Alfredo. Se han identificado las variables climáticas controlables (radiación solar, humedad, temperatura), a partir de ellas se ha establecido un modelo matemático para el cálculo de la evapotranspiración, se ha pasado luego a identificar la situación óptima del cultivo (desarrollo más o menos rápido) y se ha establecido un mecanismo de control de las variables para dirigir el cultivo en el sentido deseado. Además, ha habido que modelizar el comportamiento del sustrato en lo que respecta a la retención de agua, para así precisar la dosis de riego. Todo este proceso ha habido que completarlo con un cálculo experimental de las variables de cultivo (superficie foliar, LAI), según la edad, tipo o época del cultivo. Bien, aquí tenemos un ejemplo concreto de colaboración con una empresa pequeña dentro del Programa I+D y en él se observa la importancia de las matemáticas en un dominio que podría parecer lejano, como es el del cultivo en invernadero. El resultado de todo esto ha sido un paquete informático y una serie de mecanismos físicos de control del clima (de la temperatura, la humedad, la radiación solar) y del riego en un cultivo hidropónico en invernadero, que ya está en el mercado.

Otro aspecto que quisiera resaltar es que desde las matemáticas no podemos presentar proyectos propios de I+D, sino que tenemos que participar en proyectos que tienen que plantearse desde las empresas. Hay que resaltar que su desarrollo y resolución será muy productivo para todas las partes.

Pasemos ahora a la Estadística. Nosotros estamos convencidos que muchas grandes empresas y la Administración tienen montones de datos y no los utilizan, no extraen la información contenida en ellos. En concreto, hemos tenido relaciones con Sanidad y Osakidetza y en muchos casos ocurre así. Por ejemplo, recogen datos de contaminación en las playas y estos casi van al cajón, como mucho hacen un recuento. Lo mismo podemos decir de campañas de salud dental de los escolares de los años tal, tal y tal, los datos están en un cajón. Nos parece que esto mismo ocurre en grandes empresas. En este sentido, me parece oportuno indicar que nuestro Departamento ha establecido una colaboración permanente con Osakidetza, en concreto con el Hospital de Cruces y el Hospital de Galdácano. Esta colaboración no es sólo para tratar los datos que ya tienen, sino también para el diseño de experimentos. Muchas veces el propio experimento está mal diseñado ya que puede que el tamaño de la muestra no sea el adecuado, o que no se hagan las preguntas pertinentes, o que no se obtengan los datos adecuados para poder extraer la información que hay detrás. Tengo que notar que una de las responsables, Arantza Urkaregi, está aquí presente.

Por último está el aspecto de la matemática financiera, que nosotros no lo tocamos. Ahora bien, hay compañeros de Sarriko que están trabajando en este dominio y que tienen relaciones con instituciones financieras.

Resumiendo, nos parece que también en empresas pequeñas hay necesidad de matemáticas y que éstas son las que tienen que empezar dando pasos para localizar y perfilar las posibles aportaciones de las matemáticas en sus proyectos de I+D. Todo ello pasa por un conocimiento mutuo de la Universidad y el mundo empresarial.

Arantza Urkaregi, Departamento Matemática Aplicada, Estadística e Inv. Op., Facultad de Ciencias, UPV/EHU y Miembro del Consejo Social

Volviendo al tema de análisis de datos, ¿No crees que se puede plantear el mismo problema que antes se ha planteado con los paquetes de simulación?. Dado que hay paquetes estadísticos, existe la idea de que cualquier persona, ingeniero, economista, etc. puede utilizarlos y sin embargo, para realizar un buen análisis de datos, es necesaria buena formación en estadística.

Alfredo Bermúdez de Castro , Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Totalmente de acuerdo. Creo que lo que antes decía con carácter general, se aplica indudablemente al caso de los paquetes de análisis de datos, de los paquetes estadísticos. Utilizar un paquete es una cosa y hacerlo con capacidad crítica y con conocimiento de causa otra muy distinta; esto último requiere unos conocimientos muy importantes. No hay que confundir la facilidad en el uso formal de un paquete, con un conocimiento fundamentado de lo que está haciendo y de cómo lo está haciendo. Sólo de esta forma se puede hacer una utilización adecuada, es decir, fiable.

Javier Duoandikoetxea, Director del Departamento de Matématicas, Facultad de Ciencias, UPV/EHU y del Comité Organizador para la celebración del Año Mundial de las Matemáticas en el País Vasco

Quisiera intervenir ya que antes Alfredo ha pedido la colaboración de otros colegas académicos en las respuestas.

Mencionaré un par de cosas que se han dicho hace un momento y que me parecen interesantes. Se ha dicho que la formación de los matemáticos, de lo que específicamente se llaman matemáticos, que son los licenciados en matemáticas, es más bien teórica y también se ha indicado que el papel de las matemáticas en la empresa no sólo viene, y no sólo tiene que venir, de los licenciados en matemáticas.

Es verdad y es una tradición de muchos años que la carrera de matemáticas está sobre todo dedicada a la enseñanza. Una de las razones fundamentales es que la enseñanza demandaba muchos licenciados y eran muchos los que se colocaban, así que todos entrábamos en la Facultad con ese objetivo y no con vistas a un trabajo en la industria o en otro lugar. La formación ha sido muy teórica y cuando se ha desarrollado la posibilidad de investigación en matemáticas, que ha tenido un gran avance en España durante los últimos 20 años, seguimos reproduciendo el mismo esquema porque también la investigación matemática en la unviersidad ha sido más bien teórica. Pero ahora estamos en una situación en la que soplan vientos de cambio, sobre todo por necesidad, porque la enseñanza ya no absorbe tanto ni mucho menos. Y es una realidad que últimamente los licenciados en matemáticas han encontrado salidas distintas de la enseñanza, pero son casi siempre ante un ordenador y no trabajando directamente en matemáticas.

¿Va a producir la situación actual del mercado un cambio real en los planes de estudio?. Hace unos días, este fin de semana concretamente, ha habido una reunión de directores y de decanos de matemáticas en Santiago y era frecuente oir propuestas en esta línea, pero yo no veo que se vaya a producir a corto plazo una evolución drástica de la carrera de matemáticas. Puede haber alguna especialidad más acercada a las aplicaciones, pero se seguirá formando a la gente sobre todo en el aspecto teórico. Yo creo que existe una diferencia fundamental en la actitud de los estudiantes desde la propia entrada, el que entra en la Escuela de Ingenieros piensa que cuando termine la carrera va a ser ingeniero, el que entra a una licenciatura de matemáticas todavía posiblemente piense que va a enseñar matemáticas o algo parecido, pero no entra con la mentalidad de que se va a formar para después estar en la industria, o en una empresa, o en un banco, o en otro lugar. Seguramente tenemos que conseguir cambiar ese punto de vista para cambiar la carrera, aunque se puede discutir cuál de las dos cosas debe venir antes. Mirado desde los departamentos de matemáticas que estamos en Facultades de Ciencias formando matemáticos, una dificultad es que tendríamos que hacer los cambios personas que hemos sido formados al estilo tradicional y hay poca gente en los departamentos tiene un conocimiento de los problemas prácticos como los que ha presentado Alfredo Bermúdez de Castro. Es decir, todavía puede haber muchos profesores que se sientan más cómodos en una situación como la actual.

Entonces, ¿qué puede pasar? ¿hay salidas posibles para este problema?. En algunos sitios se están haciendo experiencias de formación de postgrado que son muy importantes y prometedoras. Seguiremos formando matemáticos teóricos y además es interesante que exista esta formación básica porque corremos el peligro que indicaba al principio el representaante de SIDENOR, de que lo mismo que ahora la gente parece haber olvidado a multiplicar por culpa de las calculadoras, si no tiene una formación básica no sepa interpretar lo que dan los aparatos. Ahora bien, después de recibir la formación básica se pueden aprender muy rápido y en poco tiempo cosas más específicas. Ya hay experiencias iniciales, por ejemplo, en matemáticas financieras en las que hay masters con bastante participación y con implicación de los bancos. En Barcelona hay uno en la Universidad Autónoma que este año tiene 25 estudiantes, pero además hay bancos dispuestos a contratar durante 4 meses en prácticas a las 25 personas inscritas y pagarles el equivalente a lo que cuesta el master. En la misma universidad se está haciendo este año por primera vez una experiencia parecida con un master de matemáticas aplicadas a la industria y ahí también hay industrias que se comprometen a recoger durante un período a las personas que realizan esa formación de postgrado y pagarles el tiempo de prácticas de modo que recuperen la inversión inicial del master. Así que también hay una implicación directamente por parte de la industria. Creo que en este tipo de formaciones de postgrado destinadas a estudiantes de matemáticas los departamentos de matemáticas tendremos una participación posiblemente pequeña porque la preparación teórica ya se la hemos dado antes. Ahí es donde deberían intervenir desde Departamentos de Economía o Departamentos de Ingeniería hasta profesionales de fuera de la Universidad que pueda presentar el aspecto práctico de cómo se trabaja en la industria, en la empresa, en las finanzas, etc. Porque, insisto, los alumnos que entran a la carrera de matemáticas no suelen tener como objetivo este tipo de actividad y, si ven que hay nuevos caminos, los estudiantes ya podrían entrar con esa mentalidad. En algún momento se podrá pensar que la profesión de matemáticas sirve para algo que no sea dar clase.

Y para terminar otro comentario: ¿quién es el que le da prestigio a la matemática? Antes, se ha hablado de los planes de estudio de escuelas técnicas y ha habido una pequeña discusión de si las matemáticas eran sólo para seleccionar o servían para otra cosa y de si se habían rebajado o no las exigencias. Yo no voy a poner un ejemplo de los ingenieros, que están en un departamento distinto del nuestro, hablaré de nuestra Facultad. Nosotros, en la Facultad de Ciencias, además de a los matemáticos damos clase a físicos, químicos, biólogos y geólogos. En biología es un hecho evidente que las matemáticas se han reducido y se han reducido drásticamente, la formación matemática de un biólogo es muy escasa y los alumnos de biología, en realidad, estarían muy contentos si incluso quitásemos ese poquito. ¿Por qué? Porque los profesores de biología o los biólogos profesionales, que son los que después deberían utilizar las matemáticas, o no ven la necesidad, o no la tienen, o por lo menos no la prestigian suficientemente. Es verdad que hemos estado enseñando muy mal esos cursos porque hemos estado dando un punto de vista demasiado teórico, inadecuado para los biólogos. Pero si hay necesidad de matemáticas, el biólogo profesional debería hacérselo sentir al alumno y él mismo o el alumno debería venir a nosotros y decirnos que el curso está mal dado pero que necesitan ese curso bien hecho.

Volviendo a los ingenieros, yo creo que la cuestión no es tanto, cuántas matemáticas enseñan los profesores de matemáticas en Ingenieros, sino cuántas matemáticas utilizan los ingenieros que no son profesores de matemáticas. Ellos deberían decir a los profesores de matemáticas “Esto es lo que queremos que enseñéis” y además decirle al alumno “Esta es la matemática que yo utilizo y esa es la que quiero que sepas y además, aprende mucha más, porque si sabes más matemática que yo, seguramente, podrás utilizarlas mejor”.

Lourdes Llorens, Directora del Instituto Vasco de Estadística EUSTAT

Me gustaría por alusiones, hablar un poquito de lo que has comentado sobre la información; sobre la cantidad de información que existe en el Gobierno. Realmente, hay muchísima información, obviamente está organizada para utilizar no solamente en la Universidad, sino en el propio Gobierno; para hacer política. Pero a mí me gustaría echar un poquito en cara a la Universidad que dedica muchos de sus esfuerzos a la investigación teórica. En el Instituto llevamos 17 ó 18 años recogiendo información, tenemos una cantidad de información impresionante y creo que se usa muy poco. Se usa muy poco por parte de la Universidad y yo animo desde aquí, a utilizarla más. Quizás sea culpa nuestra; que nosotros no sabemos vendernos; que no sabemos vender nuestra información, pero creo que la Universidad, en concreto las Facultades de Económicas y de Matemáticas, la usan muy poco, y yo creo que el tema los modelos de simulación necesitan datos. Muchas gracias.

Luis Vega, Departamento de Matemáticas, Facultad de Ciencias, UPV/ EHU

¿Cómo es de grave que en gran medida el software sea extranjero?.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Este tema siempre me ha interesado y sobre el que he reflexionado un poco. Concretamente me he preguntado cómo un país como Francia, con una investigación científica y tecnológica muy sólida, y de larga tradición, apenas ha producido software de simulación comercial. Como todos sabéis Francia es un país que tiene un gran nivel en matemáticas, y donde esta disciplina que tiene un enorme prestigio social. Por cierto, ese prestigio social en buena medida viene del hecho de que las matemáticas son la puerta para entrar en las "Grandes Écoles", lo que permite alcanzar los puestos más elevados en la escala social. Es un país donde existe una gran inversión en investigación matemática, en matemática pura, y en matemática aplicada, hay medallas Fields, en fin, es una de las grandes potencias mundiales, probablemente sea el segundo país del mundo. Sin embargo, en este campo de la simulación Francia no ha producido apenas grandes paquetes comerciales; como antes se decía la mayoría de los paquetes son norteamericanos. Por lo tanto, no hay una correlación entre potencial matemático y software, y eso es porque hay otros elementos que juegan, que son de política científica y también culturales. Muchos de los paquetes comerciales de simulación numérica, estoy pensando en paquetes de mecánica de fluidos o paquetes de cálculo estructural con elementos finitos, han surgido de grupos universitarios, de departamentos que en un momento dado desarrollaron una investigación en el campo que les llevó a generar software como un subproducto. En un momento dado, miembros de ese grupo dan un salto y crean una empresa para hacer la comercialización de ese software. Porque, efectivamente, hay un salto cualitativo, no es lo mismo estar desarrollando códigos de cálculo simplemente para ilustrar con ejemplos numéricos las publicaciones en revistas, que hacer un producto que hay que vender. Bien, pues este salto en países como Gran Bretaña o Estados Unidos se ha hecho, pero mucho menos en otros como Frania o España, y yo creo que tiene que ver con una cierta cultura de los países anglosajones que proporciona una mayor permeabilidad entre los ámbitos de lo público y de lo privado. Por eso no se ve mal que miembros de un grupo universitario, creen una empresa y, con todos los mecanismos de transparencia que sean necesarios, acabe comercializando un producto que han desarrollado, en una primera etapa, en el seno de la Universidad. En los años 80, el INRIA, (Instituto Nacional de Informática y Automática de Francia), se planteó potenciar el trasvase de sus resultados de investigación al sector productivo y entonces creó empresas filiales; pero estas empresas no las crearon los investigadores sino el Instituto como tal; fueron creadas "desde arriba" y, en cierto modo, como cualesquiera otras, con participación de sociedades de capital riesgo, de bancos, etc. de manera que se constituyeron como estructuras separadas, sobre todo de los investigadores. Ésta característica hizo que la experiencia no fuera del todo positiva, al menos en el ámbito de la simulación numérica que es el que más conozco de cerca.

El resultado es que, en este momento, un país como Francia, que ha desarrollado una investigación en el campo de la mecánica de fluidos computacional de primera magnitud, donde grupos de investigación muy solventes han desarrollado códigos para AEROSPATIALE, para Avions Marcel Dassault, etc., no ha conseguido competir en el mercado con códigos que han surgido en universidades inglesas o norteamericanas. Después está el problema del mercado. Estos productos requieren desarrollos muy costosos, pero también mantenimiento y actualización porque enseguida resultan obsoletos, y también formación y soporte a los clientes, porque son difíciles de utilizar. Por lo tanto es necesario trabajar con vistas a todo el mercado mundial. Tal es el caso de FUENT, un paquete de mecánica de fluidos norteamericano, que en este momento tiene sus clientes repartidos a partes iguales entre Asia, Europa y América.

Esto que digo es para el mercado de los grandes paquetes de propósito general, pero hay otro mercado que es el de los pequeños paquetes, o mejor el de las aplicaciones a la carta para resolver necesidades concretas de las empresas. Yo creo que los departamentos de las universidades, en particular los de matemáticas, pueden desarrollar una labor interesante en este terreno, haciendo programas de simulación, que no resuelve todos los problemas del mundo, pero que dan una respuesta mucho más ajustada a necesidades concretas.

José Antonio Lozano, Departamento de Ciencias de la Computación e Inteligencia Artificial, Facultad de Informática, UPV/EHU

¿Cuál es la sensación del ponente acerca de la disponibilidad de la empresa a colaborar con la Universidad?. Hacía esta pregunta por lo siguiente, nosotros hemos colaborado con diferentes empresas de mayor o menor tamaño en cuestión de optimización o análisis de datos y la sensación que a uno le queda es que es difícil colaborar con las empresas, en el sentido de que no se las ve muy dispuestas a ello. Parece que tienen un cierto reparo a colaborar con la Universidad y están más orientadas a resolver los problemas del día a día que a trabajar en cosas cuyo beneficio a veces no está claro. Esa es la pregunta más o menos.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Yo creo que en general existe una cierta desconfianza sobre el interés de colaborar con la Universidad, probablemente porque en algunos momentos han podido existir experiencias negativas. Antes comentábamos que ahora en los discursos institucionales, cualquier rector asume que esto de la transferencia tecnológica es un tema muy importante para la Universidad, para su financiación y para la mejora de su imagen en la sociedad; pero esto es relativamente reciente y muchos departamentos todavía no disponen de infraestructura, o bien tienen una excesiva carga de trabajo, docente, investigadora, burocrática, etc., y esto les impide dar una respuesta adecuada a las empresas. Pero creo que la situación está cambiando rápidamente y cada vez es mayor la credibilidad de que gozan entre las empresas algunos grupos universitarios.

Después hay otra cuestión que ya atañe más a las empresas: tengo la impresión de que se camina hacia un modelo en el que las empresas, incluso la grandes, no tengan que tener grandes departamentos de investigación y desarrollo. Incluso en países donde hay una gran tradición de innovación y por tanto de investigación tecnológica se camina en la dirección de encargar la investigación, e incluso el desarrollo, a Institutos y Universidades. De nuevo un ejemplo francés: como sabéis Electricidad de Francia es una empresa estatal de producción y distribución de energía que tiene en su interior un gran centro de investigación cubriendo disciplinas científicas diversas, desde la combustión al electromagnetismo, pasando por la hidráulica. Pues parece que la tendencia es a ir cambiando el modelo, reduciéndolo. Así por ejemplo, el Laboratorio Nacional de Hidráulica, que pertenecía a EDF, ha dejado de existir como tal recientemente. Yo no sé si esto es bueno o malo, pero en todo caso parece que es algo impuesto por las nuevas corrientes de la economía globalizada y está claro que es una tendencia que puede favorecer el desarrollo de la investigación en la Universidad. Aún así es imprescindible, que las empresas destinen a alguna persona para detectar y canalizar sus necesidades de I+D a las Universidades y Centros de Investigación y después asumir la interlocución con los investigadores. Además esta persona debería encargarse de una misión importante para la empresa: conseguir fondos públicos para financiar los proyectos de I+D. Como todos sabéis, tanto el Gobierno Central como las Comunidades Autónomas tienen programas para este fin, aparte de las exenciones fiscales que permiten rebajar la cuota del impuesto de sociedades entre un 30 y un 50 por ciento de los gastos en I+D.

Un caso que responde a este modelo y que está funcionando muy bien, es el de Ferroatlántica. Este grupo constituyó hace algunos años una empresa llamada Ferroatlántica I+D, para desarrollar y comercializar la tecnología que va produciendo en torno a la metalurgia del silicio y de las ferroaleaciones. En particular Ferroatlántica I+D se dedica a mejorar y a comercializar el electrodo ELSA del que os hablé. Para ello establecen relaciones con grupos universitarios con los que contratan trabajos de investigación que tienen por objeto mejorarlo.

Juán José Anza, del Departamento de Matemática Aplicada, Escuela de Ingenieros Industriales y de Telecomunicaciones de Bilbao, UPV/EHU

...en línea con lo que está explicando el profesor Bermúdez, me gustaría poner un ejemplo: sobre los equipos de investigación, la interacción Empresa-Universidad, Centros Tecnológicos, etc. que no solamente tiene que ver con la matemática aplicada, sino probablemente con el problema global de I+D, etc. Tengo una experiencia de 5 años trabajando en un Centro Tecnológico que está hoy aquí representado también y tuve en el 94 una experiencia en un proyecto europeo que voy a contar. Este proyecto europeo era en el seno de la CECA y la empresa que lideraba el proyecto era ARBED, que creo que ahora también está aquí en el País Vasco, y funcionaba de la siguiente manera: ARBED tenía un pequeño departamento de I+D, orientado al campo que abarcaba este proyecto que eran fluidos, estructuras, etc. y en ese departamento de I+D había un jefe y dos ingenieros jóvenes que trabajaban. Entonces, ARBED acudía a los proyectos de la CECA, extraía proyectos que tenían para ellos interés tecnológico, definía las especificaciones, lo que quería sacar de esos proyectos y entonces formaba un equipo donde había Centros Tecnológicos y Universidad. El núcleo fuerte de trabajo lo hacían los Centros Tecnológicos y a la Universidad en parte la tenían como consultores. Es decir, el Centro Tecnológico está organizado más como empresa, tiene más mano de obra, sin embargo, en el día a día muchas veces el Centro Tecnológico no le permite profundizar, eso se hace en las Universidades, en tesis doctorales, etc. A mí me parece que fueron proyectos que funcionaron muy bien, yo no los acabé, y precisamente la persona que los acabó del Centro Tecnológico, está también aquí, y podría, comentar algo al respecto, es Fernando Espiga de LABEIN. Me parece muy importante saber hacer equipos, y esa cultura no existe, no existe en el I+D en España. Desde la Universidad queremos hacer todo, queremos coger un proyecto, le decimos a la empresa que podemos solucionarle cualquier cosa y luego, al final, hay decepción. A veces, los Centros Tecnológicos dicen que son capaces de hacer investigación básica y tampoco. Y hay otro elemento importante que es el tema de inversión, que va un poco en la línea de lo que decía el profesor Bermúdez, al final para la empresa requiere un esfuerzo, si quiere instalar simulación mediante paquetes, tiene que comprar el paquete, tiene que contratar a gente que aprenda a utilizar esos paquetes y que se mantenga ahí, y si además quiere también interlocución con Centros Tecnológicos, con Universidad, etc. pues, esa persona no puede estar cambiando continuamente. Hay poca cultura de inversión I+D en la medida en que progresemos en esta línea es cuando las cosas serán más reales en investigación en España.

Arantza Urkaregi, Departamento Matemática Aplicada, Estadística e Inv. Op., Facultad de Ciencias, UPV/EHU y Miembro del Consejo Social

Estoy de acuerdo en la necesidad de adaptar la formación de los Matemáticos a la aplicación práctica, dado que todo el mundo no pueda saber de todo, pero ¿no crees que sería positivo impulsar equipos de investigación multidisciplinar en función del campo de trabajo?. Por ejemplo matemáticos y médicos o matemáticos, físicos e ingenieros, etc.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Indudablemente, y yo creo que la forma de hacer eso es partiendo de un proyecto, es decir, creo que esa colaboración y la creación de los equipos se puede hacer con relativa facilidad si hay un objetivo concreto desde el principio, si hay un proyecto. Se citaba la colaboración de matemáticos y médicos; es un campo clarísimo, todos los temas de tratamiento de imágenes, de medios de diagnóstico, de la llamada ingeniería biomédica en general, requieren muchos algoritmos de modo que es un campo interdisciplinar para médicos, ingenieros, físicos, matemáticos,...

Arantza Urkaregi, Departamento Matemática Aplicada, Estadística e Inv. Op., Facultad de Ciencias, UPV/EHU y Miembro del Consejo Social

He realizado esta pregunta porque creo que se ha planteado antes la dificultad que hay de formar matemáticos desde el punto de vista práctico. Javier Duoandikoetxea también ha planteado las dificultades que hay incluso entre los propios alumnos que, igual no lo ven, pero es cierto lo que tú planteas. Esos equipos tienen que estar enfocados en un problema concreto. Yo lo digo por propia experiencia, nosotros tenemos un convenio firmado con OSAKIDETZA, para un tema concreto de aplicación de la estadística al campo médico. Entonces, tú misma te vas dando cuenta de las necesidades y de las deficiencias que tienes, pero es un estímulo tanto a nivel de investigación como a nivel de docencia y lo que está claro es que las personas matemáticas tenemos una formación, tú también lo has dicho en tu exposición, que nos permite adaptarnos a un montón de cuestiones. En ese sentido creo puede ir cambiando un poco la formación matemática. Podría ser a través de esos equipos de investigación multidisciplinar que podrían llevarnos a un cambio en nuestra propia docencia. El problema que se ha planteado respecto a la reforma de los planes de estudio es algo más a medio plazo y a veces cuenta con más dificultad.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Estoy completamente de acuerdo con lo que has dicho. Yo introduciría también un elemento nuevo, que conecta con un comentario anterior. La Universidad establece una serie de objetivos recogiendo las misiones que le encarga la sociedad, pero muchas veces no es coherente y me voy a referir a un tema concreto que me parece que está relacionado con lo que tú dices. Se trata del coste que supone para un matemático incorporarse a un equipo multicisciplinar. Un matemático está trabajando en un campo de investigación, probablemente de carácter teórico; está consiguiendo una productividad que plasma en artículos, comunicaciones a congresos, etc. y que le van a permitir su promoción profesional: si se presenta a unas oposiciones de profesor, le van a pedir que tenga un curriculum investigador, que haya publicado en revistas, cuanto más prestigiosas mejor. Y de repente, un buen día se le plantea dar un giro de noventa grados y meterse en un equipo interdisciplinar, cuando eso le supone que va a estar unos años probablemente sin producir artículos. Estoy plenamente convencido que la universidad española tenía un déficit de investigación y por lo tanto ha sido muy importante que durante estos últimos años la investigación se haya primado especialmente. Pero todo es cuestión de equilibrio, de medida, de proporciones. Yo creo que la Universidad va a tener que valorar también estos otros costes que suponen las reconversiones, en este caso de los matemáticos, el que una persona esté durante unos años formándose en otro campo interdisciplinar donde probablemente va a ser muy útil, pero a corto plazo su actividad no se va a traducir en publicaciones que le vayan a permitir ganar unas oposiciones. Yo creo que la Institución, manteniendo con claridad que un profesor universitario debe hacer una investigación de calidad, debe también valorar este otro tipo de tareas. Esto creo que animaría a que, en este caso los matemáticos, se integrasen con más facilidad en otros equipos para los que van a ser muy valiosos.

Pedro Larrea, Presidente del Consejo Social

Nos estamos acercando ya al final, si hay alguna nueva pregunta …

José Antonio Lozano, Departamento de Ciencias de la Computación e Inteligencia Artificial, Facultad de Informática, UPV/EHU

Yo, como vengo de la Facultad de Informática, y aquí se está hablando de matemáticos, de ingenieros, de físicos, tengo que decir que también me gustaría que se tuviese en cuenta a los ingenieros informáticos, que tienen una base matemática bastante importante y en este tipo de grupos interdisciplinares tendrían un papel que jugar, porque el desarrollo software se hace con este tipo de aplicaciones.

Son campos en los que ellos son expertos y la base matemática les confiere también un poder de interrelación con matemáticos, con ingenieros, con físicos o con otra gente con una formación matemática.

Manuel Tello, Decano de la Facultad de Ciencias y Miembro del Consejo Social

Quisiera plantearle al ponente una reflexión final. En la discusión se han puesto de manifiesto dos opciones: primera, trabajo usual con paquetes comerciales; segunda, aplicaciones más avanzadas de los paquetes comerciales. Para la primera parece que ha quedado claro que es suficiente con incorporar la formación adecuada en matemáticas a las titulaciones de ingeniero, físico, etc. Los matemáticos entrarían en la segunda. Sin embargo las ventajas competitivas vienen de opciones nuevas no contempladas en lo que se vende. En los ejemplos de la exposición creí ver algo en este sentido.

Primera pregunta: ¿Podría darnos, en base a su experiencia, su opinión sobre el futuro en esta dirección, así como una comparación con su experiencia internacional?.

Y la segunda es: ¿Puede extender sus comentarios a la posibilidad de desarrollar tecnología avanzada con bajo coste de creación gracias a la aportación de los matemáticos? Por ejemplo, tecnología basada en lo no lineal.

Alfredo Bermúdez de Castro, Director del Departamento de Matemática Aplicada de la Universidad de Santiago de compostela, Facultad de Matemáticas

Sobre la primera parte diría que los paquetes comerciales son productos dinámicos que experimentan continuas mejoras y desarrollos, no sólo para incluir cada vez más modelos sino también nuevos algoritmos: existe una investigación de carácter estrictamente matemático, para mejorar los métodos de cálculo y cualquier paquete que se precie va a seguir vivo y abierto para incorporar todas estas mejoras. Por lo tanto en el desarrollo de los grandes paquetes también hay una contribución de los matemáticos.

Sobre la segunda pregunta creo que algo ya he comentado también en mi charla. La simulación numérica permite desarrollar tecnología con mayor rapidez y menores costes. Por ejemplo una empresa que fabrica volantes recibe un encargo para un nuevo automóvil. La forma tradicional de proceder consiste en hacer un primer diseño que se somete en laboratorio a los ensayos del “cuaderno de cargas”. Si alguno no se satisface es necesario modificar este primer diseño y volver a empezar. Este proceso lleva tiempo y tiene unos costes importantes. Por el contrario, si uno dispone de un modelo del comportamiento mecánico de ese volante, puede someterlo a todos los test del cuaderno de cargas en el ordenador y, si fuese necesario, modificarlo con ayuda de un programa de “diseño asistido por ordenador” (CAD). Por supuesto el que pase los test en el ordenador es suficiente, pues las normas de homologación requieren los ensayos en laboratorio, pero la ventaja es que el número de prototipos se reduce considerablemente.

Pedro Larrea, Presidente del Consejo Social

Muchas gracias a todos. Damos por finalizado el debate. Ahora muy brevemente, Mikel Lezaun nos presentará los actos del Año Mundial de las Matemáticas.

Mikel Lezaun, Director del Departamento de Matemática Aplicada y Estadística e Investigación Operativa, Facultad de Ciencias, UPV/EHU y Coordinador del Comité en el País Vasco para la celebración del Año Mundial de las Matemáticas

Como casi todos ustedes ya saben, este año 2000 es el Año Mundial de las Matemáticas. El inicio de esta conmemoración se remonta a mayo del 92, fecha en la que la Unión Matemática Internacional, reunida en Río, declaró el año 2000 como el Año Mundial de las Matemáticas, con los objetivos de determinar los grandes desafíos matemáticos del siglo XXI, proclamar a las matemáticas como una de las claves fundamentales para el desarrollo e impulsar la presencia sistemática de las matemáticas en la sociedad de la información. A su vez la UNESCO, en su Conferencia General de noviembre de 1997, acordó su apoyo y patrocinio del año 2000 como el Año Mundial de las Matemáticas, señalando además el papel clave de las matemáticas en todos los niveles del sistema educativo.

A comienzos del año pasado, se reunieron en Madrid todas las sociedades matemáticas españolas y tomaron una actitud decidida a utilizar el Año Mundial de las Matemáticas, para dar a conocer las matemáticas a la sociedad. Una de las primeras propuestas fue la aprobación en febrero de un Proyecto no de Ley en el que el Congreso de los Diputados resaltaba la importancia de las matemáticas y apoyaba todos los actos conmemorativos del año 2000. El 22 de enero de este año hubo una Jornada Matemática en el Congreso de los Diputados que fue presidida por D. Federico Trillo y en la que intervino el profesor francés Jacques Louis Lions, que en el año 92 era Presidente de la Unión Matemática Internacional. Por su parte el Senado ha montado una exposición que fue inaugurada por su Presidenta Dª Esperanza Aguirre, titulada “Las Medidas y las Matemáticas. La Introducción del Sistema Métrico Decimal en España” y ha editado en facsímil “El libro de los relojes solares”. Siguiendo con actividades institucionales, el Parlamento de Cataluña, el Parlamento de Galicia, el Parlamento de Valencia y el Parlamento de Andalucía han aprobado declaraciones referentes al Año Mundial de las Matemáticas 2000.

Desde aquí, desde el País Vasco, el año pasado nos reunimos profesores de diferentes departamentos que tienen relación con las matemáticas y convenimos en la necesidad de organizar distintos actos conmemoraticos de este Año Mundial de las Matemáticas. Desde un principio pensamos que en los actos conmemorativos había que desarrollar tres aspectos: la presencia de las matemáticas en la sociedad y sus relaciones con otros sectores de la sociedad, la investigación matemática y por último la enseñanza de las matemáticas.

Dentro del primer apartado, el primer acto es éste que ha organizado el Consejo Social. Tengo que hacer notar que desde el momento en que vinimos a informar al Consejo Social de que éste es el Año Mundial de las Matemáticas, su interés ha sido constante y de ellos ha partido esta iniciativa. También en este apartado, hemos organizado un ciclo de conferencias de carácter divulgativo en la Biblioteca de Bidebarrieta titulado “La irrazonable eficacia de las Matemáticas”. Este ciclo va a empezar el martes y durará cinco martes consecutivos. Con respecto de este ciclo, pensamos que teníamos que ofrecer este marco a personas que hablaran de matemáticas pero que no fueran del mundo académico de las matemáticas. Así, empezaremos con una conferencia de “Matemáticas y Física”, tendremos otra que se titula “El Uso de las Matemáticas en los Mercados Financieros”, una tercera se titula “La Concepción Matemática en la Música del Siglo XX”, otra conferencia será “Algunas Aplicaciones de las Matemáticas a la Ingeniería” y la última se titula “La Eficacia de la Programación

Matemática en el Mundo Empresarial". Está claro que la primera estará dada por un físico, Enrique Alvarez, la segunda la impartirá el responsable de la Mesa de Nuevos Productos del BBVA, Eloy Fontecha, el tercer conferenciante es un compositor y musicógrafo, Carlos Villasol, el cuarto es Enrique Castillo Ron y el último el Director de Sistemas de Apoyo a la Decisión, Laureano Escudero. Tendremos pues cinco conferencias divulgativas con el enfoque que ya he comentado antes. También, dentro de este apartado, se han programado para finales de agosto dos cursos de verano en la Universidad de Verano de San Sebastián que se titulan "Matemáticas en el Mundo Real". Voy a leer varios títulos para que vean la orientación de estos cursos: "Sistemas de reacción-difusión, una clase de modelos matemáticos en biología", "Procesos Estocásticos, ¿realmente son útiles en finanzas?", "La Estadística, problemas y métodos", "Caos en el movimiento del Sistema Solar", "Análisis Numérico, aplicación a problemas reales", "Codificación de la Información".

En el apartado de las matemáticas desde las matemáticas, el acontecimiento más importante que va a haber este año se celebrará en Barcelona y es el tercer Congreso Europeo de Matemáticas, que tendrá lugar el mes de julio. Ya aquí, en Bilbao, va a haber un Congreso muy importante de Geometría Diferencial en memoria de Alfred Gray, que era un profesor de Maryland que estando de visita en la Universidad del País Vasco murió de un infarto. Su viuda quiso que el congreso homenaje se realizara en Bilbao y éste se va a celebrar en septiembre. Hay que resaltar que vienen dos "medallas Fields", que podríamos decir que es el máximo galardón que puede obtener un matemático.

Por último, en lo referente a las matemáticas y la educación, lo vamos a dejar para el comienzo del curso que viene. Dos o tres personas tenemos el compromiso de organizar una o dos jornadas-debate con profesores de enseñanzas medias y también alguna jornada universitaria en el ámbito, por lo menos, de la Facultad de Ciencias. Esta última jornada sería alrededor de San Alberto, que es nuestro patrón, ya que, debido a los exámentes, es más fácil hacerla a principio que al final de curso.

Estos serían, en resumen, los actos que tenemos previstos para conmemorar el Año Mundial de las Matemáticas 2000. Muchas Gracias.

Pedro Larrea, Presidente del Consejo Social

Muchas gracias Mikel. Nos gustaría terminar estos Encuentros con algún resultado tangible, que sea fruto del debate. Para ello, el Consejo Social va a trasladar al Parlamento Vasco la necesidad de que inste al Gobierno, en línea con otras iniciativas adoptadas por otros Parlamentos a los que acaba de hacer referencia Mikel Lezaun, para que favorezca programas de investigación en el ámbito de las matemáticas, sean didácticos o de aplicación científica e industrial, empresarial o tecnológica, y segundo, para que se divulguen las matemáticas en los medios de comunicación de titularidad pública. Por otra parte, a mí me gustaría hacer una reflexión final. Es muy fácil querer trasladar la responsabilidad a las instancias sociales diciendo: "Es que no nos piden, es que están muy distantes de nosotros. No somos todo lo útiles que podríamos ser porque desde el otro lado no hay ningún intento de acercamiento. De la misma manera, este mismo esquema puede operar, y a veces opera, desde las empresas o desde estamentos sociales: "Es que la Universidad no se acerca a nosotros. La Universidad no es consciente de los problemas reales que tiene la sociedad".

Bien, yo creo que una forma, si no la única por lo menos la más expeditiva para solucionar este distanciamiento entre unos y otros, es aproximar la oferta a la demanda. En este sentido, la idea de una Fundación, como la que el Consejo Social viene proponiendo con poco éxito, desde hace seis años, sigue siendo una idea válida.

Sé que en los equipos de las tres candidaturas que pasado mañana compiten por el rectorado, hay sensibilidades muy distintas a este respecto. Desde el Consejo tenemos que ser exquisitamente neutrales ante el proceso pero en cualquier caso, sí os diré que al nuevo equipo rectoral le vamos a plantear la fundación Universidad-Sociedad. Con esto prentendo haceros partícipes de nuestras intenciones. Me dirijo a todos los académicos, pero también a la representación empresarial, para que cuando nosotros tomemos la iniciativa nos apoyéis. Porque repito, creo que es una herramienta realmente útil como se ha demostrado en otras Universidades de Galicia, Madrid o Cataluña.

Para que esta aproximación sea más estrecha, y para que realmente la Universidad aporte más a la sociedad y al mismo tiempo resuelva el problema de financiación que tiene planteado, todas estas iniciativas pueden ser realmente útiles.

Muchas gracias a todos por su participación y por sus aportaciones a este debate.